Journal of Mathematics Research; Vol. 4, No. 6; 2012
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

The Space of Coefficients in a Linear Topological Space

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Received: August 28, 2012 Accepted: September 14, 2012 Online Published: November 26, 2012

doi:10.5539/jmr.v4n6p83 URL: http://dx.doi.org/10.5539/jmr.v4n6p83

Abstract

It is proved that the arbitrary nondegenerate system in a linear complete topological space has a correspondence complete topological space of coefficients with canonical basis. Basicity criterion for systems in such spaces is given in terms of coefficient operator.

Keywords: topological space, nondegenerate system, space of coefficients

Classification 2000: 41A65; 46B15

1. Introduction

The concept of the space of coefficients belongs to the theory of bases. As is known, every basis in a Banach space has a Banach space of coefficients which is isomorphic to an initial one (see, e.g., Dremin, Ivanov, & Nechitailo, 2001; Singer, 1970; Singer, 1981). Every nondegenerate system (to be defined later) in a Banach space generates the corresponding Banach space of coefficients with canonical basis (see, e.g., Bilalov & Najafov, 2011; Dremin, Ivanov, & Nechitailo, 2001). Similar results are obtained in fuzzy structures (Bilalov, Farahani, & Guliyeva, 2012) which have recently gained a great scientific interest. Therefore, space of coefficients plays an important role in the study of approximative properties of systems. It has very important applications in various fields of science, such as solid body physics, molecular physics, multiple production of particles, aviation, medicine, biology, data compression, etc (see, e.g., Chui, 1992; Edwards, 1969 and references within). All these applications are closely related to wavelet analysis, and there arose a great interest in them lately (see, e.g., Chui, 1992; Christensen, 2003 & 2004; Dremin, Ivanov, & Nechitailo, 2001). It is well known that many topological spaces are nonnormable. Therefore, the study of various properties of the space of coefficients in topological and, in particular, in metric spaces is of special scientific interest.

Our work is dedicated to the study of topological properties of the space of coefficients generated by nondegenerate system in a Hausdorff linear topological spaces. It is structured as follows. In Section 2 we state basic concepts and facts to be used later. In Section 3 we state and prove our main results. We prove that the arbitrary nondegenerate system in a linear complete topological space generates complete linear topological space of coefficients with canonical basis. Basicity criterion for systems in such spaces is given in terms of coefficient operator.

2. Needful Concepts and Facts

We will use the usual notations: $\mathbb{N}$ will be the set of all positive integers, $\mathbb{R}$ will denote the set of all real numbers, $\mathbb{C}$ will stand for the set of all complex numbers, $L(X;Y)$ will be the linear space of continuous linear operators from X to Y, and F will denote the field of scalars. By the linear topological space $(X;\ \tau)$ (LTS in short) we mean the linear space X over field F ($F \equiv \mathbb{R}$ or $F \equiv \mathbb{C}$) with a topology τ, where linear operations are continuous and every point is a closed set. Set $M \subset X$ is said to be bounded if for an arbitrary neighborhood of zero O_ε, $\exists\delta > 0 : \lambda M \subset O_\varepsilon$, $\forall\lambda : |\lambda| < \delta$. By the neighborhood $O_\varepsilon(x_0)$ of point $x_0 \in X$ we mean an open set $x_0 + O_\varepsilon$, where O_ε is a neighborhood of zero. Local base in X is a family of the neighborhoods of zero $\mathscr{B}$ such that the every neighborhood of zero contains some neighborhood belonging to $\mathscr{B}$. $(X;\ \tau)$ is called a metrizable space if its topology τ is generated some metric. Metrizable and complete space is called Frechet space or F -space. By $L[M]$ we denote the linear span of the set $M \subset X$. The closure of this set in topology τ is denoted by $\bar{M}$. We also state some facts from the theory of LTS which will be used later in this work.

Let M be some set and " $\leq$ " an order relation (partial) in it. $(M;\leq)$ is called directed set if $\forall\alpha,\beta \in M$, $\exists\gamma \in M$: $\alpha \leq \gamma \wedge \beta \leq \gamma$. $\{x_\lambda\}_{\lambda\in M} \subset X$ is called net in X if M is a directed set. Let $(X;\tau)$ be some LTS. Net $\{x_\lambda\}_{\lambda\in M}$ converges to $k_0 \in X$ over M, if for any neighborhood O_{x_0} of the point x_0, $\exists \lambda_0 \in M : x_\lambda \in O_{x_0}, \forall \lambda \geq \lambda_0$. This fact we will denote by $\lim_{\lambda \in M} x_\lambda = x_0$ or $x_\lambda \to x_0, \lambda \in M$. The net $\{x_\lambda\}_{\lambda \in M}$ is called Cauchy net, if for arbitrary neighborhood of zero O_ε in X, $\exists \lambda_0 \in M : (x_\lambda - x_\mu) \in O_\varepsilon$, $\forall \lambda.\mu \geq \lambda_0$. If any Cauchy net converges in $(X; \tau)$, then this space is called complete.

Definition 1 System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called complete in X if $\overline{L[\{x_n\}_{n\in\mathbb{N}}]} \equiv X$.

Definition 2 System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n\neq k}]}$, $\forall k \in \mathbb{N}$.

Definition 3 System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called ω-linearly independent in X if $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ in X implies that $\lambda_n = 0$, $\forall n \in \mathbb{N}$.

Definition 4 System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called a basis for X if $\forall x \in X$, $\exists! \{\lambda_n\}_{n\in\mathbb{N}} \subset F$:$x = \sum_{n=1}^{\infty} \lambda_n x_n$.

We will use the following concept.

Definition 5 System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called nondegenerate if $x_n \neq 0$, $\forall n \in \mathbb{N}$.

Also recall some facts which will be needed to obtain our main results.

Theorem 1 (Chui, 1992) *If $\mathscr{B}$ is a local base in LTS $(X; \tau)$, then every neighborhood of zero belonging to it contains the closure of some other neighborhood of zero in $\mathscr{B}$.*

Theorem 2 (Rudin, 1975) *Every LTS $(X; \tau)$ has a balanced local base.*

Just recall that the set $M \subset X$ is called balanced if $\alpha M \subset M$ for $\forall \alpha \in F : |\alpha| \leq 1$. Set $M \subset X$ is compact if every open covering of it has a finite sub-covering. Set $M \subset X$ is pre-compact if its closure is compact. The arbitrary family of sets $\{M_\alpha\}_{\alpha\in A} : M \subset \bigcup_{\alpha\in A} M_\alpha$ is called the covering of the set M, where A is an index set. More details of these and other facts and concepts can be found in the monographs (Bourbaki, 1959 & 1968; Edwards, 1969; Heil, 2011; Rudin, 1975).

3. Space of Coefficients

Let $(X; \tau)$ be a complete linear topological space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some nondegenerate system. Assume

$$\mathscr{K}_{\bar{x}} \equiv \left\{ \{\lambda_n\}_{n\in\mathbb{N}} \subset F : \text{ the series } \sum_{n=1}^{\infty} \lambda_n x_n \text{ is convergent in } X \right\}.$$

Obviously, $\mathscr{K}_{\bar{x}}$ is a linear space with regard to usual operations of component-specific addition and multiplication by a scalar. Every neighborhood of zero O_ε in X generates corresponding neighborhood of zero $O_\varepsilon^{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$:

$$O_\varepsilon^{\mathscr{K}} \equiv \left\{ \bar{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}} : \sum_{n=1}^{m} \lambda_n x_n \in O_\varepsilon ,\ \forall m \in \mathbb{N} \right\}.$$

The set of neighborhoods of zero $O_\varepsilon^{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$ generates corresponding topology $\tau_{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$. Every linear topological space is a Hausdorff space. Consequently, every compact in such spaces is bounded. Therefore, every Cauchy sequence is pre-compact and, consequently, is bounded in such spaces. More details about these facts can be found in Bourbaki (1959) and Heil (2011).

Let us show that $(\mathscr{K}_{\bar{x}}; \tau_{\mathscr{K}})$ is complete. We will need the following:

Lemma 1 *Let $(X; \tau)$ be an LTS, $x \in X$, $x \neq 0$,$\{f_\lambda\}_{\lambda\in M} \subset F-$ be some net and $f_\lambda x \to 0$ as $\lambda \in M$. Then $f_\lambda \to 0$ as $\lambda \in M$.*

Proof. Suppose that the net $\{f_\lambda\}_{\lambda\in M}$ does not converge to zero. Also suppose that $\{f_\lambda\}_{\lambda\in M}$ has a bounded subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Then it is possible to derive a convergent subsequence from it, and, without loss of generality, we will assume that $\lambda_{n_k} \to \lambda_0$, $k \to \infty$. We have $\lambda_{n_k} x \to \lambda_0 x$ as $k \to \infty$. Hence, $\lambda_0 = 0$, because $(X; \tau)$ is a Hausdorff space (see, e.g., Bourbaki, 1968). Thus, every bounded subsequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converges to zero. It follows from the assumption made above that $\{\lambda_n\}_{n\in\mathbb{N}}$ contains unbounded subsequence. Let $\lambda_{n_k} \to \infty$ as $k \to \infty$. Then $\mu_k = \frac{1}{\lambda_{n_k}} \to 0$, $k \to \infty$. As a result, $\lim_{k\to\infty} \mu_k (\lambda_{n_k} x) = \lim_{k\to\infty} \mu_k \lim_{k\to\infty} (\lambda_{n_k} x) = 0$. On the other hand, $\mu_k (\lambda_{n_k} x) = \frac{1}{\lambda_{n_k}} \lambda_{n_k} x = x \neq 0$. So we came upon a contradiction which proves the lemma.

Take an arbitrary Cauchy net $\left\{\bar{f}_\lambda\right\}_{\lambda\in M} \subset \mathscr{K}_{\bar{x}}$ with $\bar{f}_\lambda \equiv \left\{f_k^{(\lambda)}\right\}_{k\in\mathbb{N}}$. Take a neighborhood of zero $O_\varepsilon^{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$ generated by an arbitrary neighborhood of zero O_ε in X. Definitionally this means that $\exists \lambda_0 \in M : \bar{f}_\lambda - \bar{f}_\mu \in O_\varepsilon^{\mathscr{K}}$, $\forall \lambda, \mu \geq \lambda_0$.

Consequently

$$\sum_{k=1}^{r}\left(f_k^{(\lambda)}-f_k^{(\mu)}\right)x_k\in O_\varepsilon,\ \forall r\in\mathbb{N},\ \forall\lambda,\mu\geq\lambda_0.$$

And this, in turn, means that the net $\left\{\sum_{k=1}^{r}f_k^{(\lambda)}x_k\right\}_{\lambda\in M}$ is convergent for $\forall r\in\mathbb{N}$. In particular, $\left\{f_1^{(\lambda)}x_1\right\}_{\lambda\in M}$ is convergent. From Lemma 1 we obtain that $\left\{f_1^{(\lambda)}\right\}_{\lambda\in M}$ is convergent. As $\left\{\sum_{k=1}^{2}f_k^{(\lambda)}x_k\right\}_{\lambda\in M}$is convergent, we obtain from $f_2^{(\lambda)}x_2=\sum_{k=1}^{2}f_k^{(\lambda)}x_k-f_1^{(\lambda)}x_1$ that the net $\left\{f_2^{(\lambda)}x_2\right\}_{\lambda\in M}$ is convergent in X, and, as a result, $\left\{f_2^{(\lambda)}\right\}_{\lambda\in M}$is convergent. Continuing this process, we obtain that $\left\{f_k^{(\lambda)}\right\}_{\lambda\in M}$ is convergent for $\forall k\in\mathbb{N}$. Let $f_k^{(\lambda)}\to\lambda_k,\ \ \lambda\in M$. Assume $\bar{f}\equiv\{f_\lambda\}_{\lambda\in M}$. Let us show that $\bar{f}\in\mathscr{K}_{\bar{x}}$ and $\bar{f}_n\ \to\ \bar{f}$ as $\lambda\in M$. Let $\mathscr{B}$ be some local base of the neighborhoods of zero in X. By virtue of Theorem 2, such base always exists in LTS. Take $\forall O_{\varepsilon_1}\in\mathscr{B}$. Due to Theorem 1, $\exists O_{\varepsilon_2}\in\mathscr{B}:\overline{O_{\varepsilon_2}}\subset O_{\varepsilon_1}$. Let $O_{\varepsilon_2}^{\mathscr{K}}$ be the neighborhood of zero in $\mathscr{K}_{\bar{x}}$ corresponding to O_{ε_2}. Obviously, $\exists\lambda_0\in M:$ $\left(\bar{f}_\lambda-\bar{f}_\mu\right)\in O_{\varepsilon_2}^{\mathscr{K}},\ \forall\lambda,\mu\geq\lambda_0$. According to the definition, this implies

$$\sum_{k=1}^{r}\left(f_k^{(\lambda)}-f_k^{(\mu)}\right)x_k\in O_{\varepsilon_2}\subset O_{\varepsilon_1},\ \forall r\in\mathbb{N},\ \forall\lambda,\mu\geq\lambda_0. \tag{1}$$

It follows from the arbitrariness of O_{ε_1} that the net $\left\{\sum_{k=1}^{r}f_k^{(\lambda)}x_k\right\}_{\lambda\in M}$ is Cauchy net in X for $\forall r\in\mathbb{N}$. Passing to the limit in (1) as $\mu\in M$ yields

$$\sum_{k=1}^{r}\left(f_k^{(n)}-f_k\right)x_k\in\overline{O_{\varepsilon_2}}\subset O_{\varepsilon_1},\ \forall r\in\mathbb{N},\ \forall\lambda\geq\lambda_0. \tag{2}$$

In what follows, we will use the following assertion:

Assertion 1 (Rudin, 1975) *For every neighborhood of zero W in X there exists a symmetric neighborhood of zero U(in the sense that $U=-U$) which satisfies the relation $U+U+U\subset W$.*

By virtue of Theorem 1, we have directly from this assertion the following.

Corollary 1 *For every neighborhood of zero W in X there exists a neighborhood of zero Usuch that $\bar{U}+\bar{U}+\bar{U}\subset W$.*

Taking neighborhood W as O_{ε_1} and assuming $O_{\varepsilon_2}=U$, we have

$$\overline{O_{\varepsilon_2}}+\overline{O_{\varepsilon_2}}+\overline{O_{\varepsilon_2}}\subset O_{\varepsilon_1}. \tag{3}$$

As $\bar{f}_\lambda\in\mathscr{K}_{\bar{x}}$, it is clear that $\exists r_0\in\mathbb{N}:$

$$\sum_{k=r}^{r+p}f_k^{(\lambda)}x_k\in O_{\varepsilon_2},\ \forall r\geq r_0,\ \forall p\in\mathbb{N}. \tag{4}$$

We have

$$\sum_{k=r}^{r+p}f_kx_k=\sum_{k=r}^{r+p}f_k^{(\lambda)}x_k+\sum_{k=r}^{r+p}\left(f_k-f_k^{(\lambda)}\right)x_k=\sum_{k=r}^{r+p}f_k^{(\lambda)}x_k+\sum_{k=1}^{r+p}\left(f_k-f_k^{(\lambda)}\right)x_k+\sum_{k=1}^{r-1}\left(f_k^{(\lambda)}-f_k\right)x_k.$$

Taking $n\geq m_0$, by virtue of inclusions (2)-(4) we get

$$\sum_{k=r}^{r+p}f_kx_k\in\overline{O_{\varepsilon_2}}+\overline{O_{\varepsilon_2}}+\overline{O_{\varepsilon_2}}\subset O_{\varepsilon_1}.$$

Consequently, the series $\sum_{k=1}^{\infty}f_kx_k$ is convergent in X. Hence, $\bar{f}\in\mathscr{K}_{\bar{x}}$. It follows directly from (2) that $\bar{f}_\lambda\to\bar{f}$ as $\lambda\in M$ in $\mathscr{K}_{\bar{x}}$.

Let us show that the linear operations are continuous in $\mathscr{K}_{\bar{x}}$. Take $\forall\bar{f}\in\mathscr{K}_{\bar{x}}$ and assume $a\to a_0$ in F. Let us show that $(a-a_0)\bar{f}\to0$ as $a\to a_0$. Let $S_m=\sum_{n=1}^{m}f_nx_n$, with $\bar{f}\equiv\{f_n\}_{n\in\mathbb{N}}$. As the sequence $\{S_m\}_{m\in\mathbb{N}}$ is convergent in X, it is evidently bounded there. Take an arbitrary neighborhood of zero O_ε in X. Then $\exists\delta>0:tS_m\in O_\varepsilon$, $\forall m\in\mathbb{N}$, $\forall t:|t|<\delta$. Consequently, $(a-a_0)S_m\in O_\varepsilon$, $\forall m\in\mathbb{N}$, $\forall a:\ |a-a_0|<\delta$, i.e.

$$\sum_{n=1}^{m}(a-a_0)\lambda_nx_n\in O_\varepsilon,\forall m\in\mathbb{N},\forall a:\ |a-a_0|<\delta.$$

Definitionally this means that $a\bar{f} \to a_0\bar{f}$ as $a \to a_0$ in $\mathscr{K}_{\bar{x}}$. Now let $\bar{f}^{(\lambda)} \to \bar{f}$, $\bar{g}^{(\lambda)} \to \bar{g}$, $\lambda \in M$ in $\mathscr{K}_{\bar{x}}$, where $\bar{f}^{(\lambda)} \equiv \left\{f_k^{(\lambda)}\right\}_{k\in\mathbb{N}}$, $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}}$, $\bar{g}^{(\lambda)} \equiv \left\{g_k^{(\lambda)}\right\}_{k\in\mathbb{N}}$, $\bar{g} \equiv \{g_n\}_{n\in\mathbb{N}}$. Let $S_m^{(\lambda)}(f) = \sum_{k=1}^m f_k^{(\lambda)} x_k$, $S_m(f) = \sum_{k=1}^m f_k x_k$, $S_m^{(\lambda)}(g) = \sum_{k=1}^m g_k^{(\lambda)} x_k$, $S_m(g) = \sum_{k=1}^m g_k x_k$, $\forall m \in \mathbb{N}$. Take an arbitrary neighborhood of zero O_{ε_1} in X and consider another neighborhood of zero O_{ε_2} such that $O_{\varepsilon_2} + O_{\varepsilon_2} \subset O_{\varepsilon_1}$. It is clear that $\exists \lambda_0 : (S_m^{(\lambda)}(f) - S_m(f)) \in O_{\varepsilon_2}$, $S_m^{(\lambda)}(g) - S_m(g) \in O_{\varepsilon_2}$, $\forall \lambda \geq \lambda_0$, $\forall m \in \mathbb{N}$. Consequently, $(S_m^{(\lambda)}(f) + S_m^{(\lambda)}(g) - (S_m(f) + S_m(g))) \in O_{\varepsilon_2} + O_{\varepsilon_2} \subset O_{\varepsilon_1}, \forall \lambda \geq \lambda_0$, $\forall m \in \mathbb{N}$. And this, in turn, means that $\bar{f}^{(\lambda)} + \bar{g}^{(\lambda)} \to \bar{f} + \bar{g}$, $\lambda \in M$ in $\mathscr{K}_{\bar{x}}$. Thus, linear operations are continuous in $\mathscr{K}_{\bar{x}}$. Let $\bar{f} \neq 0$, $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}$. Obviously, $\exists r \in \mathbb{N} : f_r \neq 0$. Set $n_0 = \min\{r : f_r \neq 0\}$. As $f_{n_0} x_{n_0} \neq 0$, there exists a neighborhood of zero O_ε in X such that $f_{n_0} x_{n_0} \notin O_\varepsilon$. Let this neighborhood be corresponded by a neighborhood of zero $O_\varepsilon^{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$:

$$O_\varepsilon^{\mathscr{K}} \equiv \left\{ \bar{g} \in \mathscr{K}_{\bar{x}} : \sum_{n=1}^m g_n x_n \in O_\varepsilon \, , \, \forall m \in \mathbb{N}, \, \bar{g} \equiv \{g_n\}_{n\in\mathbb{N}} \right\}.$$

It is absolutely obvious that $\sum_{n=1}^{n_0} f_n x_n = f_{n_0} x_n \notin O_\varepsilon$. As a result, $\bar{f} \notin O_\varepsilon^{\mathscr{K}}$. It follows directly that the space $\mathscr{K}_{\bar{x}}$ is a Hausdorff space. So we obtain that every one-point set in $\mathscr{K}_{\bar{x}}$ is closed. Thus, we have proved the following:

Theorem 3 *Space $\mathscr{K}_{\bar{x}}$ with a topology $\tau_{\mathscr{K}}$ has the following properties: 1) it is complete; 2) every one-point set in it is closed; 3) linear operations are continuous in it.*

Let's consider the operator $T : \mathscr{K}_{\bar{x}} \to X$ defined by

$$T\bar{f} = \sum_{n=1}^\infty f_n x_n, \bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}.$$

Let $\bar{f}^{(\lambda)} \to \bar{f}$, $\lambda \in M$ in $\mathscr{K}_{\bar{x}}$, where $\bar{f}^{(\lambda)} \equiv \left\{f_k^{(\lambda)}\right\}_{k\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}$. We have

$$T\bar{f}^{(\lambda)} - T\bar{f} = \sum_{k=1}^\infty \left(\lambda_k^{(\lambda)} - f_k\right) x_k, \forall \lambda \in M.$$

Take an arbitrary neighborhood of zero O_{ε_1} in X and consider another neighborhood of zero O_{ε_2} such that $\overline{O_{\varepsilon_2}} \subset O_{\varepsilon_1}$. Let $O_{\varepsilon_2}^{\mathscr{K}}$ be a neighborhood of zero in $\mathscr{K}_{\bar{x}}$ generated by O_{ε_2}. Obviously, $\exists \lambda_0 \in M$:$\left(\bar{f}^{(\lambda)} - \bar{f}\right) \in O_{\varepsilon_2}^{\mathscr{K}}$, $\forall \lambda \geq \lambda_0$, i.e.

$$\sum_{k=1}^m \left(f_k^{(\lambda)} - f_k\right) x_k \in O_{\varepsilon_2}, \forall \lambda \geq \lambda_0, \forall m \in \mathbb{N}.$$

Passing to the limit in this relation as $m \to \infty$ yields

$$\sum_{k=1}^\infty \left(f_k^{(\lambda)} - f_k\right) x_k \in \overline{O_{\varepsilon_2}} \subset O_{\varepsilon_1}, \forall \lambda \geq \lambda_0.$$

Thus, $T\bar{f}^{(\lambda)} \to T\bar{f}$, $\lambda \in M$ in X. It follows directly that T is a continuous operator. Let $\bar{f} \in KerT$, i.e. $T\bar{f} = 0 \Rightarrow \sum_{n=1}^\infty f_n x_n = 0$, where $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}$. It is clear that if the system $\{x_n\}_{n\in\mathbb{N}}$ is ω-linearly independent, then $f_n = 0$, $\forall n \in \mathbb{N}$, and, as a result, $KerT = \{0\}$. In this case there exists an inverse operator $T^{-1} : ImT \to \mathscr{K}_{\bar{x}}$.

Denote by $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{K}_{\bar{x}}$ a canonical system with $e_n = \{\delta_{nk}\}_{k\in\mathbb{N}}$, where δ_{nk} is the Kronecker symbol. Let us show that $\{e_n\}_{n\in\mathbb{N}}$ forms a basis for $\mathscr{K}_{\bar{x}}$. Take $\forall \bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}$ and prove that the series $\sum_{n=1}^\infty f_n e_n$ is convergent in $\mathscr{K}_{\bar{x}}$. Take an arbitrary neighborhood of zero $O_\varepsilon^{\mathscr{K}}$ in $\mathscr{K}_{\bar{x}}$, generated by a neighborhood of zero O_ε in X. As the series $\sum_{n=1}^\infty f_n x_n$ is convergent in X, we have $\exists m_0 \in \mathbb{N}$:

$$\sum_{n=m}^{m+p} f_n x_n \in O_\varepsilon, \ \forall m \geq m_0, \ \forall p \in \mathbb{N}.$$

Consequently

$$\sum_{n=m}^{m+p} f_n e_n \in O_\varepsilon^{\mathscr{K}}, \ \forall m \geq m_0, \ \forall p \in \mathbb{N}.$$

For partial sums S_r corresponding to the element $\sum_{n=m}^{m+p} f_n e_n$ we have

$$S_r = \begin{cases} 0, & 1 \le r \le m-1, \\ \sum_{n=m}^{r} f_n x_n\,, & m \le r \le m+p, \\ \sum_{n=m}^{m+p} f_n x_n\,, & \forall r \ge m+p. \end{cases}$$

Therefore it is evident that $S_r \in O_\varepsilon$, $\forall r \in \mathbb{N}$. As a result, the series $\sum_{n=1}^{\infty} f_n e_n$ is fundamental in $\mathscr{K}_{\bar{x}}$ and, consequently, is convergent in it. Assume $\bar{f}_m = \bar{f} - \sum_{n=1}^{m} \lambda_n e_n = \{...; 0; f_{m+1}; ...\}$. Let $O_\varepsilon^{\mathscr{K}}$ be an arbitrary neighborhood of zero $\mathscr{K}_{\bar{x}}$, generated by the neighborhood of zero O_ε in X. The convergence of the series $\sum_{n=1}^{\infty} f_n x_n$ implies that $\exists m_0 \in \mathbb{N}$: $\sum_{n=m}^{m+p} f_n x_n \in O_\varepsilon$, $\forall m \ge m_0$, $\forall p \in \mathbb{N}$. For partial sums S_r corresponding to the element $\bar{f}_m$ we have

$$S_r = \begin{cases} 0\,, & 1 \le r \le m, \\ \sum_{n=m+1}^{r} f_n x_n\,, & \forall r \ge m+1. \end{cases}$$

As a result, we get $\bar{f}_m \in O_\varepsilon^{\mathscr{K}}$, $\forall m \ge m_0$. Consequently, $\lim_{m\to\infty} \sum_{n=1}^{m} f_n e_n = \bar{f}$ in $\mathscr{K}_{\bar{x}}$. Consider linear functionals $e_n^*\left(\bar{f}\right) = f_n$, $\forall n \in \mathbb{N}$. Let us show that they are continuous. Let $\bar{f}^{(\lambda)} \to \bar{f}$, $\lambda \in M$ in $\mathscr{K}_{\bar{x}}$, where $\bar{f}^{(\lambda)} \equiv \left\{f_k^{(\lambda)}\right\}_{k\in\mathbb{N}} \subset \mathscr{K}_{\bar{x}}$. As already established above, $f_k^{(\lambda)} \to f_k$, $\lambda \in M$. Consequently, $e_k^*\left(\bar{f}^{(\lambda)}\right) = f_k^{(\lambda)} \to f_k = e_k^*\left(\bar{f}\right)$ as $\lambda \in M$, and therefore, e_k^* is continuous $\forall k \in \mathbb{N}$. It is easy to see that $e_n^*(e_k) = \delta_{nk}$, $\forall n, k \in \mathbb{N}$. As a result, we obtain that $\{e_n^*\}_{n\in\mathbb{N}}$ is a system biorthogonal to $\{e_n\}_{n\in\mathbb{N}}$. This proves the basicity of system $\{e_n\}_{n\in\mathbb{N}}$ in $\mathscr{K}_{\bar{x}}$. So the following theorem is true.

Theorem 4 *Let $(X;\tau)$ be complete LTS and let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some nondegenerate system. Then the corresponding space of coefficients $(\mathscr{K}_{\bar{x}}; \tau_{\mathscr{K}_{\bar{x}}})$ is also a complete LTS with canonical basis.*

Now suppose that the system $\{x_n\}_{n\in\mathbb{N}}$ is *ω-linearly independent* and the value area of operator T is closed, i.e. $ImT = \overline{ImT}$. It is easily seen that $Te_n = x_n$, $\forall n \in \mathbb{N}$. Then it is clear that the system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis for ImT. Indeed, it directly follows from the definition of the basis and from the expresiion of the operator T. If this system is complete in X, then it forms a basis also for X. We will call T a coefficient operator. Converse is also true, i.e. if the system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis for X, then it is clear that it is complete in it and $w-$linearly independent. It is evident that in this case $ImT = \overline{ImT}$. Hence, it is easy to see that $ImT = X$. Thus, the following theorem is true.

Theorem 5 *Let $(X;\tau)$ be complete LTS, $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a nondegenerate system, $(\mathscr{K}_{\bar{x}}; \tau_{\mathscr{K}_{\bar{x}}})$ be the corresponding space of coefficients, and $T : \mathscr{K}_{\bar{x}} \to X$ be the corresponding coefficient operator. System $\{x_n\}_{n\in\mathbb{N}}$ forms a basis for X if and only if the following conditions are satisfied:*

1) it is complete in X; 2) it is ω-linearly independent; 3) $ImT = \overline{ImT}$.

4. Conclusion

Summing up, we arrive at the following conclusions:

1) In a complete topological vector space every non-degenerate system generates a similar space of coefficients;

2) The space of coefficients has a canonical basis, regardless of whether the system is complete, minimal or forms a basis;

3) A basicity criterion, different from the one for classical case, is obtained in terms of coefficient operator.

Acknowledgement

The authors are thankful to the referees for their valuable comments.

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