Anisotropic Fractional Maximal Operator in Anisotropic Generalized Morrey Spaces

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Abstract

In this paper it is proved that anisotropic fractional maximal operator \( M_{\alpha,\sigma} f \), \( 0 \leq \alpha < |\sigma| \) is bounded on anisotropic generalized Morrey spaces \( M_{p,\varphi,\sigma,}\ ), where \( |\sigma| = \sum_{i=1}^{n} \sigma_i \) is the homogeneous dimension of \( \mathbb{R}^n \). We find the conditions on the pair \((\varphi, \psi)\) which ensure the Spanne-Guliyev type boundedness of the operator \( M_{\alpha,\sigma} \) from anisotropic generalized Morrey space \( M_{p,\varphi,\sigma,}\ ) to \( M_{q,\psi,\sigma,}\ ) for \( 1 < p < q < \infty \) and from \( M_{1,\varphi,\sigma,} \) to the weak space \( WM_{q,\psi,\sigma,}\ ) for \( 1 < q < \infty \). We also find conditions on the \( \varphi \) which ensure the Adams-Guliyev type boundedness of \( M_{\alpha,\sigma} \) from \( M^{\alpha/\sigma}_{p,\varphi,\sigma,}\ ) to \( M^{\beta/\gamma}_{q,\psi,\sigma,}\ ) for \( 1 < q < \infty \), and from \( M_{1,\varphi,\sigma,} \) to \( WM^{\alpha/\sigma}_{q,\psi,\sigma,}\ ) for \( 1 < q < \infty \).

As applications, we establish the boundedness of some Schrödinger type operators on anisotropic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

Keywords: anisotropic fractional maximal function, anisotropic generalized Morrey space

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1. Introduction

In the present paper we will prove the boundedness of the anisotropic fractional maximal operator in the anisotropic generalized Morrey spaces.

For \( x \in \mathbb{R}^n \) and \( t > 0 \), let \( B(x, t) \) denote the open ball centered at \( x \) of radius \( t \) and \( \overline{B(x, t)} = \mathbb{R}^n \setminus B(x, t) \). Let \( 0 \leq b \leq 1 \), \( \sigma = (\sigma_1, ..., \sigma_n) \) with \( \sigma_i > 0 \) for \( i = 1, ..., n \), \( |\sigma| = \sigma_1 + ... + \sigma_n \) and \( r^\sigma x = (r^{\sigma_1} x_1, ..., r^{\sigma_n} x_n) \) for \( t > 0 \). For \( x \in \mathbb{R}^n \) and \( t > 0 \), let \( E_\sigma(x, t) = \prod_{i=1}^{n} (x_i - t^{\sigma_i} x_i + t^{\sigma_i}) \) denote the open parallelepiped centered at \( x \) of side length \( t^{\sigma_i} \) for \( i = 1, ..., n \).

By Besov, Il’in and Lizorkin (1966) and Fabes and Rivière (1966), the function \( F(x, \rho) = \sum_{i=1}^{n} x_i^2 \rho^{-2\sigma_i} \), considered for any fixed \( x \in \mathbb{R}^n \), is a decreasing one with respect to \( \rho > 0 \) and the equation \( F(x, \rho) = 1 \) is uniquely solvable. This unique solution will be denoted by \( \rho(x) \). Define \( \rho(x) = \rho \) and \( \rho(0) = 0 \). It is a simple matter to check that \( \rho(x - y) \) defines a distance between any two points \( x, y \in \mathbb{R}^n \). Thus \( \mathbb{R}^n \), endowed with the metric \( \rho \), defines a homogeneous metric space (Besov, Il’in, & Lizorkin, 1966; Bramanti & Cerutti, 1996; Fabes & Rivière, 1966). Note that \( \rho(x) \) is equivalent to \( |x|_\rho = \max_{1 \leq i \leq n} |x_i|^{2/\sigma_i} \).

One of the most important variants of the anisotropic maximal function is the so-called anisotropic fractional maximal function defined by the formula

\[
M_{\alpha,\sigma} f(x) = \sup_{t>0} \left| E_\sigma(x, t)^{1+\alpha/|\sigma|} \right| \int_{E_\sigma(x, t)} |f(y)| dy, \quad 0 \leq \alpha < |\sigma|,
\]

where \( |E_\sigma(x, t)| = 2^n t^{\sigma} \) is the Lebesgue measure of the parallelepiped \( E_\sigma(x, t) \).

It coincides with the anisotropic maximal function \( M_\sigma f \equiv M_{0,\sigma} f \) and is intimately related to the anisotropic Riesz potential operator

\[
I_{\alpha,\sigma} f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x - y|^{n+\alpha}}, \quad 0 < \alpha < |\sigma|.
\]

109
If \( \sigma = 1 \), then \( M_\sigma \equiv M_{a,1} \) and \( I_\sigma \equiv I_{a,1} \) is the fractional maximal operator and Riesz potential, respectively. The operators \( M_\sigma, M_{a,\sigma}, I_\sigma \) and \( I_{a,\sigma} \) play important role in real and harmonic analysis (see, for example Besov, Il’in, & Nikol’skii, 1996; Stein, 1993).

**Definition 1.1** Let \( 0 \leq b \leq 1 \) and \( 1 \leq p < \infty \). We denote by \( L_{p,b,\sigma} \equiv L_{p,b,\sigma}(\mathbb{R}^n) \) anisotropic Morrey space, the set of locally integrable functions \( f(x), x \in \mathbb{R}^n \), with the finite norm

\[
\|f\|_{L_{p,b,\sigma}} = \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{r^{-b} r^\sigma} \int_{E_{x,r}} |f(y)|^p \, dy \right)^{1/p}.
\]

**Remark 1.1** Note that \( L_{p,0,\sigma} = L_p(\mathbb{R}^n) \) and \( L_{p,1,\sigma} = L_\infty(\mathbb{R}^n) \). If \( b < 0 \) or \( b > 1 \), then \( L_{p,b,\sigma} = \emptyset \), where \( \emptyset \) is the set of all functions equivalent to 0 on \( \mathbb{R}^n \). In the case \( \sigma \equiv 1 = (1, \ldots, 1) \) and \( b = \frac{1}{n} \) we get the classical Morrey space \( L_{p,1}(\mathbb{R}^n) = L_{p,\frac{1}{n}}(\mathbb{R}^n), 0 \leq \lambda \leq n \).

In the theory of partial differential equations, together with weighted \( L_{p,n}(\mathbb{R}^n) \) spaces, Morrey spaces \( L_{p,\lambda}(\mathbb{R}^n) \) play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems of the Riesz potential. An exposition of the Morrey spaces can be found in the book Kufner, John and Fucik (1977).

**Definition 1.2** (Burenkov, Guliev, H. V., & Guliev, V. S., 2007) Let \( 1 \leq p < \infty \) and \( 0 \leq b \leq 1 \). We denote by \( WL_{p,b,\sigma} \equiv WL_{p,b,\sigma}(\mathbb{R}^n) \) the weak anisotropic Morrey space as the set of locally integrable functions \( f(x), x \in \mathbb{R}^n \) with finite norm

\[
\|f\|_{WL_{p,b,\sigma}} = \sup_{r > 0} \sup_{x \in \mathbb{R}^n} r^{\sigma} \left( \frac{1}{(r^{1+b})} \int_{E_{x,r}} |f(y)|^p \, dy \right)^{1/p}.
\]

Note that

\[
WL_p(\mathbb{R}^n) = WL_{p,0,\sigma}(\mathbb{R}^n),
\]

\[
WL_{p,b,\sigma}(\mathbb{R}^n) \subset WL_{p,b,\sigma}(\mathbb{R}^n)
\]

and

\[
\|f\|_{WL_{p,b,\sigma}} \leq \|f\|_{L_{p,b,\sigma}}.
\]

The anisotropic result by Hardy-Littlewood-Sobolev states that if \( 1 < p < q < \infty \), then \( L_{p,\alpha} \) is bounded from \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \) if and only if \( \alpha = |\sigma| \left( \frac{1}{p} - \frac{1}{q} \right) \) and for \( p = 1 < q < \infty \), \( L_{1,\alpha} \) is bounded from \( L_1(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \) if and only if \( \alpha = |\sigma| \left( 1 - \frac{1}{q} \right) \). Spanne (see Spanne, 1966) and Adams (1975) studied boundedness of the Riesz potential \( I_\sigma \) for \( 0 < \alpha < n \) in Morrey spaces \( L_{p,\alpha} \). Later on Chiarenza and Frasca (1987) was reproved boundedness of the Riesz potential \( I_\sigma \) in these spaces. By more general results of Guliyev (1994) (see also 1999 & 2009) one can obtain the following generalization of the results in Adams (1975), Chiarenza and Frasca (1987) and Spanne (1966) to the anisotropic case.

**Theorem A** Let \( 0 < \alpha < |\sigma| \) and \( 0 \leq b < 1 \), \( 1 \leq p < \frac{1}{b + |\sigma|} \).

1) If \( 0 < p = \frac{1}{b + |\sigma|} \), then condition \( \frac{1}{p} - \frac{1}{q} = \frac{a}{(1 + b)|\sigma|} \) is necessary and sufficient for the boundedness of the operators \( M_{a,\sigma} \) and \( I_{a,\sigma} \) from \( L_{p,b,\sigma}(\mathbb{R}^n) \) to \( L_{q,b,\sigma}(\mathbb{R}^n) \).

2) If \( p = 1 \), then condition \( 1 - \frac{1}{q} = \frac{a}{(1 + b)|\sigma|} \) is necessary and sufficient for the boundedness of the operators \( M_{a,\sigma} \) and \( I_{a,\sigma} \) from \( L_{1,\sigma}(\mathbb{R}^n) \) to \( WL_{q,b,\sigma}(\mathbb{R}^n) \).

It is known that the anisotropic maximal operator \( M_\sigma \) is also bounded from \( L_{p,b,\sigma} \) to \( L_{p,b,\sigma} \), for all \( 1 < p < \infty \) and \( 0 < b < 1 \) (see, for example, Guliyev, 1994 & 1999), which isotropic case proved by Chiarenza and Frasca (1987).

In this work, we prove the boundedness of the fractional maximal operator \( M_{a,\sigma} \) for \( 0 \leq \alpha < |\sigma| \) from one generalized Morrey space \( M_{p,q,\sigma} \) to \( M_{q,2,\sigma} \), \( 1 < p \leq q < \infty \), \( 1/p - 1/q = \alpha/|\sigma| \), and from \( M_{1,q,\sigma} \) to the weak space \( WM_{q,\sigma} \), \( 1 < q < \infty \), \( 1 - 1/q = \alpha/|\sigma| \). We also prove the Adams-Guliyev type boundedness of the operator \( M_{a,\sigma} \) from \( M_{p,q,\sigma} \) to \( M_{q,2,\sigma} \), \( 1 < p < q < \infty \), and from \( M_{1,q,\sigma} \) to \( WM_{q,\sigma} \), for \( 1 < q < \infty \). In all the cases the conditions for the boundedness are given it terms of supremal-type inequalities on \( (\varphi_1, \varphi_2) \) and \( \varphi \) in \( r \). By using the \( (M_{p,q,\sigma}, M_{q,2,\sigma}) \) boundedness of the fractional maximal operators we establish the boundedness of some Schrödinger type operators on anisotropic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.
By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

2. Notations

Everywhere in the sequel the functions $\varphi(x, r)$, $\varphi_1(x, r)$ and $\varphi_2(x, r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{R}^n \times (0, \infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

**Definition 2.3** Let $1 \leq p < \infty$. The anisotropic generalized Morrey space $M_{p, \varphi, \sigma}$ is defined of all functions $f \in L^p_{\varphi}(\mathbb{R}^n)$ by the finite norm

$$
\|f\|_{M_{p, \varphi, \sigma}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |E_{\varphi}(x, r)|^{-\frac{1}{p}} \|f\|_{L^p(E_{\varphi}(x, r))}.
$$

According to this definition, when $\varphi(x, r) = r^{\frac{1}{p} - \frac{r}{n}}$, we can see that

$$
M_{p, \varphi, \sigma}(\mathbb{R}^n) = L_{p, \varphi, \sigma}(\mathbb{R}^n).
$$

There are many papers discussed the conditions on $\varphi$ to obtain the boundedness of integral operators on the generalized Morrey spaces (see Guliyev, 1994, 1999 & 2009; Mizuhara, 1991; Nakai, 1994 & 2006; Softova, 2006).

In Nakai (2006) the following statements were proved.

**Theorem 2.1** Let $1 \leq p < \infty$, $0 < \alpha < \frac{|b|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x, \tau)$ satisfy conditions

$$
c^{-1} \varphi(x, r) \leq \varphi(x, \tau) \leq c \varphi(x, r),
$$

whenever $r \leq \tau \leq 2r$, where $c$ does not depend on $r$, $\tau$ and $x \in \mathbb{R}^n$,

$$
\int_{r}^{\infty} r^{\alpha} \varphi(x, \tau) \frac{d\tau}{\tau} \leq C r^{\alpha} \varphi(x, r)^{\alpha}.
$$

Then for $p > 1$ the operator $M_{q, \tau}$ is bounded from $M_{p, \varphi, \sigma}$ to $M_{q, \varphi, \sigma}$ and for $p = 1$ $M_{1, \varphi, \sigma}$ is bounded from $M_{1, \varphi, \sigma}$ to $WM_{q, \varphi, \sigma}$.


**Theorem 2.2** Let $1 \leq p < \infty$, $0 < \alpha < \frac{|b|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $(\varphi_1, \varphi_2)$ satisfy the condition

$$
\int_{\tau}^{\infty} r^{\alpha} \varphi_1(x, r) \frac{dr}{r} \leq C \varphi_2(x, \tau),
$$

where $C$ does not depend on $x$ and $\tau$. Then the operator $M_{p, \varphi, \sigma}$ is bounded from $M_{p, \varphi, \sigma}$ to $M_{q, \varphi, \sigma}$ for $p > 1$ and from $M_{1, \varphi, \sigma}$ to $WM_{q, \varphi, \sigma}$ for $p = 1$.

In Guliyev, Aliyev, Karaman and Shukurov (2011) obtained sufficient conditions on the pair $(\varphi_1, \varphi_2)$

$$
\int_{\tau}^{\infty} \text{ess inf}_{s \leq \tau} \frac{\varphi_1(x, s)}{s^{\frac{n}{2} + 1}} ds \leq C \varphi_2(x, \tau),
$$

for the boundedness of $M_{\sigma}$ from $M_{p, \varphi, 1}$ to $M_{q, \varphi, 1}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

3. Boundedness of the Fractional Maximal Operator in the Spaces $M_{p, \varphi, \sigma}$

3.1 Spanne-Guliyev Type Result

Sufficient conditions on $\varphi$ for the boundedness of $M_{\sigma}$ and $M_{p, \varphi, \sigma}$ in generalized Morrey spaces $M_{p, \varphi, \sigma}$ have been obtained in Akbulut, Guliyev and Mustafayev (2012), Burenkov, Gogatishvili, Guliyev and Mustafayev (2010), Guliyev (2009), Guliyev and Mustafayev (1997 & 1998) and Nakai (2006).

The following lemma is true.
Lemma 3.1 Let \( 1 \leq p < \infty, \ 0 \leq \alpha < \frac{|\sigma|}{p}, \ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|} \). Then for \( p > 1 \) and any ball \( E_\sigma = E_\sigma(x, r) \) the inequality

\[
\| M_{\alpha, \sigma} f \|_{L_q(E_\sigma(x, r))} \lesssim \| f \|_{L_p(E_\sigma(x, 2r))} + r^{\frac{\alpha}{|\sigma|}} \sup_{r > 2r} \| f \|_{L_q(E_\sigma(x, r))}
\]

holds for all \( f \in L_p^\text{loc}(\mathbb{R}^n) \).

Moreover for \( p = 1 \) the inequality

\[
\| M_{\alpha, \sigma} f \|_{W_{L_q}(E_\sigma(x, r))} \lesssim \| f \|_{L_1(E_\sigma(x, 2r))} + r^{\frac{\alpha}{|\sigma|}} \sup_{r > 2r} \| f \|_{L_q(E_\sigma(x, r))}
\]

holds for all \( f \in L_1^\text{loc}(\mathbb{R}^n) \).

Proof. Let \( 1 < p < q < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|} \). For arbitrary ball \( E_\sigma = E_\sigma(x, r) \) let \( f = f_1 + f_2 \), where \( f_1 = f\chi_{E_\sigma} \) and \( f_2 = f\chi_{\overline{E_\sigma}} \).

\[
\| M_{\alpha, \sigma} f \|_{L_q(E_\sigma)} \leq \| M_{\alpha, \sigma} f_1 \|_{L_q(E_\sigma)} + \| M_{\alpha, \sigma} f_2 \|_{L_q(E_\sigma)}.
\]

By the continuity of the operator \( M_{\alpha, \sigma} : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n) \) we have

\[
\| M_{\alpha, \sigma} f_1 \|_{L_q(E_\sigma)} \lesssim \| f \|_{L_q(2E_\sigma)}.
\]

Let \( y \) be an arbitrary point from \( E_\sigma \). If \( E_\sigma(y, t) \cap \overline{E_\sigma(2E_\sigma)} \neq \emptyset, \) then \( t > r \). Indeed, if \( z \in E_\sigma(y, t) \cap \overline{E_\sigma(2E_\sigma)} \), then \( t \geq |y - z|_\sigma \geq |x - z|_\sigma - |x - y| \geq 2^{\tau_\text{max}} r - r = 2^{\tau_\text{max}} - 1 \).

On the other hand, \( E_\sigma(y, t) \cap \overline{E_\sigma(2E_\sigma)} \subset E_\sigma(x, 2r) \). Indeed, \( z \in E_\sigma(y, t) \cap \overline{E_\sigma(2E_\sigma)} \), then we get \( |x - z|_\sigma \leq 2^{\tau_\text{max}}(|y - z|_\sigma + |x - y|) < 2^{\tau_\text{max}}(t + r < 2^{\tau_\text{max}} + 1)t \).

Hence

\[
M_{\alpha, \sigma} f_2(y) = \sup_{r > 0} \frac{1}{|E_\sigma(y, t)|^{1 - \alpha/|\sigma|}} \int_{E_\sigma(y, t) \cap \overline{E_\sigma(2E_\sigma)}} |f(z)|dz
\]

\[
\leq 2^{\alpha(1 - \frac{1}{q})} \sup_{r > r} \frac{1}{|E_\sigma(x, 2t)|^{1 - \alpha/|\sigma|}} \int_{E_\sigma(x, 2t)} |f(z)| dz
\]

\[
= 2^{\alpha(1 - \frac{1}{q})} \sup_{r > 2r} \frac{1}{|E_\sigma(x, r)|^{1 - \alpha/|\sigma|}} \int_{E_\sigma(x, r)} |f(z)| dz.
\]

Therefore, for all \( y \in E_\sigma \) we have

\[
M_{\alpha, \sigma} f_2(y) \leq 2^{\alpha(1 - \frac{1}{q})} \sup_{r > 2r} \frac{1}{|E_\sigma(x, r)|^{1 - \alpha/|\sigma|}} \int_{E_\sigma(x, r)} |f(z)| dz.
\]

Thus

\[
\| M_{\alpha, \sigma} f \|_{L_q(E_\sigma)} \lesssim \| f \|_{L_q(2E_\sigma)} + |E_\sigma|^{\frac{1}{q}} \left( \sup_{r > 2r} \frac{1}{|E_\sigma(x, r)|^{1 - \alpha/|\sigma|}} \int_{E_\sigma(x, r)} |f(z)|dz \right).
\]

Let \( p = 1 \). It is obvious that for any ball \( E_\sigma = E_\sigma(x, r) \)

\[
\| M_{\alpha, \sigma} f \|_{W_{L_q}(E_\sigma)} \leq \| M_{\alpha, \sigma} f_1 \|_{W_{L_q}(E_\sigma)} + \| M_{\alpha, \sigma} f_2 \|_{W_{L_q}(E_\sigma)}.
\]

By the continuity of the operator \( M_{\alpha, \sigma} : L_1(\mathbb{R}^n) \to WL_q(\mathbb{R}^n) \) we have

\[
\| M_{\alpha, \sigma} f_1 \|_{W_{L_q}(E_\sigma)} \lesssim \| f \|_{L_q(2E_\sigma)}.
\]

Then by (3.3) we get the inequality (3.2).

Lemma 3.2 Let \( 1 \leq p < \infty, \ 0 \leq \alpha < \frac{|\sigma|}{p}, \ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|} \). Then for \( p > 1 \) and any ellipsoid \( E_\sigma = E_\sigma(x, r) \) in \( \mathbb{R}^n \), the inequality

\[
\| M_{\alpha, \sigma} f \|_{L_q(E_\sigma(x, r))} \lesssim r^{\frac{\alpha}{|\sigma|}} \sup_{r > 2r} \| f \|_{L_q(E_\sigma(x, r))}
\]

(3.4)
holds for all $f \in L^p_\text{loc}(\mathbb{R}^n)$.  

Moreover for $p = 1$ the inequality  

$$\|M_{a,r}f\|_{L_q(E_r(x,l))} \lesssim r^{w \frac{\alpha}{\alpha + \sigma}} \|f\|_{L_1(E_r(x,l))}$$  

(3.5)  

holds for all $f \in L^1_\text{loc}(\mathbb{R}^n)$.  

Proof. Let $1 < p < \infty$, $0 \leq \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$. Denote 

$$M_1 = |E_{\sigma}|^{\frac{1}{2}} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \left( \int_{E_{\sigma}(x,t)} |f(z)|^q dz \right)^{\frac{1}{q}}.$$  

$$M_2 = \|f\|_{L_p(E_{\sigma})}.$$  

Applying H"older’s inequality, we get  

$$M_1 \lesssim |E_{\sigma}|^{\frac{1}{2}} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \left( \int_{E_{\sigma}(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}}.$$  

On the other hand,  

$$|E_{\sigma}|^{\frac{1}{2}} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \left( \int_{E_{\sigma}(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \lesssim |E_{\sigma}|^{\frac{1}{2}} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \|f\|_{L_p(E_{\sigma})} \approx M_2.$$  

Since by Lemma 3.1 

$$\|M_{a,r}f\|_{L_p(E_{\sigma})} \leq M_1 + M_2,$$  

we arrive at (3.4).  

Let $p = 1$. The inequality (3.5) directly follows from (3.2).  

Theorem 3.3 Let $1 \leq p < \infty$, $0 \leq \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, and $(\varphi_1, \varphi_2)$ satisfies the condition  

$$\sup_{r < q < \infty} r^p \varphi_1(x,t) \leq C \varphi_2(x,r),$$  

(3.6)  

where $C$ does not depend on $x$ and $r$. Then for $p > 1$, $M_{a,r}$ is bounded from $M_{p,\varphi_1,\varphi_2}$ to $M_{q,\varphi_2,\varphi_2}$ and for $p = 1$, $M_{a,r}$ is bounded from $M_{1,\varphi_1,\varphi_2}$ to $WM_{q,\varphi_2,\varphi_2}$.  

Proof. By Lemma we get  

$$\|M_{a,r,f}\|_{L_{q_1}(x,l)} \lesssim \sup_{x \in \mathbb{R}^n \cap 0} \varphi_2(x,r)^{-1} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \left( \int_{E_{\sigma}(x,t)} |f(z)|^q dz \right)^{\frac{1}{q}}$$  

$$\lesssim \|f\|_{M_{q,\varphi_2,\varphi_2}} \sup_{x \in \mathbb{R}^n \cap 0} \varphi_2(x,r)^{-1} \sup_{\sigma \cdot r > 0} \varphi_1(x,t)$$  

$$\lesssim \|f\|_{M_{q,\varphi_2,\varphi_2}}$$  

if $p \in (1, \infty)$ and  

$$\|M_{a,r,f}\|_{WM_{q_1}(x,l)} \lesssim \sup_{x \in \mathbb{R}^n \cap 0} \varphi_2(x,r)^{-1} \left( \sup_{\sigma \cdot r > 0} \frac{1}{|E_{\sigma}(x,t)|^{\frac{1}{2}}} \right) \left( \int_{E_{\sigma}(x,t)} |f(z)|^q dz \right)^{\frac{1}{q}}$$  

$$\lesssim \|f\|_{M_{q,\varphi_2,\varphi_2}} \sup_{x \in \mathbb{R}^n \cap 0} \varphi_2(x,r)^{-1} \sup_{\sigma \cdot r > 0} \varphi_1(x,t)$$  

$$\lesssim \|f\|_{M_{q,\varphi_2,\varphi_2}}$$  

if $p = 1$.  

113
In the case $\alpha = 0$ and $p = q$ from Theorem 3.3 we get the following corollary, which proven in Akbulut, Guliyev and Mustafayev (2012) on $\mathbb{R}^n$.

**Corollary 3.1** Let $1 \leq p < \infty$ and $(\varphi_1, \varphi_2)$ satisfies the condition

$$\sup_{r \in \mathbb{R}^n} \varphi_1(x, t) \leq C \varphi_2(x, r), \quad (3.7)$$

where $C$ does not depend on $x$ and $r$. Then for $p > 1$, $M_\alpha$ is bounded from $M_{p, \varphi_1, \alpha}$ to $M_{p, \varphi_2, \alpha}$ and for $p = 1$, $M_\alpha$ is bounded from $M_{1, \varphi_1, \alpha}$ to $WM_{1, \varphi_2, \alpha}$.

**Corollary 3.2** Let $p \in [1, \infty)$ and let $\varphi : (0, \infty) \to (0, \infty)$ be an decreasing function. Assume that the mapping $r \mapsto \varphi(r) r^{\frac{\alpha}{p}}$ is almost increasing (there exists a constant $c$ such that for $s < r$ we have $\varphi(s) s^{\frac{\alpha}{p}} \leq c \varphi(r) r^{\frac{\alpha}{p}}$). Then there exists a constant $C > 0$ such that

$$\|Mf\|_{M_{p, \varphi}} \leq C \|f\|_{M_{p, \varphi}} \quad \text{if } p > 1,$$

and

$$\|Mf\|_{WM_{p, \varphi}} \leq C \|f\|_{M_{p, \varphi}}.$$

### 3.2 Adams-Guliyev Type Result

The following is a result of Adams-Guliyev type for the fractional maximal operator.

**Theorem 3.4** Let $1 \leq p < q < \infty$, $0 < \alpha < \frac{\mu}{p}$ and let $\varphi(x, t)$ satisfy the condition

$$\sup_{r < r_0} \varphi(x, t) \leq C \varphi(x, r), \quad (3.8)$$

and

$$\sup_{r < r_0} t^{\alpha} \varphi(x, t)^{\frac{1}{p}} \leq Cr^{\frac{\alpha}{q/p}}, \quad (3.9)$$

where $C$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $M_{\alpha, \sigma}$ is bounded from $M_{p, \varphi_1, \alpha}$ to $M_{q, \varphi_1, \alpha}$ for $p > 1$ and from $M_{1, \varphi_1, \alpha}$ to $WM_{q, \varphi_1, \alpha}$.

**Proof.** Let $1 \leq p < q < \infty$, $0 < \alpha < \frac{\mu}{p}$ and $f \in M_{p, \varphi_1, \alpha}$. Write $f = f_1 + f_2$, where $E_{\sigma} = E_{\sigma}(x, r)$, $f_1 = f \chi_{2E_\sigma}$ and $f_2 = f \chi_{(2E_\sigma)^c}$.

For $M_{\alpha, \sigma}f_2(y)$ for all $y \in E_\sigma$ from (3.3) we have

$$M_{\alpha, \sigma}(f_2)(y) \leq 2^{\mu - \alpha} \sup_{t > 2} \frac{1}{E_{\sigma}(x, t)} \int_{E_{\sigma}(x, t)} |f(z)|dz \leq \sup_{t > 2} \frac{1}{t^{\alpha/q\cdot p}} \|f\|_{L_{\varphi_1}(E_{\sigma}(x, t))}.$$

Then from conditions (3.9) and (3.10) for all $y \in E_\sigma$ we get

$$M_{\alpha, \sigma}f(y) \leq r^{\alpha} Mf(y) + \sup_{t > 2} \frac{1}{t^{\alpha/q\cdot p}} \|f\|_{L_{\varphi_1}(E_{\sigma}(x, t))} \leq r^{\alpha} Mf(y) + \|f\|_{M_{p, \varphi_1, \alpha}} \sup_{t > 2} r^{\alpha} \varphi(x, t)^{\frac{1}{p}} \leq r^{\alpha} Mf(y) + r^{\frac{\mu}{p}} \|f\|_{M_{p, \varphi_1, \alpha}}.$$

Hence choose $r = \left(\frac{\|f\|_{M_{p, \varphi_1, \alpha}}}{Mf(y)}\right)^{\frac{\mu}{\alpha}}$ for every $y \in E_\sigma$, we have

$$|M_{\alpha, \sigma}f(y)| \leq (Mf(y))^\frac{\mu}{\alpha} \|f\|_{M_{p, \varphi_1, \alpha}}.$$

114
Hence the statement of the theorem follows in view of the boundedness of the anisotropic maximal operator $M_{\sigma}$ in $M_{p, q^{\infty}, \sigma^{\infty}}$ provided by Corollary 3.1 in virtue of condition (3.8).

$$
\|M_{\sigma, \tau} f\|_{M_{p, \tau}} \leq \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_{\sigma, \tau} f\|_{L_p(E_{\tau}(x, \tau))}) \\
\lesssim \frac{1}{\|f\|_{M_{p, \tau}}} \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_f\|_{L_p(E_{\tau}(x, \tau))}) \\
= \frac{1}{\|f\|_{M_{p, \tau}}} \left( \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_f\|_{L_p(E_{\tau}(x, \tau))}) \right)^{\frac{1}{\gamma}} \\
= \frac{1}{\|f\|_{M_{p, \tau}}} \|M_f\|_{L_p(E_{\tau}(x, \tau))} \\
\lesssim \|f\|_{L_p(E_{\tau}(x, \tau))}
$$

if $1 < p < q < \infty$ and

$$
\|M_{\sigma, \tau} f\|_{W_{p, \tau}} \leq \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_{\sigma, \tau} f\|_{W_{L_p}(E_{\tau}(x, \tau))}) \\
\lesssim \frac{1}{\|f\|_{W_{p, \tau}}} \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_f\|_{W_{L_p}(E_{\tau}(x, \tau))}) \\
= \frac{1}{\|f\|_{W_{p, \tau}}} \left( \sup_{x \in \mathbb{R}^n, \tau > 0} \varphi(x, \tau^{-\frac{1}{\gamma}} \tau^{-\frac{1}{\tau}} \|M_f\|_{W_{L_p}(E_{\tau}(x, \tau))}) \right)^{\frac{1}{\gamma}} \\
= \frac{1}{\|f\|_{W_{p, \tau}}} \|M_f\|_{W_{L_p}(E_{\tau}(x, \tau))} \\
\lesssim \|f\|_{W_{p, \tau}}
$$

if $1 < q < \infty$.

In the case $\varphi(x, \tau) = \tau^{(b-1)\frac{|\tau|}{\gamma}}$, $0 < b < 1$ from Theorem we get the following Adams type result for the fractional maximal operator.

**Corollary 3.3** Let $0 < \alpha < |\sigma|$, $1 \leq p < \frac{|\sigma|}{\alpha}$, $0 < \lambda < |\sigma| - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\sigma}{|\sigma|-1}$. Then for $p > 1$, the operator $M_{\alpha, \sigma}$ is bounded from $L_{p, b, \sigma}$ to $L_{q, b, \sigma}$ and for $p = 1$, $M_{\alpha, \sigma}$ is bounded from $L_{1, b, \sigma}$ to $W_{1, q, b, \sigma}$.

4. **Parabolic Schrödinger Type Operators** $V^\gamma(\frac{\partial}{\partial t} - \Delta + V)^{\beta}$ and $V^\gamma\nabla^2(\frac{\partial}{\partial t} - \Delta + V)^{\beta}$

In this section we consider the parabolic Schrödinger operator

$$
\frac{\partial}{\partial t} - \Delta + V \quad \text{on} \quad \mathbb{R}^{n+1},
$$

where $V = V(x, t)$ is a nonnegative potential which belongs to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$. Examples of such potentials are all positive polynomials but also singular functions like max functions. To extend the results to the parabolic case, we consider the parabolic operator $\frac{\partial}{\partial t} - \Delta + V$ where $V \in B_q(\mathbb{R}^{n+1})$ is a nonnegative potential depending only on the space variables and, under the assumptions $n \geq 3$ and $p > (n + 2)/2$, they proved the boundedness of $V(\frac{\partial}{\partial t} - \Delta + V)^{-1}$ in $L_p(\mathbb{R}^{n+1})$. 

115
The main purpose of this section is to investigate the parabolic generalized Morrey $M_{p,q;\sigma,n}(\mathbb{R}^{n+1})$ boundedness of the operators

$$T_1 = V^\gamma (\frac{\partial}{\partial t} - \Delta + V)^{\beta}, \quad 0 \leq \gamma \leq 1,$$

$$T_2 = V^\gamma \nabla^2 (\frac{\partial}{\partial t} - \Delta + V)^{\beta}, \quad 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1, \quad \beta - \gamma \geq \frac{1}{2}.$$  

Note that the operator $\nabla^2 (\frac{\partial}{\partial t} - \Delta + V)^{-1}$ in Gao and Jiang (2005) is the special case of $T_2$.

It is worth pointing out that we need to establish pointwise estimates for $T_1$, $T_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^{n+1}$ in Gao and Jiang (2005). And we prove the parabolic generalized Morrey estimates by using $M_{p,q;\sigma,n}(\mathbb{R}^{n+1}) \to M_{p,q;\sigma,n}(\mathbb{R}^{n+1})$ boundedness of the parabolic fractional maximal operators.

**Definition 4.4** 1) A nonnegative locally $L_q$ integrable function $V$ on $\mathbb{R}^{n+1}$ is said to belong to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$ ($1 < q < \infty$) if there exists $C > 0$ such that the reverse Hölder inequality

$$\left( \frac{1}{|K|} \int_K V(y, \tau)^q dyd\tau \right)^{\frac{1}{q}} \leq C \frac{|K|}{\int_K V(y, \tau) dyd\tau}$$

holds for every parabolic cylinder

$$K = K((x, t), r) = \{ (y, \tau) \in \mathbb{R}^{n+1} : |x_i - y_i| < r, |t - \tau| < r^2, i = 1, \ldots, n \}$$

of center $(x, t)$ and radius $r$ in $\mathbb{R}^{n+1}$.

2) Let $V = V(x, t) \geq 0$. We say $V \in B_{\infty}(\mathbb{R}^{n+1})$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_{\infty}(K)} \leq C \frac{|K|}{\int_K V(y, \tau) dyd\tau}$$

holds for every parabolic cylinder $K = K((x, t), r)$ in $\mathbb{R}^{n+1}$.

Clearly, $B_{\infty}(\mathbb{R}^{n+1}) \subset B_q(\mathbb{R}^{n+1})$ for $1 < q < \infty$. But it is important that the $B_q(\mathbb{R}^{n+1})$ class has a property of “self-improvement”; that is, if $V \in B_q(\mathbb{R}^{n+1})$, then $V \in B_{q+\varepsilon}(\mathbb{R}^{n+1})$ for some $\varepsilon > 0$ (see Li, 1999).

By the functional calculus, we may write, for all $0 < \beta < 1$,

$$\left( \frac{\partial}{\partial t} - \Delta + V \right)^{\beta} = \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} \left( \frac{\partial}{\partial t} - \Delta + V + \lambda \right)^{-1} d\lambda.$$  

Let $f \in C_0^\infty(\mathbb{R}^{n+1})$. From

$$\left( \frac{\partial}{\partial t} - \Delta + V + \lambda \right)^{-1} f(x, t) = \int_{\mathbb{R}^{n+1}} \Gamma(x, t; y, \tau; \lambda) f(y, \tau) dyd\tau,$$

it follows that

$$T_1 f(x, t) = \int_{\mathbb{R}^{n+1}} K_1(x, t; y, \tau) f(y, \tau) dyd\tau,$$

where

$$K_1(x, t; y, \tau) = \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} \Gamma(x, t; y, \tau; \lambda) d\lambda \quad \text{for} \quad 0 < \beta < 1,$$

$$K_1(x, t; y, \tau) = \Gamma(x, t; y, \tau; 0) \quad \text{for} \quad \beta = 1.$$  

The following two pointwise estimates for $T_1$ and $T_2$ which proven in Zhong (1993), Lemma 3.2 with the potential $V \in B_{\infty}(\mathbb{R}^{n+1})$.

**Theorem A** Suppose $V \in B_{\infty}(\mathbb{R}^{n+1})$ and $0 \leq \gamma \leq \beta \leq 1$. Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$|T_1 f(x, t)| \lesssim M_{\alpha;\sigma,n} f(x, t),$$

where $\alpha = 2(\beta - \gamma)$.  

116
Theorem B Suppose \( V \in B_\alpha(\mathbb{R}^{n+1}), 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1 \) and \( \beta - \gamma \geq \frac{1}{2} \). Then, for any \( f \in C_0^\infty(\mathbb{R}^{n+1}) \)

\[
|T_2f(x,t)| \lesssim M_{\alpha,\gamma}\gamma f(x,t),
\]

where \( \alpha = 2(\beta - \gamma) - 1 \).

Note that the similar estimates for the adjoint operators \( T_1^* \) and \( T_2^* \) with the potential \( V \in B_\alpha \) for some \( q_1 > \frac{n+2}{2} \) also valid (see Liu, 2009).

Theorem C Suppose \( V \in B_{q_1}(\mathbb{R}^{n+1}) \) for some \( q_1 > \frac{n+2}{2} \), \( 0 \leq \gamma \leq \beta \leq 1 \) and let \( \frac{1}{q_1} = 1 - \frac{1}{q_1} \). Then there exists a constant \( C > 0 \) such that

\[
|T_1^*f(x,t)| \leq C \left( M_{\alpha,2q_1}(|f|^{q_1})(x,t) \right)^{\frac{1}{q_1}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),
\]

where \( \alpha = 2(\beta - \gamma) \).

Theorem D Suppose \( V \in B_{q_1}(\mathbb{R}^{n+1}) \) for some \( q_1 > \frac{n+2}{2} \), \( 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1 \) and \( \beta - \gamma \geq \frac{1}{2} \). And let

\[
\frac{1}{q_2} = \left\{ \begin{array}{ll}
\frac{1}{q_2} = 1 - \frac{1}{q_1} & \text{if } q_1 > n + 2, \\
1 - \frac{1}{q_1} + \frac{1}{n+2} & \text{if } \frac{n+2}{2} < q_1 < n + 2.
\end{array} \right.
\]

Then there exists a constant \( C > 0 \) such that

\[
|T_2^*f(x,t)| \leq C \left( M_{\alpha,2q_2}(|f|^{q_2})(x,t) \right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),
\]

where \( \alpha = 2(\beta - \gamma) - 1 \).

The above theorems will yield the parabolic generalized Morrey estimates for \( T_1 \) and \( T_2 \).

Corollary 4.4 Assume that \( V \in B_\alpha(\mathbb{R}^{n+1}) \), and \( 0 \leq \gamma \leq \beta \leq 1 \). Let \( 1 \leq p \leq q < \infty \), \( 2(\beta - \gamma) = (n + 2) \left( \frac{1}{p} - \frac{1}{q} \right) \) and the condition (3.6) be satisfied for \( \alpha = 2(\beta - \gamma) \).

Then, for any \( f \in C_0^\infty(\mathbb{R}^{n+1}) \)

\[
\|T_1f\|_{M_{\alpha,2q_1}} \lesssim \|f\|_{M_{\alpha,2q_1}}, \quad \text{for } p > 1
\]

and

\[
\|T_1f\|_{WM_{\alpha,2q_1}} \lesssim \|f\|_{M_{\alpha,2q_1}} \quad \text{for } p = 1
\]

Corollary 4.5 Assume that \( V \in B_\alpha(\mathbb{R}^{n+1}) \), \( 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1 \) and \( \beta - \gamma \geq \frac{1}{2} \). Let \( 1 \leq p \leq q < \infty \), \( 2(\beta - \gamma) - 1 = (n + 2) \left( \frac{1}{p} - \frac{1}{q} \right) \) and the condition (3.6) be satisfied for \( \alpha = 2(\beta - \gamma) - 1 \).

Then, for any \( f \in C_0^\infty(\mathbb{R}^{n+1}) \)

\[
\|T_2f\|_{M_{\alpha,2q_2}} \lesssim \|f\|_{M_{\alpha,2q_2}}, \quad \text{for } p > 1
\]

and

\[
\|T_2f\|_{WM_{\alpha,2q_2}} \lesssim \|f\|_{M_{\alpha,2q_2}} \quad \text{for } p = 1
\]

Corollary 4.6 Assume that \( V \in B_{q_1}(\mathbb{R}^{n+1}) \) for \( q_1 > \frac{n+2}{2} \), and \( 0 \leq \gamma \leq \beta \leq 1 \). Let \( 1 \leq p < \frac{1}{n+2}, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2} \), \( \frac{1}{q_1} = 1 - \frac{1}{q_1} \) and the condition (3.6) be satisfied for \( \alpha = 2(\beta - \gamma) \).

Then, for any \( f \in C_0^\infty(\mathbb{R}^{n+1}) \)

\[
\|T_1f\|_{M_{\alpha,2q_1}} \lesssim \|f\|_{M_{\alpha,2q_1}}, \quad \text{for } p > 1
\]

and

\[
\|T_1f\|_{WM_{\alpha,2q_1}} \lesssim \|f\|_{M_{\alpha,2q_1}} \quad \text{for } p = 1
\]

Corollary 4.7 Assume that \( V \in B_{q_1}(\mathbb{R}^{n+1}) \) for \( q_1 > \frac{n+2}{2} \), and

\[
\left\{ \begin{array}{ll}
0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1, & \text{if } q_1 > n + 2, \\
0 \leq \gamma \leq \frac{1}{2} < \beta \leq 1, & \text{if } \frac{n+2}{2} < q_1 < n + 2.
\end{array} \right.
\]
Let $\beta - \gamma \geq \frac{1}{2}$, $1 \leq p < \frac{1}{\frac{1}{q_1} + \frac{1}{q_2}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2}$, $\frac{1}{q_1} = 1 - \frac{\alpha}{q_1}$, and the condition (3.6) be satisfied for $\alpha = 2(\beta - \gamma) - 1$, where

$$
\frac{1}{p_1} = \begin{cases} \frac{\gamma_1}{q_1} - \frac{1}{n+2}, & \text{if } q_1 > n+2, \\ \frac{\gamma_1}{q_1} - \frac{\alpha}{q_2}, & \text{if } \frac{n+2}{2} < q_1 < n+2. 
\end{cases}
$$

Then, for any $f \in C^0_0(\mathbb{R}^{n+1})$

$$
\|T_2 f\|_{M_{\frac{q_1}{q_1-\alpha}, q_1}} \lesssim \|f\|_{M_{\frac{q_1}{q_1-\alpha}, q_1}}, \quad \text{for } p > 1
$$

and

$$
\|T_2 f\|_{WM_{\frac{q_1}{q_1-\alpha}, q_1}} \lesssim \|f\|_{M_{\frac{q_1}{q_1-\alpha}, q_1}} \quad \text{for } p = 1
$$

References


