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Multipoint Boundary Value Problem for the Adjoint Equation and Its Green's Function

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Abstract

Connection between the adjoint family of functions has been defined for the first and the adjoint characteristic equation of the (n-1)-th order of the Riccati type has been studied. The nonhomogeneous multipoint problem for the adjoint differential equation of the n-th order has been solved and the Green's function has been constructed and its new properties have been determined.

Keywords: adjoint multipoint problem, adjoint boundary conditions, Green's function, new properties

1. Introduction

A generalized multipoint problem was not studied in a theory of boundary value problem, i.e. researching of the non-trivial solution of the homogeneous multipoint boundary problem and construction of the generalized Green's function (Bryns, 1971; Levin, 1985; Pokornyi, 1968; Das & Vatsala, 1973; Eloe & Grimm, 1980; Peterson, 1976). An adjoint multipoint problem is not so simple. What multipoint boundary value problem will correspond to the adjoint operator L^+z? Would be there only one adjoint problem and would the linear differential and adjoint operators be Noetherian or Fredholm? These are important questions as well. The multipoint problem with a small parameter in the boundary conditions $(T_k y)(x_i)$ have not been considered, degeneration of the equation order, boundary conditions and correctness of the problem itself have not been studied (Klokov, 1967; Maksimov & Rakhmatullina, 1977).

Among the boundary value problem which are topical due to their different applications, the multipoint problem and problems for the differential equations have been studied least of all when a separate differential equation is set on each segment and solutions of different equations are connected through the boundary conditions. Such statement generalizes usual multipoint problems. The adjoint multipoint problem, uniqueness of solution and Noetherian or Fredholm property of the linear differential and adjoint operators have not been studied yet. The problems where impulse jumps of the function or its derivatives are set non-linearly in the boundary conditions and the problems with a small parameter have not been considered (Samoilenko & Perestyuk, 1987; Azbelev, Maksimov, & Rakhmatullina, 1982; Krall, 1969).

2. Adjoint Family of Functions

Let us consider a linear differentiation operator

$$Ly = \sum_{\nu=0}^{n} b_\nu(x) y^{(\nu)}, \quad b_n(x) \equiv 1, \tag{1}$$

with coefficients $b_\nu(x) \in C^\nu[x_1, x_m]$, $\nu = 0, 1, 2, \dots, n-1$. Let us put operator L^+ adjoint under Lagrange in compliance with operator L, i.e.

$$L^+z = \sum_{\nu=0}^{n} (-1)^\nu [b_\nu(x) z(x)]^{(\nu)}, \quad b_n(x) \equiv 1. \tag{2}$$

Let $\{x_i\}_1^m$ be a partition of segment $[x_1, x_m]$. In (Trenogin & Khasseinov, 1991), the adjoint boundary conditions with corresponding operators of boundary conditions were produced in a more general case for $(T_k y)(x_i)$

$$(T^+_\chi z)(x_i) = \sum_{\nu=0}^{\chi} (-1)^{\chi-\nu}[b_{n-\nu}(x)z(x)]^{(\chi-\nu)} - \rho_{k,n-\chi}(s)\Big|_{x=x_i} = 0\ ,$$

$\chi = 0, 1, 2, \dots, n-1$; $k = 1, 2, \dots, r_i$; $i = 1, 2, \dots, m$, and $\sum_{i=1}^{m} r_i = n$. Then we tried to solve m-point problem for the linear differential Equation (1) with a nonhomogenous part $f(x)$, create a Green's function using adjoint family of functions $\{\varphi_{jl}(x)\}$, $j = 1, 2, \dots, m$; $l = 1, 2, \dots, r_j$ and $\{z_{ik}(x)\}$, $i = 1, 2, \dots, m$; $k = 1, 2, \dots, r_i$.

Let us introduce a linear differentiation operator of the boundary conditions

$$\left(T^+_k z\right)(x) = \sum_{\nu=1}^{n} r_{k\nu}(x) z^{(\nu-1)}(x),$$

where $r_{k\nu}(x) \in C[x_1, x_m]$, $k = 1, 2, \dots, r_i$; $i = 1, 2, \dots, m$.

Assuming, that domain of operator L^+ consists of functions $z(x) \in C^{n-1}[x_1, x_m]$ complying with boundary conditions

$$\left(T^+_k z\right)(x_i) = \sum_{\nu=1}^{n} r_{k\nu}(x_i) z^{(\nu-1)}(x_i) = 0, \tag{3}$$

and coefficients $r_{k\nu}(x)$ comply with the condition of absence of degeneracy in points x_i

$$\sum_{\nu=1}^{n} r^2_{k\nu}(x_i) \neq 0, i = 1, 2, \dots, m.$$

Let us identify connection between the solutions of linear $Ly = 0$ and adjoint equation $L^+ z = 0$ and find boundary conditions for solutions y which are "adjoint" to the conditions (3). Let us create a Green's function of m-point problem for the adjoint equation and solve the corresponding nonhomogenous boundary value problem.

Lemma 1 *Let the following be executed for points $\{x_i\}_1^m$ and fundamental system of solutions $\{z_\nu(x)\}_1^n$ adjoint equation, $L^+ z = 0$*

$$\Delta = \det\left\|\left(T^+_k z_\nu\right)(x_i)\right\| = \begin{vmatrix} (T^+_1 z_1)(x_1) & (T^+_1 z_2)(x_1) & \dots & (T^+_1 z_n)(x_1) \\ (T^+_2 z_1)(x_1) & (T^+_2 z_2)(x_1) & \dots & (T^+_2 z_n)(x_1) \\ \vdots & \vdots & & \vdots \\ (T^+_{r_1} z_1)(x_1) & (T^+_{r_1} z_2)(x_1) & \dots & (T^+_{r_1} z_n)(x_1) \\ \vdots & \vdots & & \vdots \\ (T^+_1 z_1)(x_m) & (T^+_1 z_2)(x_m) & \dots & (T^+_1 z_n)(x_m) \\ (T^+_2 z_1)(x_m) & (T^+_2 z_2)(x_m) & \dots & (T^+_2 z_n)(x_m) \\ \vdots & \vdots & & \vdots \\ (T^+_{r_m} z_1)(x_m) & (T^+_{r_m} z_2)(x_m) & \dots & (T^+_{r_m} z_n)(x_m) \end{vmatrix} \neq 0.$$

Then there is a fundamental system of solutions $\{\psi_{jl}(x)\}$, $j = 1, 2, \dots, m$; $l = 1, 2, \dots, r_j$ of the homogenous adjoint equation $L^+ z = 0$ which results in

$$\left(T^+_k \psi_{jl}\right)(x_i) = \delta_{ij} \cdot \delta_{kl},\ i = 1, 2, \dots, m;\ k = 1, 2, \dots, r_i; \sum_{i=1}^{m} r_i = n. \tag{4}$$

Proof. Since coefficients $b_{\nu-1}(x) \in C^{\nu-1}[x_1, x_m]$, there is a fundamental system of solutions $\{z_i(x)\}_1^n$ for the linear adjoint homogenous equation $L^+ z = 0$. Let us find solutions $\psi_{jl}(x)$ for the adjoint equation that comply with the conditions (4), in a form of

$$\psi_{jl}(x) = C_{1jl} z_1(x) + C_{2jl} z_2(x) + \dots + C_{njl} z_n(x). \tag{5}$$

To make it clear, let us prove the lemma for $j = 1$, $l = 1$. Let us apply an operator of the boundary conditions T_k^+ where $k = 1, 2, \ldots, r_i$ in the corresponding points x_i to function (5). Having considered these expressions together with the boundary conditions (4) and jointly with (5), we produce a system (n+1) of the homogenous linear algebraic equations with respect to $C_1, C_2, \ldots, C_{n-1}$.

$$\begin{cases} C_1 z_1(x) + C_2 z_2(x) + ... + C_n z_n(x) - \psi_{11}(x) & = 0, \\ C_1 \left(T_1^+ z_1\right)(x_1) + C_2 \left(T_1^+ z_2\right)(x_1) + ... + C_n \left(T_1^+ z_n\right)(x_1) - 1 & = 0, \\ C_1 \left(T_2^+ z_1\right)(x_1) + C_2 \left(T_2^+ z_2\right)(x_1) + ... + C_n \left(T_2^+ z_n\right)(x_1) & = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ C_1 \left(T_{r_1}^+ z_1\right)(x_1) + C_2 \left(T_{r_1}^+ z_2\right)(x_1) + ... + C_n \left(T_{r_1}^+ z_n\right)(x_1) & = 0, \\ \quad\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\quad \vdots & \\ C_1 \left(T_1^+ z_1\right)(x_m) + C_2 \left(T_1^+ z_2\right)(x_m) + ... + C_n \left(T_1^+ z_n\right)(x_m) & = 0, \\ C_1 \left(T_2^+ z_1\right)(x_m) + C_2 \left(T_2^+ z_2\right)(x_m) + ... + C_n \left(T_2^+ z_n\right)(x_m) & = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ C_1 \left(T_{r_m}^+ z_1\right)(x_m) + C_2 \left(T_{r_m}^+ z_2\right)(x_m) + ... + C_n \left(T_{r_m}^+ z_n\right)(x_m) & = 0. \end{cases}$$

Let us decompose the determinant by the elements of the last column, since a determinant in the left part is different from zero under conditions of Lemma 1, there is a solution specified

$$\psi_{11}(x) = \frac{\Delta_{11}(x)}{\Delta}, \quad \Delta = \det \left\| \left(T_k^+ z_\nu\right)(x_i) \right\| \neq 0.$$

Similarly, we will find the other solutions of the homogenous adjoint equation $L^+ z = 0$ complying with boundary conditions (4)

$$\psi_{jl}(x) = \frac{\Delta_{jl}(x)}{\Delta}, \Delta \neq 0, j = 1, 2, \ldots, m; l = 1, 2, ..., r_j. \tag{6}$$

Determinants $\Delta_{jl}(x)$ are produced from Δ by substitution of the elements of the line $\left(l + \sum_{\mu=1}^{j-1} r_\mu\right)$ of the fundamental system of solutions $z_1(x), z_2(x), \ldots, z_n(x)$.

Let us prove a linear independence of the function (6).

Having written a homogenous system without proportions (5) for $j = 1, 2, ..., m; l = 1, 2, ..., r_j$, we produce a system consisting of n^2 equations, which can be represented in a matrix form

$$\begin{pmatrix} (T_1^+ z_1)(x_1) & \ldots & (T_1^+ z_n)(x_1) \\ \vdots & & \vdots \\ (T_{r_1}^+ z_1)(x_1) & \ldots & (T_{r_1}^+ z_n)(x_1) \\ \vdots & & \vdots \\ (T_1^+ z_1)(x_m) & \ldots & (T_1^+ z_n)(x_m) \\ \vdots & & \vdots \\ (T_{r_m}^+ z_1)(x_m) & \ldots & (T_{r_m}^+ z_n)(x_m) \end{pmatrix} \cdot \begin{pmatrix} C_{1_{11}} & \ldots & C_{1_{1r_1}} & \ldots & C_{1_{m1}} & \ldots & C_{1_{mr_m}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ C_{n_{11}} & \ldots & C_{n_{1r_1}} & \ldots & C_{n_{m1}} & \ldots & C_{n_{mr_m}} \end{pmatrix} = \mathrm{E},$$

where E is an identity matrix $n \times n$.

Due to condition $\Delta \neq 0$, it follows that $\det \left\| C_{s_{jl}} \right\| \neq 0, \ s = 1, 2, \ldots, n$.

Let us differentiate ($n-1$) times the functions (5) and proportions produced at $j = 1, 2, \ldots, m; \ l = 1, 2, \ldots, r_j$ and write it in the following way

$$\begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) & \dots & \psi_{1r_1}(x) & \cdots & \psi_{m1}(x) & \psi_{m2}(x) & \dots & \psi_{mr_m}(x) \\ \psi'_{11}(x) & \psi'_{12}(x) & \dots & \psi'_{1r_1}(x) & \cdots & \psi'_{m1}(x) & \psi'_{m2}(x) & \dots & \psi'_{mr_m}(x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \psi^{(n-1)}_{11}(x) & \psi^{(n-1)}_{12}(x) & \dots & \psi^{(n-1)}_{1r_1}(x) & \cdots & \psi^{(n-1)}_{m1}(x) & \psi^{(n-1)}_{m2}(x) & & \psi^{(n-1)}_{mr_m}(x) \end{pmatrix} =$$

$$= \begin{pmatrix} z_1(x) & z_2(x) & \dots & z_n(x) \\ z'_1(x) & z'_2(x) & \dots & z'_n(x) \\ \vdots & \vdots & & \vdots \\ z^{(n-1)}_1(x) & z^{(n-1)}_2(x) & \dots & z^{(n-1)}_n(x) \end{pmatrix} \cdot \begin{pmatrix} C_{1_{11}} & \dots & C_{1_{1r_1}} & \dots & C_{1_{m1}} & \dots & C_{1_{mr_m}} \\ C_{2_{11}} & \dots & C_{2_{1r_1}} & \dots & C_{2_{m1}} & \dots & C_{2_{mr_m}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ C_{n_{11}} & \dots & C_{n_{1r_1}} & \dots & C_{n_{m1}} & \dots & C_{n_{mr_m}} \end{pmatrix}.$$

Then Wronskian for the function $\{\psi_{jl}(x)\}$ is different from zero, since $W(\psi_{11},...,\psi_{1r_1},...,\psi_{m1},...,\psi_{mr_m}) = W(z_1, z_2, ..., z_n)\det\left\|C_{s_{jl}}\right\| \neq 0$, and it proves linear independence of the function $\{\psi_{jl}(x)\}$.

Now, let us identify functions $\{y_{ik}(x)\}, i = 1, 2, \dots, m;\ k = 1, 2, \dots, r_i$ as a solution to the algebraic system of equations

$$\begin{cases} \psi_{11}(x)y_{11}(x) + ... + \psi_{1r_1}(x)y_{1r_1}(x) + ... + \psi_{m1}(x)y_{m1}(x) + ... + \psi_{mr_m}(x)y_{mr_m}(x) = 0, \\ (b_2\psi_{11})'y_{11}(x) + ... + (b_2\psi_{1r_1})'y_{1r_1}(x) + ... + (b_2\psi_{m1})'y_{m1}(x) + ... + (b_2\psi_{mr_m})'y_{mr_m}(x) = 0, \\ \dots \\ (b_{n-1}\psi_{11})^{(n-2)}y_{11}(x) + ... + (b_{n-1}\psi_{1r_1})^{(n-2)}y_{1r_1}(x) + ... + (b_{n-1}\psi_{m1})^{(n-2)}y_{m1}(x) + ... + (b_{n-1}\psi_{mr_m})^{(n-2)}y_{mr_m}(x) = 0, \\ \psi^{(n-1)}_{11}y_{11}(x) + ... + \psi^{(n-1)}_{1r_1}y_{1r_1}(x) + ... + \psi^{(n-1)}_{m1}y_{m1}(x) + ... + \psi^{(n-1)}_{mr_m}y_{mr_m}(x) = (-1)^n. \end{cases} \quad (*)$$

To make it simple, let us use a proportion

$$\begin{vmatrix} \psi_{11} & \dots & \psi_{1r_1} & \dots & \psi_{m1} & \dots & \psi_{mr_m} \\ (b_2\psi_{11})' & \dots & (b_2\psi_{1r_1})' & \dots & (b_2\psi_{m1})' & \dots & (b_2\psi_{mr_m})' \\ \vdots & & \vdots & & \vdots & & \vdots \\ (b_{(n-1)}\psi_{11})^{(n-2)} & \dots & (b_{(n-1)}\psi_{1r_1})^{(n-2)} & \dots & (b_{(n-1)}\psi_{m1})^{(n-2)} & \dots & (b_{(n-1)}\psi_{mr_m})^{(n-2)} \\ \psi^{(n-1)}_{11} & \dots & \psi^{(n-1)}_{1r_1} & \dots & \psi^{(n-1)}_{m1} & \dots & \psi^{(n-1)}_{mr_m} \end{vmatrix} =$$

$$= b_2 \cdot b_3 \cdot ... \cdot b_{n-1} W(\psi_{11}, ..., \psi_{mr_m}),$$

which is easy proved based on the consequent use of the determinant properties. Then it follows that determinant of the system is different from zero. Moreover, it is not difficult to show that system (*) is equivalent to the following system in brief

$$\begin{cases} \sum_{i=1}^{m}\sum_{k=1}^{r_i} y_{ik}(x)\dfrac{d^s\psi_{ik}(x)}{dx^s} = 0, \quad s = 0, 1, 2, \dots, n-2, \\ \sum_{i=1}^{m}\sum_{k=1}^{r_i} y_{ik}(x)\dfrac{d^{n-1}\psi_{ik}(x)}{dx^{n-1}} = (-1)^n. \end{cases} \quad (8)$$

Solving the system by the Cramer method and writing determinant by the elements of the column with number $\left(k + \sum_{\mu=1}^{i-1} r_\mu\right) = p$, we obtain

$$y_{ik}(x) = \frac{(-1)^{2n+p}}{W(x_i)} \begin{vmatrix} \psi_{11}(x) & \dots & \psi_{ik-1}(x) & \psi_{ik+1}(x) & \dots & \psi_{mr_m}(x) \\ \psi'_{11}(x) & \dots & \psi'_{ik-1}(x) & \psi'_{ik+1}(x) & \dots & \psi'_{mr_m}(x) \\ \vdots & & \vdots & \vdots & & \vdots \\ \psi^{(n-3)}_{11}(x) & \dots & \psi^{(n-3)}_{ik-1}(x) & \psi^{(n-3)}_{ik+1}(x) & \dots & \psi^{(n-3)}_{mr_m}(x) \\ \psi^{(n-2)}_{11}(x) & \dots & \psi^{(n-2)}_{ik-1}(x) & \psi^{(n-2)}_{ik+1}(x) & \dots & \psi^{(n-2)}_{mr_m}(x) \end{vmatrix} e^{-\int_{x_i}^{x} b_{n-1}(t)dt}. \quad (9)$$

Bilinear Lagrange form

$$\Phi(y,z)=\sum_{\nu=1}^{n}\sum_{q=0}^{\nu-1}(-1)^{\nu-1-q}y^{(q)}[b_\nu(x)z]^{(\nu-1-q)},\, b_n(x)\equiv 1$$

in work (Trenogin & Khasseinov, 1987), through the complicated and crockish manipulation, is brought to the linear forms with respect to $y(x)$ and $z(x)$ and their derivatives

$$\Phi(y,z)=\sum_{\chi=0}^{n-1}y^{(n-1-\chi)}(x)\cdot\left(T_\chi^+ z\right)(x), \tag{10}$$

where

$$\left(T_\chi^+ z\right)(x)=\sum_{\nu=0}^{\chi}(-1)^{\chi-\nu}[b_{n-\nu}(x)z(x)]^{(\chi-\nu)}, \tag{10*}$$

$\chi=0,1,\dots,n-1;\; b_n(x)\equiv 1$, and

$$\Phi(z,y)=\sum_{\chi=0}^{n-1}z^{(n-1-\chi)}(x)\cdot\left(T_\chi y\right)(x), \tag{11}$$

where

$$\left(T_\chi y\right)(x)=\sum_{\nu=n-\chi}^{n}\sum_{\substack{p+q=\nu-1\\ p\geq n-1-\chi,q\geq 0}}(-1)^p C_p^{n-1-\chi}b_\nu^{(p-n+1+\chi)}(x)y^{(q)}(x). \tag{11*}$$

Connections (10) and (11) are interesting, first of all, because they can help to produce adjoint conditions for the boundary conditions $(Ty)(x_i)=0$

$$\left(T_\chi^+ z\right)(x)=\rho_{n-\chi}(x_i)$$

and vice versa.

Lemma 2 *Let us assume that* $\{\psi_{jl}(x)\},\ j=1,2,\dots,m;\ i=1,2,\dots,r_j$ *is a fundamental system of solutions of the equation* $L^+z=0$*, like in Lemma 1, and* $\{y_{ik}(x)\},\ i=1,2,\dots,m;\ k=1,2,\dots,r_i$ *is a system of functions specified by the formula (9). Then:*

A. $\{y_{ik}(x)\}$ *is a fundamental system of solutions for the linear differential equation*

$$Ly=0,\quad \forall(x_\mu,x_{\mu+1}),\ \mu=1,2,\dots,m-1,$$

B. *Functions* $y_{ik}(x)$ *comply with the boundary conditions*

$$\left(T_k y\right)(x_i)=\sum_{\nu=n-\chi}^{n}\sum_{\substack{p+q=\nu-1\\ p\geq n-1-\chi,q\geq 0}}(-1)^p C_p^{n-1-\chi}b_\nu^{(p-n+1+\chi)}(x)y^{(q)}(x)\Big|_{x=x_i}=r_{k,n-\chi}(x_i), \tag{12}$$

$\chi=0,1,\dots,n-1;\ b_n(x)\equiv 1.$

C. *The following proportions are true*

$$\Phi[\psi_{jl}(x),y_{ik}(x)]=\delta_{ij}\cdot\delta_{kl}\quad \forall x\in[x_1,x_m], \tag{13}$$

$i,j=1,2,\dots,m;\ k=1,2,\dots,r_i;\ l=1,2,\dots,r_j,\ \sum_{i=1}^m r_i=n.$

Proof. For brevity we prove only the third part **C** of the lemma. The bilinear form with respect to $\psi_{jl}(x),\psi'_{jl}(x)$, ..., $\psi_{jl}^{(n-1)}(x)$ and $y_{ik}(x_i),y'_{ik}(x_i),\dots,y_{ik}^{(n-1)}(x_i)$ is equal to the constant

$$\Phi[\psi_{jl}(x),y_{ik}(x)]=const,\qquad \forall x\in[x_1,x_m].$$

Actually, as we can see from the Lagrange identity, its derivative is equal to zero

$$\frac{d}{dx}\Phi[\psi_{jl}(x),y_{ik}(x)]=\psi_{jl}(x)Ly_{ik}-y_{ik}L^+\psi_{jl}=0,$$

because $Ly_{ik}(x) = 0$ and $L^{+}\psi_{jl}(x) = 0$.

Therefore, to prove the proportions (13), it is enough to show that the following is true for the points $x_i \in [x_1, x_m]$

$$\Phi[\psi_{jl}(x_i), y_{ik}(x_i)] = \delta_{ij} \cdot \delta_{kl}, \tag{13*}$$

$i, j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, r_i$; $l = 1, 2, \ldots, r_j$.

Let us take the bilinear form $\Phi(z, y)$ (11) for the functions $z = \psi_{jl}(x)$, $y = y_{ik}(x)$ and write it at $x = x_i$:

$$\Phi[\psi_{jl}(x_i), y_{ik}(x_i)] = \sum_{\chi=0}^{n-1} \psi_{jl}^{(n-1-\chi)}(x_i) \cdot \left(T_{\chi} y_{ik}\right)(x_i).$$

Then due to (12), we have

$$\Phi[\psi_{jl}(x_i), y_{ik}(x_i)] = \sum_{\chi=0}^{n-1} r_{k,n-\chi}(x_i)\psi_{jl}^{(n-1-\chi)}(x_i) = \left(T_{k}^{+}\psi_{jl}\right)(x_i) = \delta_{ij} \cdot \delta_{kl},$$

which demonstrates that (13*) is true, thereby, the proportions (13) are true as well. Lemma is completely proved.

It results from the equality

$$\Phi[\psi_{jl}(x), y_{ik}(x)] = 0, \ \forall x \in [x_1, x_m] \text{ at } i \neq j \text{ or } k \neq l$$

that family of functions $\{\psi_{jl}(x)\}$ and $\{y_{ik}(x)\}$ at $i \neq j$ or $k \neq l$ are adjoint.

2. Adjoint Characteristic Equation of the $(n-1)$-th Order of the Riccati Type

Let us consider the characteristic equation of the $(n-1)$-th order of the Riccati type

$$R_{n-1}(x) = \sum_{\nu=1}^{n} b_{\nu}(x)[p + r(x)]^{\nu-1} \cdot r(x) + b_0(x) = 0, \tag{14}$$

where $b_{\nu}(x) \in C[x_1, x_m], \nu = 0, 1, ..., n-1; b_n(x) \equiv 1$.

Here $[p + r(x)]^k \cdot r(x)$ means consequent application of the operator $[p+r(x)]$, $p = \frac{d}{dx}$ to the function $r(x)$ used k-times. If $r_i(x)$ is smooth or impaired solutions of the Equation (14), the partial solutions of the linear differential equation

$$L_n(y) = \sum_{\nu=0}^{n} b_{\nu}(x) y^{(\nu)} = 0 \tag{15}$$

are represented in a following form

$$y_i(x) = e^{\int_{x_0}^{x} r_i(t)dt}, x_0 \in \Omega = [x_1, x_n] \backslash \Sigma.$$

And if the following condition is met for $r_i(x)$

$$D(x) = \begin{vmatrix} 1 & \ldots & 1 \\ r_1(x) & \ldots & r_n(x) \\ [p + r_1(x)] \cdot r_1(x) & \ldots & [p + r_n(x)] \cdot r_n(x) \\ \vdots & & \vdots \\ [p + r_1(x)]^{n-2} \cdot r_1(x) & \ldots & [p + r_n(x)]^{n-2} \cdot r_n(x) \end{vmatrix} \neq 0, \quad \forall x \in \Omega,$$

where $\Sigma = \{c_k \in [x_1, x_n]\colon\ y_i(c_k) = 0, i = 1, 2, \ldots, n\}$, then $\{y_i(x)\}_1^n$ are a fundamental system of solutions of the equation $L_n y = 0$.

Studying the characteristic equation of the Riccati type (14), we have managed to produce some new results. In particular, we have defined the multiple solutions of the Equation (14) and described the fundamental system of solutions. We have also developed an algebraic way of solving one class of linear differential equation of the$n$-th order with floating factors and found conditions for exponential solution existence. We have introduced a concept of the reciprocal linear differential equation of random order and studied its properties. It is proved in (Khasseinov, 1984) that the shift formula keeps its form for the exponent $\exp \int_{x_0}^{x} r(t)dt$, i.e.

$$L_n(p, x) = e^{\int_{x_0}^{x} r(t)dt} \cdot f(x) = e^{\int_{x_0}^{x} r(t)dt} \cdot L_n[p + r(x), x] \cdot f(x),$$

where $L_n(p, x) = \sum_{\nu=0}^{n} b_\nu(x)p^\nu$; $f(x) \in C^n[x_1, x_n]$.

Let us put the differentiation operator L^+ adjoint by Lagrange in compliance with operator Ly, i.e.

$$L^+_n z = \sum_{\nu=0}^{n} (-1)^\nu [b_\nu(x)z]^{(\nu)}; \quad b_n(x) \equiv 1 \tag{16}$$

with coefficients $b_\nu(x) \in C^\nu[x_1, x_n]$, $\nu = 0, 1, \ldots, n-1$.

To solve the adjoint multipoint boundary problem and other problems, it is necessary to find the fundamental system of solutions $\{z_i(x)\}_1^n$ of the homogenous adjoint differential equation $L_n^+ z = 0$. To produce a corresponding linear differential equation of the Riccati type for the adjoint Equation (16), similarly to the linear differential Equation (15), we try to find a solution in the following form

$$z(x) = e^{\int_{x_0}^{x} u(t)dt}.$$

Let us use the n-derivative formula

$$p^n e^{\int_{x_0}^{x} r(t)dt} \cdot f(x) = e^{\int_{x_0}^{x} r(t)dt} \cdot [p + r(x)]^n \cdot f(x),$$

then we have

$$L^+_n z = \sum_{\nu=0}^{n} (-1)^\nu p^\nu [z(x)b_\nu(x)] = \sum_{\nu=0}^{n} (-1)^\nu p^\nu e^{\int_{x_0}^{x} u(t)dt} \cdot b_\nu(x) = e^{\int_{x_0}^{x} u(t)dt} \cdot \sum_{\nu=0}^{n} (-1)^\nu [p + u(x)]^\nu \cdot b_\nu(x) = 0.$$

Therefore, we will have the adjoint characteristic equation of the $(n-1)$-th order of the Riccati type

$$R^+_{n-1}(u) = \sum_{\nu=0}^{n} (-1)^\nu [p + u(x)]^\nu \cdot b_\nu(x) = 0, \tag{17}$$

here

$$[p + u]b_\nu(x) = b_\nu(x)u + b'_\nu(x),$$
$$[p + u]b_n(x) = (p + u)1 = u(x).$$

Theorem *Let $u_i(x), i = 1, 2, \ldots, n$ be the impaired solutions around points $\Sigma = \{c_k \in [x_1, x_n]: u_i(c_k) = 0\}$ of the adjoint characteristic equation of the Riccati type (17) $D[u(x)] \neq 0 \ \ \forall x \in \Omega = [x_1, x_n]\backslash\Sigma$.*

Then functions

$$z_i(x) = e^{\int_{x_0}^{x} u_i(t)dt}, x_0 \in \Omega, \ i = 1, 2, ..., n,$$

are a fundamental system of solutions of the adjoint differential equation $L^+_n z = 0$ (16), and the inverse is also true.

Thus, to find fundamental system of solutions of the equation $L^+_n z = 0$, it is sufficient to find the impaired solutions of the adjoint characteristic equation of the Riccati type $R^+_{n-1}(u) = 0$.

3. Solution of the Multipoint Problem for the Adjoint Equation, Green's Function and Its New Properties

Let us assume that we need to solve a nonhomogenous m-point problem for the adjoint differential equation

$$L^+ z = F(s), \tag{18}$$

where $F(s) \in C[x_1, x_m], b_{\nu-1}(s) \in C^{\nu-1}[x_1, x_m], \nu = 1, 2, ..., n$, with boundary conditions

$$\left(T^+_k z\right)(x_i) = \sum_{\nu=1}^{n} r_{k\nu}(x_i) z^{(\nu-1)}(x_i) = A_{ik}, \tag{19}$$

here $i = 1, 2, ..., m; k = 1, 2, ..., r_i, \sum_{i=1}^{m} r_i = n$. Regarding the task it is true.

Theorem 1 *Let $\{\psi_{jl}(x)\}$ and $\{y_{ik}(x)\}$ be a system of functions specified in Lemmas 1, 2. Then there is the unique solution to the problem (18)-(19)*

$$z(s) = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\psi_{ik}(s) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}(s) \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx. \tag{20}$$

Proof. Let us multiply the terms (20) by the coefficient $b_1(s)$ and find a derivative $(b_1 z)'$

$$(b_1(s)z)' = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(b_1\psi_{ik}(s))' + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_1\psi_{ik}(s))' \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx + F(s)b_1(s) \cdot \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}(s)y_{ik}(s).$$

Since functions $\{y_{ik}(x)\}$ are defined as solutions of the algebraic Equations (7) or (8), the last double sum is equal to zero due to the first equation of this system, then

$$(b_1 z)' = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(b_1\psi_{ik}(s))' + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_1\psi_{ik}(s))' \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx.$$

Let us find the second derivative $(b_2 z)''$. To do that, we will take $(b_2 z)'$ instead of the previous expression $(b_1 z)'$ and differentiate it once

$$(b_2 z)'' = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(b_2\psi_{ik}(s))'' + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_2\psi_{ik}(s))'' \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx + F(s) \cdot \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_2\psi_{ik}(s))'\, y_{ik}(s).$$

The last double sum is equal to zero due to the second equation of the system (*). Ergo

$$(b_2 z)'' = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(b_2\psi_{ik}(s))'' + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_2\psi_{ik}(s))'' \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx.$$

Similarly, at $(n-2)$-differentiation time, taking into account the last but one equation of the system (*), we have

$$(b_{n-1} z)^{(n-1)} = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(b_{n-1}\psi_{ik}(s))^{(n-1)} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (b_{n-1}\psi_{ik}(s))^{(n-1)} \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx.$$

Setting $b_{n-1}(x) \equiv 1$, we have

$$z^{(n-1)} = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\psi_{ik}^{(n-1)}(s) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}^{(n-1)}(s) \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx.$$

Let us differentiate this expression

$$z^{(n)} = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\psi_{ik}^{(n)}(s) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}^{(n)}(s) \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx + F(s) \cdot \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}^{(n-1)}(s)y_{ik}(s).$$

Here the third double sum is the last equation of the system (*) and is equal to $(-1)^n$, therefore,

$$z^{(n)} = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\psi_{ik}^{(n)}(s) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \psi_{ik}^{(n)}(s) \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx + (-1)^n \cdot F(s).$$

By placing the derivatives found (multiplied by (-1) in corresponding degrees) on the left part of the adjoint nonhomogenous Equation (18), we have

$$\begin{aligned}
(L^{+}z)(s) &= (-1)^n z^{(n)} + (-1)^{n-1}(b_{n-1}z)^{(n-1)} + ... + (-1)^2(b_2 z)'' - (b_1 z)' + b_0 z \\
&= \textstyle\sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}[(-1)^n\psi_{ik}^{(n)} + (-1)^{n-1}(b_{n-1}\psi_{ik})^{(n-1)} + ... + (-1)^2(b_2\psi_{ik})'' - (b_1\psi_{ik})' + b_0\psi_{ik}] \\
&\quad + \textstyle\sum_{i=1}^{m}\sum_{k=1}^{r_i}[(-1)^n\psi_{ik}^{(n)} + (-1)^{n-1}(b_{n-1}\psi_{ik})^{(n-1)} + ... + (-1)(b_1\psi_{ik})' + b_0\psi_{ik}] \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx \\
&\quad + (-1)^{2n}F(s) \\
&= \textstyle\sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\,(L^{+}\psi_{ik})(s) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} (L^{+}\psi_{ik})(s) \cdot \int_{x_i}^{s} F(x)y_{ik}(x)dx + F(s) \equiv F(s),
\end{aligned}$$

because $\{\psi_{ik}(s)\}$ are the solutions of the homogenous adjoint equation $L^+\psi_{ik}(s) = 0$. Therefore, function $z(s)$ designated by the formula (20) is a solution of the nonhomogenous adjoint differential Equation (18).

Let us show that solution (20) complies with boundary conditions (19). Let us apply the operator of the boundary conditions T_l^+ to the function (20)

$$\begin{aligned}\left(T_l^+ z\right)(s) &= \textstyle\sum_{\nu=1}^n r_{l\nu}(s) z^{(\nu-1)} \\ &= \textstyle\sum_{\nu=1}^n r_{l\nu}(s)\cdot\left[\sum_{i=1}^m\sum_{k=1}^{r_i} A_{ik}\psi_{ik}^{(\nu-1)}(s) + \sum_{i=1}^m\sum_{k=1}^{r_i}\psi_{ik}^{(\nu-1)}(s)\cdot\int_{x_i}^s F(x)y_{ik}(x)dx\right] \\ &= \textstyle\sum_{i=1}^m\sum_{k=1}^{r_i} A_{ik}\sum_{\nu=1}^n r_{l\nu}(s)\psi_{ik}^{(\nu-1)}(s) + \sum_{i=1}^m\sum_{k=1}^{r_i}\sum_{\nu=1}^n r_{l\nu}(s)\psi_{ik}^{(\nu-1)}(s)\cdot\int_{x_i}^s F(x)y_{ik}(x)dx \\ &= \textstyle\sum_{i=1}^m\sum_{k=1}^{r_i} A_{ik}\left(T_l^+\psi_{ik}\right)(s) + \sum_{i=1}^m\sum_{k=1}^{r_i}\left(T_l^+\psi_{ik}\right)(s)\cdot\int_{x_i}^s F(x)y_{ik}(x)dx.\end{aligned}$$

Let us consider this expression at points $x_j,\ j = 1, 2, \ldots, m$

$$\left(T_l^+ z\right)(x_j) = \sum_{i=1}^m\sum_{k=1}^{r_i} A_{ik}\left(T_l^+\psi_{ik}\right)(x_j) + \sum_{i=1}^m\sum_{k=1}^{r_i}\left(T_l^+\psi_{ik}\right)(x_j)\cdot\int_{x_i}^{x_j} F(x)y_{ik}(x)dx.$$

When $j \neq i$ and $l \neq k$, it results in the following, based on the boundary conditions (4)

$$\left(T_l^+\psi_{ik}\right)(x_j) = 0,$$

and therefore, terms of the sum will remain on the right only at j=i and l=k, i.e.

$$\left(T_k^+ z\right)(x_i) = A_{ik}\left(T_k^+\psi_{ik}\right)(x_i) + \left(T_k^+\psi_{ik}\right)(x_i)\cdot\int_{x_i}^{x_i} F(x)y_{ik}(x)dx.$$

Since $\left(T_k^+\psi_{ik}\right)(x_i) = 1$ based on (4), and an integral with identical lower and upper limits are equal to zero, the second addend is also transformed to zero.

Then finally we get

$$\left(T_k^+ z\right)(x_i) = A_{ik}, i = 1, 2, ..., m; k = 1, 2, ..., r_i,$$

and it is congruent with the boundary conditions (21). Theorem is proved.

In conclusion, we have a formula of solution to the nonhomogenous adjoint m-point problem through the preset fundamental system of solutions $z_1(s), z_2(s), z_3(s), ..., z_n(s)$ of the homogenous adjoint equation $L^+z = 0$. This form can be helpful for practical use.

Similarly to the Householder method (Householder, 1956), it is not difficult to find out that

$$y_{ik}(x) = \frac{W_n\left[\left(T_k^+ z_r\right)(x_i), x\right]}{W(x)}, i = 1, 2, ..., m; k = 1, 2, ..., r_i, \tag{21}$$

and in the last line of the determinant within the numerator, we take values of points $\left(T_k^+ z_r\right)(x_i)$ for fundamental system of solutions $z_1(s), z_2(s), ..., z_n(s)$ instead of $\{z_k(x)\}_1^n$.

It should be noted that the functions $y_{ik}(x)$ have the properties proved in Lemma 2.

Similarly to Khasseinov (1984), we can produce a formula of solution to the nonhomogenous multipoint problem (18), (19) through the fundamental system of solutions $z_1(s), z_2(s), \ldots, z_n(s)$

$$z(s) = \sum_{i=1}^m\sum_{k=1}^{r_i} A_{ik}\psi_{ik}(s) + \sum_{i=1}^m\sum_{k=1}^{r_i}\psi_{ik}(s)\cdot\int_{x_i}^s F(x)\frac{W_n\left[\left(T_k^+ z_r\right)(x_i), x\right]}{W(x)}dx.$$

4. Green's Function of the Multipoint Problem for the Adjoint Equation and Its New Properties

By analogy with definition (Yerugin, 1974; Kiguradze, 1987; Lando, 1969; Levin, 1961; Jackson, 1977) of the Green function for the adjoint differential operator L^+z and boundary conditions

$$\left(T_k^+ z\right)(x_i) = A_{ik}, i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, r_i,\ \sum_{i=1}^m r_i = n,$$

Let us name a function of two variables $G^{+}(x,s)$ complying with the following conditions:

1) Derivatives $\dfrac{\partial^{r} G^{+}(x,s)}{\partial s^{r}}$, $r = 0, 1, 2, \ldots, n-2$ are uninterrupted by variables s, x in the entire area $x_1 \le s, x \le x_m$ except the lines $x = x_\mu, \mu = 2, 3, \ldots, m-1$.

2) Derivative $\dfrac{\partial^{n-1} G^{+}(x,s)}{\partial s^{n-1}}$ is uninterrupted by variables s, x at $x \ne x_\mu$. Besides, function $G^{+}(x,s)$ and its derivatives under s to $(n-2)$ at $s = x$ uninterrupted, $(n-1)$-derivative has a saltus equal to $(-1)^n$, i.e.

$$\begin{array}{lcll} G^{+}(x,x+0) & - & G^{+}(x,x-0) & = \ 0 \\ \dfrac{\partial G^{+}(x,x+0)}{\partial s} & - & \dfrac{\partial G^{+}(x,x-0)}{\partial s} & = \ 0 \\ \ldots & & \ldots & \quad\ \ldots \\ \dfrac{\partial^{n-2} G^{+}(x,x+0)}{\partial s^{n-2}} & - & \dfrac{\partial^{n-2} G^{+}(x,x-0)}{\partial s^{n-2}} & = \ 0 \\ \dfrac{\partial^{n-1} G^{+}(x,x+0)}{\partial s^{n-1}} & - & \dfrac{\partial^{n-1} G^{+}(x,x-0)}{\partial s^{n-1}} & = \ (-1)^n. \end{array} \tag{22}$$

3) $G^{+}(x,s)$ under the variable s complies with the homogenous adjoint equation $L^{+}G^{+}(x,s) = 0$ at $x \ne x_\mu$, $x \ne s$.

4) At $x \ne x_\mu$, $G^{+}(x,s)$ complies with homogenous boundary conditions

$$\left(T_k^{+} G^{+}\right)(x, x_i) = 0, i = 1, 2, ..., m.$$

Let us show that the Green's function of the m-point boundary value problem for the adjoint differential equation actually exists and it can be useful to solve the nonhomogenous boundary value problem (18), (19). Creating the Green's function of the m-point problem for the adjoint equation, let us use an attribute of the adjoint operators (Trenogin, 1980), i.e. the variables shall be changed in the course of the function creating.

Let us consider the function at $x_\mu \le s \le x_{\mu+1}$

$$G^{+}(x,s) = \begin{cases} \sum_{l=1}^{r_1} \psi_{1l}(s) y_{1l}(x) & x_1 \le x < x_2 \\ \sum_{l=1}^{r_1} \psi_{1l}(s) y_{1l}(x) + \sum_{l=1}^{r_2} \psi_{2l}(s) y_{2l}(x), & x_2 \le x < x_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ \sum_{l=1}^{r_1} \psi_{1l}(s) y_{1l}(x) + \ldots + \sum_{l=1}^{r_\mu} \psi_{\mu l}(s) y_{\mu l}(x), & x_\mu \le x \le s \\ -\sum_{l=1}^{r_{\mu+1}} \psi_{\mu+1 l}(s) y_{\mu+1 l}(x) - \ldots - \sum_{l=1}^{r_m} \psi_{ml}(s) y_{ml}(x), & s \le x \le x_{\mu+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ -\sum_{l=1}^{r_{m-1}} \psi_{m-1 l}(s) y_{m-1 l}(x) - \sum_{l=1}^{r_m} \psi_{ml}(s) y_{ml}(x), & x_{m-2} < x \le x_{m-1} \\ -\sum_{l=1}^{r_m} \psi_{ml}(s) y_{ml}(x), & x_{m-1} < x \le x_m \end{cases} \tag{23}$$

where $\mu = 1, 2, \ldots, m-1$.

Here the linearly independent functions $\psi_{jl}(s)$ comply with a homogenous adjoint equation $L^{+}z(s) = 0$ and boundary conditions $\left(T_k^{+}\psi_{jl}\right)(x_i) = \delta_{ij} \cdot \delta_{kl}$, and $y_{ik}(x)$ complies with the homogenous linear equation $Ly = 0$ and "adjoint" boundary conditions (10). On the issue of the problem, it is true

Theorem 1 *Let coefficients of the equation (18) be* $b_{\nu-1}(s) \in C^{\nu-1}[x_1, x_m]$, $\upsilon = 1, 2, \ldots, n$, *the right part F(s) is uninterrupted at* $[x_1,x_m]$ *and* $\{\psi_{jl}(s)\}$, $\{y_{ik}(x)$*–a system of functions specified in Lemmas 1, 2. Then* $G^{+}(x,s)$ *(25) is the Green's function of the m- point problem for the adjoint Equations (18), (19).*

Proof. We will try to find the Green's function of the m-point boundary value problem for $x_\mu \le s \le x_{\mu+1}$,

$\mu = 1, 2, \ldots, m-1$ in the following form

$$G^+(x,s) = \begin{cases} \sum_{l=1}^{r_1} \psi_{1l}(s)\chi_{1l}(x) & x_1 \le x < x_2 \\ \sum_{l=1}^{r_1} \psi_{1l}(s)\chi_{1l}(x) + \sum_{l=1}^{r_2} \psi_{2l}(s)\chi_{2l}(x), & x_2 \le x < x_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ \sum_{l=1}^{r_1} \psi_{1l}(s)\chi_{1l}(x) + \ldots + \sum_{l=1}^{r_\mu} \psi_{\mu l}(s)\chi_{\mu l}(x), & x_\mu \le x \le s \\ -\sum_{l=1}^{r_{\mu+1}} \psi_{\mu+1l}(s)\chi_{\mu+1l}(x) - \ldots - \sum_{l=1}^{r_m} \psi_{ml}(s)\chi_{ml}(x), & s \le x \le x_{\mu+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ -\sum_{l=1}^{r_{m-1}} \psi_{m-1l}(s)\chi_{m-1l}(x) - \sum_{l=1}^{r_m} \psi_{ml}(s)\chi_{ml}(x), & x_{m-2} < x \le x_{m-1} \\ -\sum_{l=1}^{r_m} \psi_{ml}(s)\chi_{ml}(x), & x_{m-1} < x \le x_m \end{cases} \tag{24}$$

where $\psi_{jl}(s)$ are the linearly independent functions complying with the theorem conditions.

Let us select the unknown functions $\chi_{ik}(x)$ to fulfill the second condition of the Green's function definition $G^+(x,s)$. It is easy to notice that the first proportion for $x_\mu \le s \le x_{\mu+1}$ are as follows

$$\begin{aligned} & G^+(x,x+0) - G^+(x,x-0) \\ = \; & \sum_{l=1}^{r_1} \psi_{1l}(s)\chi_{1l}(x) + \ldots + \sum_{l=1}^{r_\mu} \psi_{\mu l}(s)\chi_{\mu l}(x) + \sum_{l=1}^{r_{\mu+1}} \psi_{\mu+1l}(s)\chi_{\mu+1l}(x) + \ldots + \sum_{l=1}^{r_m} \psi_{ml}(s)\chi_{ml}(x) \\ = \; & 0. \end{aligned}$$

Or, by writing the sum, we have

$$\psi_{11}(x)\chi_{11}(x) + \ldots + \psi_{1r_1}(x)\chi_{1r_1}(x) + \ldots + \psi_{m1}(x)\chi_{m1}(x) + \ldots + \psi_{mr_m}(x)\chi_{mr_m}(x) = 0.$$

We will have the similar expressions at the other vertical strips $x_\nu \le s \le x_{\nu+1}$. Thus, writing the second condition of the Green's function definition of the m-point boundary problem for the adjoint equation $G^+(x,s)$ (22) for all strips $x_\mu \le s \le x_{\mu+1}$, we have the same system of linear equations regarding the unknown functions $\chi_{ik}(x)$

$$\begin{cases} \psi_{11}(x)\chi_{11}(x) + \ldots + \psi_{1r_1}(x)\chi_{1r_1}(x) + \ldots + \psi_{m1}(x)\chi_{m1}(x) + \ldots + \psi_{mr_m}(x)\chi_{mr_m}(x) = 0 \\ \psi'_{11}(x)\chi_{11}(x) + \ldots + \psi'_{1r_1}(x)\chi_{1r_1}(x) + \ldots + \psi'_{m1}(x)\chi_{m1}(x) + \ldots + \psi'_{mr_m}(x)\chi_{mr_m}(x) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{11}^{(n-2)}(x)\chi_{11}(x) + \ldots + \psi_{1r_1}^{(n-2)}(x)\chi_{1r_1}(x) + \ldots + \psi_{m1}^{(n-2)}(x)\chi_{m1}(x) + \ldots + \psi_{mr_m}^{(n-2)}(x)\chi_{mr_m}(x) = 0 \\ \psi_{11}^{(n-1)}(x)\chi_{11}(x) + \ldots + \psi_{1r_1}^{(n-1)}(x)\chi_{1r_1}(x) + \ldots + \psi_{m1}^{(n-1)}(x)\chi_{m1}(x) + \ldots + \psi_{mr_m}^{(n-1)}(x)\chi_{mr_m}(x) = (-1)^n \end{cases}$$

Solving a system of the algebraic equations by the Cramer method, we find that

$$\chi_{ik}(x) = \frac{(-1)^n \cdot (-1)^{n+()}}{W(x_i)} \cdot \begin{vmatrix} \psi_{11}(x) & \ldots & \psi_{ik-1}(x) & \psi_{ik+1}(x) & \ldots & \psi_{mr_m}(x) \\ \psi'_{11}(x) & \ldots & \psi'_{ik-1}(x) & \psi'_{ik+1}(x) & \ldots & \psi'_{mr_m}(x) \\ \vdots & & \vdots & \vdots & & \vdots \\ \psi_{11}^{(n-3)}(x) & \ldots & \psi_{ik-1}^{(n-3)}(x) & \psi_{ik+1}^{(n-3)}(x) & \ldots & \psi_{mr_m}^{(n-3)}(x) \\ \psi_{11}^{(n-2)}(x) & \ldots & \psi_{ik-1}^{(n-2)}(x) & \psi_{ik+1}^{(n-2)}(x) & \ldots & \psi_{mr_m}^{(n-2)}(x) \end{vmatrix} e^{-\int_{x_i}^{x} b_{n-1}(t)dt}.$$

By comparing the functions with (7), we can see that $\chi_{ik}(x) = y_{ik}(x)$.

Execution of the third condition of the definition is obvious from the formula (24). Let us show that the function (24) complies with the fourth condition as well. Let us apply the operator of the boundary conditions T_k^+ to the function $G^+(x,s)$ and set $s = x_i$ to set $\mu = i$ in (24). So, in representation of the function (24), the sum disappears at $x_i < x \le s$, since an interval comes to the point $x_i < x \le x_i$. Considering this and linearity of the operator T_k^+, we have

$$\left(T_k^+G\right)(x,x_i)=\begin{cases}\sum_{l=1}^{r_1} y_{1l}(x)\left(T_k^+\psi_{1l}\right)(x_i), & x_1\le x<x_2\\ \dots\dots\dots\dots & \dots\dots\\ \sum_{l=1}^{r_1} y_{1l}(x)\left(T_k^+\psi_{1l}\right)(x_i)+\ldots+\sum_{l=1}^{r_{i-1}} y_{i-1l}(x)\left(T_k^+\psi_{i-1l}\right)(x_i), & x_{i-1}\le x\le x_i\\ -\sum_{l=1}^{r_{i+1}} y_{i+1l}(x)\left(T_k^+\psi_{i+1l}\right)(x_i)-\ldots-\sum_{l=1}^{r_m} y_{ml}(x)\left(T_k^+\psi_{ml}\right)(x_i), & x_i\le x\le x_{i+1}\\ \dots\dots\dots\dots & \dots\dots\\ -\sum_{l=1}^{r_m} y_{ml}(x)\left(T_k^+\psi_{ml}\right)(x_i). & x_{m-1}<x\le x_m\end{cases}$$

Since the first index of value of the function $\psi_{jl}(x_i)$ is not congruent with the index of operator $T_k^+(x_i)$, i.e. $j\neq i$, all sums of the right part transform to zero based on (5).

Thus,

$$\left(T_k^+G^+\right)(x,x_i)=0,\forall x\neq x_\mu,\mu=2,3,\ldots,m-1.$$

Therefore, the function created in a form (23) is a Green's function of the m-point boundary value problem for the adjoint Equations (18), (19). Theorem is proved.

As far as we know, the adjoint forms (which existence was proved by Y. Tamarkin long ago) when ranges of functions $\{z\}$ and $\{y\}$ would be adjoint, have not been found in works (Levin, 1985; Maksimov & Rakhmatullina, 1977; Lando, 1969; Jackson, 1977; Kiguradze, 1975; Grimm & Eloe, 1984). In Khasseinov (1988), there were found adjoint boundary conditions $\left(T_k^+z\right)(x_i)$ for a line of the multipoint task for the linear differential equation, and in Khasseinov (1984)-$(T_ky)(x_i)$ for the n-point task for the adjoint differential equation.

But here we have a new result that it was done for the adjoint differential equation $L^+z=F(s)$ with common boundary conditions in the m-points $\left(T_k^+z\right)(x_i)$. We have found corresponding "adjoint" boundary conditions $(T_ky)(x_i)$ for such a problem and solved a nonhomogenous multipoint problem for the adjoint equation. Herewith, especially these adjoint family of functions $\{\psi_{jl}(s)\}$ and $\{y_{ik}(x)\}$ are used in the course of creating of the Green's function $G^+(x,s)$. Thereby, we give particular new properties of the Green's function of the m-point problem for the adjoint equation.

1) Green's function $G^+(x,s)$ of the m-point problem for the adjoint equation under the variable x in the rectangle $x_1\le x,\ s\le x_m$, except lines $x=x_\mu$, $x=s$, complies with the homogenous differential equation, i.e. $LG^+(x,s)=0$.

Proof. Function $G^+(x,s)$ represents the sum of products $\forall\ s\in[x_\mu,x_{\mu+1}]$, $\mu=1,2,\ldots,m-1$

$$G^+(x,s)=\pm\sum_{k=1(\mu+1)}^{\mu(m)}\sum_{l=1}^{r_k}\psi_{kl}(s)y_{kl}(x)=0;\quad\text{for } x_\mu<x<s\ (s<x<x_{\mu+1}).$$

By applying the linear differentiation operator L under the variable x and considering its linearity, we get

$$LG^+(x,s)=\pm\sum_{k=1(\mu+1)}^{\mu(m)}\sum_{l=1}^{r_k}\psi_{kl}(s)Ly_{kl}(x)=0,\quad x_\mu<x<s\ (s<x<x_{\mu+1}),$$

because the functions $\{y_{kl}(x)\}$, due to Lemma 2, are solutions of the homogenous equation $Ly=0$.

2) Green's function $G^+(x,s)$ of the m-point problem for the adjoint equation at lines $x=x_\mu$, $\mu=2,3,\ldots,m-1$ has a gap of the first kind, and the following proportion is executed for $\forall\ s\in[x_1,x_m]$

$$G^+(x_\mu+0,s)-G^+(x_\mu-0,s)=\sum_{l=1}^{r_\mu}\psi_{\mu l}(s)y_{\mu l}(x)\Big|_{x=x_\mu}=(-1)^n\sum_{l=1}^{r_\mu}r_{ln}(x_\mu)\cdot\psi_{\mu l}(s),\tag{25}$$

i.e. the Green's function saltus is equal to the sum of products of the coefficients at the higher derivatives in the boundary conditions by the corresponding functions.

Proof. It is obvious from the Green's function structure (23) that to define the necessary difference, it is sufficient to subtract expression of the previous line at $x<x_\mu$ from the expression $G^+(x,s)$ at $x\ge x_\mu$, i.e.

$$\begin{aligned}G^+(x_\mu+0,s)-G^+(x_\mu-0,s)&=\left(\sum_{k=1}^{\mu}\sum_{l=1}^{r_k}\psi_{kl}(s)y_{kl}(x)-\sum_{k=1}^{\mu-1}\sum_{l=1}^{r_k}\psi_{kl}(s)y_{kl}(x)\right)\Big|_{x=x_\mu}\\&=\sum_{l=1}^{r_\mu}\psi_{\mu l}(s)y_{\mu l}(x)\Big|_{x=x_\mu}=(-1)^n\sum_{l=1}^{r_\mu}r_{ln}(x_\mu)\cdot\psi_{\mu l}(s),\end{aligned}$$

based on value $y_{il}(x_i)$ from (7)

$$y_{il}(x_i) = \frac{(-1)^n \cdot (-1)^{2\left[n+(l+\sum_{\mu=1}^{i-1} r_\mu)\right]}}{W(x_i)} \times \begin{vmatrix} \psi_{11}(x_i) & \dots & \psi_{il}(x_i) & \dots & \psi_{mr_m}(x_i) \\ \psi'_{11}(x_i) & \dots & \psi'_{il}(x_i) & \dots & \psi'_{mr_m}(x_i) \\ \vdots & & \vdots & & \vdots \\ \psi_{11}^{(n-2)}(x_i) & \dots & \psi_{il}^{(n-2)}(x_i) & \dots & \psi_{mr_m}^{(n-2)}(x_i) \\ \left(T_{il}^+\psi_{11}\right)(x_i) & \dots & \left(T_{il}^+\psi_{il}\right)(x_i) & \dots & \left(T_{il}^+\psi_{mr_m}\right)(x_i) \end{vmatrix} e^{-\int_{x_i}^{x_i} b_{n-1}(t)dt} = (-1)^n r_{ln}(x_i),$$

because

$$\left(T_{il}^+\psi_{jk}\right)(x_i) = r_{ln}(x_i)\psi_{jk}^{(n-1)}(x_i) + \dots + r_{l1}(x_i)\psi_{jk}(x_i) = \delta_{ij}\delta_{kl}.$$

The similar is set at $x \geq s$ as well.

Green's function $G^+(x,s)$ of the m-point problem for the adjoint equation and correspondingly its derivatives on s to $(n-2)$-th order are uninterrupted at the lines $x = x_\mu$, $\mu = 2, 3, \dots, m-1$, if only coefficients $r_{ln}(x_\mu)$, $l = 1, 2, \dots, r_\mu$ at senior derivatives in all boundary conditions (21) in points x_μ transform to zero. In other cases, there is no continuity.

Actually, if the sum

$$\sum_{l=1}^{r_\mu} r_{ln}(x_\mu) \cdot \psi_{\mu l}(s)$$

is equal to zero $\forall\ s \in [x_1, x_m]$, the functions $\left\{\psi_{\mu l}(s)\right\}_{l=1}^{r_\mu}$ would be linearly dependent. These function are a subsystem of the linearly independent functions $\psi_{jl}(s)$, $j = 1, 2, \dots, m$; $l = 1, 2, \dots, r_j$, which in our assumption should be linearly dependent but it contradicts the conditions of Lemma 2.

Thus, the Green's function $G^+(x,s)$ and its derivatives are uninterrupted at the lines $x = x_\mu$ in the only case when there are no $(n-1)$-th order derivatives in all the boundary conditions (19) in the point $s = x_\mu$. Such an attribute simply results from the proportion (25).

A. Levin produced the similar attribute for the Green's function of the direct multipoint problem under the other considerations (Levin, 1961).

Let us specify the Green's function saltus of the adjoint boundary task $G^+(x,s)$ at the lines $x = x_\mu$, $\mu = 2, 3, \dots, m-1$,

$$\delta G^+(x_\mu, s) = G^+(x_\mu + 0, s) - G^+(x_\mu - 0, s) = \psi_\mu(s) y_\mu(x)\big|_{x=x_\mu}.$$

To make it comfortable, supposing $G^+(x_1 - 0, s) = 0$, we can find

$$\delta G^+(x_1, s) = G^+(x_1 + 0, s) = G^+(x, s)\big|_{x=x_1} = \psi_1(s) y_1(x)\big|_{x=x_1}.$$

Similarly, if we take $G^+(x_n + 0, s) = 0$, so

$$\delta G^+(x_n, s) = -G^+(x_n - 0, s) = -G^+(x, s)\big|_{x=x_n} = \psi_n(s) y_n(x)\big|_{x=x_n}.$$

Now, we can take $i = 1, 2, \dots, n$ instead of index μ, i.e.

$$\delta G^+(x_i, s) = G^+(x_i + 0, s) - G^+(x_i - 0, s) = \psi_i(s) y_i(x)\big|_{x=x_i}. \tag{26}$$

Let us specify the difference of derivatives

$$\delta G^{+(k)}(x_i, s) = \frac{\partial^k G^+(x_i + 0, s)}{\partial x^k} - \frac{\partial^k G^+(x_i - 0, s)}{\partial x^k} = \psi_i(s) \frac{d^k y_i(x)}{dx^k}\big|_{x=x_i}. \qquad (26)$$

Supposing the first corresponding coefficients (different from zero) at the senior derivatives are $r_{n-k_i}(x_i)$ in the adjoint boundary conditions (19), i.e.

$$r_n(x_i) = r_{n-1}(x_i) = \dots = r_{n-k_i+1}(x_i) = 0.$$

Then it is easy to determine from the boundary conditions for the operator Ly (5) considering (d) that

$$\frac{d^{k_i}y_i(x)}{dx^{k_i}}\Big|_{x=x_i} = W_{n-k_i-1,n-1}(x_i)e^{-\int_{x_i}^{x_i} b_{n-1}(t)dt} = \frac{(-1)^{2n+i}}{W(x_i)}(-1)^{n-k_i+i}W(x_i)r_{n-k_i}(x_i) = (-1)^{n-k_i}r_{n-k_i}(x_i) \neq 0, \tag{28}$$

where k_i is a natural number corresponding to each point x_i.

Let us consider saltus k_i-derivatives of the Green's function of the adjoint boundary value problem at the corresponding lines $x = x_i$:

$$\delta G^{+(k_i)}(x_i, s) = \frac{\partial^{k_i}G^+(x_i+0,s)}{\partial x^{k_i}} - \frac{\partial^{k_i}G^+(x_i-0,s)}{\partial x^{k_i}} = \psi_i(s)\frac{d^{k_i}y_i(x)}{dx^{k_i}}\Big|_{x=x_i} = \psi_i(s)(-1)^{n-k_i}r_{n-k_i}(x_i).$$

So, we can define

$$\psi_i(s) = (-1)^{n-k_i}\frac{1}{r_{n-k_i}(x_i)}\delta G^{+(k_i)}(x_i, s). \tag{29}$$

Applying this expression and considering (20), we can produce a solution of the homogenous adjoint equation $L^+z(s) = 0$ at the boundary conditions (19) for the case (28):

$$z(s) = \sum_{i=1}^{n}(-1)^{n-k_i}\frac{A_{ik}}{r_{n-k_i}(x_i)}\delta G^{+(k_i)}(x_i, s), \quad r_{n-k_i}(x_i) \neq 0. \tag{30}$$

Theorem 2 *Let $G^+(x,s)$ be a Green's function of the m-point boundary value problem for the adjoint Equations (18), (19). Then the only solution to the nonhomogenous boundary problem (18), (19) is defined by the following formula*

$$z(s) = \sum_{i=1}^{m}\sum_{k=1}^{r_i} A_{ik}\psi_{ik}(s) + \int_{x_i}^{x_m} F(x)G^+(x, s)dx. \tag{31}$$

Proof. It is obvious from Theorem 1 of the previous paragraph that the double sum is a solution of the homogenous adjoint equation $L^+z(s) = 0$ at condition (19).

Therefore, it is sufficient to prove that an integral is a solution of the nonhomogenous adjoint Equation (18) at the zero boundary conditions $\left(T_k^+z\right)(x_i) = 0$. Let us take a random strip $x_\mu \leq s \leq x_{\mu+1}$ and write an integral out for the sum of m integrals by considering assignment of the Green's function (23):

$$\int_{x_i}^{x_m} F(x)G^+(x, s)dx = \int_{x_1}^{x_2} F(x)\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x)dx + \int_{x_2}^{x_3} F(x)[\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x)+$$

$$+\sum_{l=1}^{r_2}\psi_{2l}(s)y_{2l}(x)]dx + ... + \int_{x_{\mu-1}}^{x_\mu} F(x)[\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x) + ... + \sum_{l=1}^{r_{\mu-1}}\psi_{\mu-1l}(s)y_{\mu-1l}(x)]dx+$$

$$+\int_{x_\mu}^{s} F(x)[\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x)dx + \sum_{l=1}^{r_2}\psi_{2l}(s)y_{2l}(x)dx + ... + \sum_{l=1}^{r_\mu}\psi_{\mu l}(s)y_{\mu l}(x)]dx-$$

$$-\int_{s}^{x_{\mu+1}} F(x)[\sum_{l=1}^{r_{\mu+1}}\psi_{\mu+1l}(s)y_{\mu+1l}(x) + \sum_{l=1}^{r_{\mu+2}}\psi_{\mu+2l}(s)y_{\mu+2l}(x) + ... + \sum_{l=1}^{r_m}\psi_{ml}(s)y_{ml}(x)]dx-$$

$$-\int_{x_{\mu+1}}^{x_{\mu+2}} F(x)[\sum_{l=1}^{r_{\mu+2}}\psi_{\mu+2l}(s)y_{\mu+2l}(x) + \sum_{l=1}^{r_{\mu+3}}\psi_{\mu+3l}(s)y_{\mu+3l}(x) + ... + \sum_{l=1}^{r_m}\psi_{ml}(s)y_{ml}(x)]dx - ...-$$

$$-\int_{x_{m-2}}^{x_{m-1}} F(x)[\sum_{l=1}^{r_{m-1}}\psi_{m-1l}(s)y_{m-1l}(x) + \sum_{l=1}^{r_m}\psi_{ml}(s)y_{ml}(x)]dx - \int_{x_{m-1}}^{x_m} F(x)\sum_{l=1}^{r_m}\psi_{ml}(s)y_{ml}(x)dx.$$

For the first, let us consider only positive terms. Since the sum $\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x)$ is under all the integrals beginning with the first and including the integral with variables of the upper limit s, and there is $\sum_{l=1}^{r_2}\psi_{2l}(s)y_{2l}(x)$ beginning from the second integral and including the integral with variables upper limit s and so on, then we have

$$\int_{x_1}^{s} F(x)\sum_{l=1}^{r_1}\psi_{1l}(s)y_{1l}(x)dx + \int_{x_2}^{s} F(x)\sum_{l=1}^{r_2}\psi_{2l}(s)y_{2l}(x)dx + ...+$$

$$+\int_{x_\mu}^{s} F(x)\sum_{l=1}^{r_\mu}\psi_{\mu l}(s)y_{\mu l}(x)dx=\sum_{i=1}^{\mu}\sum_{l=1}^{r_i}\psi_{il}(s)\int_{x_i}^{s}F(x)y_{il}(x)dx.$$

It is true for the negative integrals as well. Having applied addition of integrals and mobbing from the last to the integral with variable lower limit s, we will have

$$-\int_{s}^{x_{\mu+1}} F(x)\sum_{l=1}^{r_{\mu+1}}\psi_{\mu+1l}(s)y_{\mu+1l}(x)dx-\int_{s}^{x_{\mu+2}} F(x)\sum_{l=1}^{r_{\mu+2}}\psi_{\mu+2l}(s)y_{\mu+2l}(x)dx-...-$$

$$-\int_{s}^{x_m} F(x)\sum_{l=1}^{r_m}\psi_{ml}(s)y_{ml}(x)dx=-\sum_{i=\mu+1}^{m}\sum_{l=1}^{r_i}\psi_{il}(s)\int_{s}^{x_i}F(x)y_{il}(x)dx.$$

Having changed integration limits and having summed up the integrals, finally we have

$$\int_{x_1}^{x_m} F(x)G^{+}(x,s)dx=\sum_{i=1}^{m}\sum_{l=1}^{r_i}\psi_{il}(s)\int_{x_i}^{s}F(x)y_{il}(x)dx.$$

This integral is congruent with the second sum of the proportion (20), therefore, Theorem 1 of the previous paragraph proves that the formula (31) is true.

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