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Poly-Bergman Type Spaces on the Siegel Domain: Quasi-parabolic Case

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Abstract

We introduce poly-Bergman type spaces on the Siegel domain $D_n \subset \mathbb{C}^n$, and we prove that they are isomorphic to tensorial products of one-dimensional spaces generated by orthogonal polynomials of two kinds: Laguerre polynomials and Hermite type polynomials. The linear span of all poly-Bergman type spaces is dense in the Hilbert space $L^2(D_n, d\mu_\lambda)$, where $d\mu_\lambda = (\operatorname{Im} z_n - |z_1|^2 - \cdots - |z_{n-1}|^2)^\lambda dx_1 dy_1 \cdots dx_n dy_n$, with $\lambda > -1$.

Keywords: Siegel Domain, Poly-Bergman Space, Laguerre polynomials, Hermite polynomials

1. Introduction

In this paper we generalized the concept of polyanalytic function on the Siegel domain $D_n \subset \mathbb{C}^n$, which is the unbounded realisation of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

The spaces of polyanalytic functions on the unit disc $\mathbb{D}$, or the upper half-plane as its unbounded realisation, were introduced and studied in Balk (1997), Balk and Zuev (1970), Dzhuraev (1985) and Dzhuraev (1992). Recall some preliminaries known facts. Let $\Pi \subset \mathbb{C}$ be the upper half-plane and let $l \in \mathbb{N}$. We denote by $\mathcal{A}^2_l(\Pi)$ $[\tilde{\mathcal{A}}^2_l(\Pi)]$ the subspace of $L^2(\Pi)$ consisting of all l-analytic functions [l-anti-analytic functions], i.e., the functions satisfying the equation $(\partial/\partial\bar{z})^l\varphi = 0$ $[(\partial/\partial z)^l\varphi = 0]$. The function space $\mathcal{A}^2_l(\Pi)$ is called poly-Bergman space of Π. Let $\mathcal{A}^2_{(l)}(\Pi) = \mathcal{A}^2_l(\Pi) \ominus \mathcal{A}^2_{l-1}(\Pi)$ and $\tilde{\mathcal{A}}^2_{(l)}(\Pi) = \tilde{\mathcal{A}}^2_l(\Pi) \ominus \tilde{\mathcal{A}}^2_{l-1}(\Pi)$ be the spaces of true-l-analytic functions and true-l-anti-analytic functions, respectively. Let $\chi_\pm$ stand for the characteristic function of $\mathbb{R}_\pm = \mathbb{R}_{\pm 1} = \{x \in \mathbb{R} : \pm x \geq 0\}$. The main result of Vasilevski (1999) says that the space $L^2(\Pi)$ admits the decomposition

$$L^2(\Pi) = \bigoplus_{l=1}^{\infty} \mathcal{A}^2_{(l)}(\Pi) \oplus \bigoplus_{l=1}^{\infty} \tilde{\mathcal{A}}^2_{(l)}(\Pi),$$

and that there exists an unitary operator $W : L^2(\Pi) \to L^2(\Pi)$ such that the restriction mappings

$$W : \mathcal{A}^2_{(l)}(\Pi) \to L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l-1},$$

$$W : \tilde{\mathcal{A}}^2_{(l)}(\Pi) \to L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l-1},$$

are isometric isomorphisms, where $\mathcal{L}_l$ is the one-dimensional space generated by $\ell^\lambda_l(y) = (-1)^l c_l L^\lambda_l(y) e^{-y/2}\chi_+(y)$, with $L^\lambda_l(y)$ the Laguerre polynomial of order λ and degree l. Note that the above restriction mappings from poly-Bergman spaces and anti-poly-Bergman spaces are the analogue of the Bargmann type transform.

For the Bergman space $\mathcal{A}^2_\lambda(D_n)$ of the Siegel domain D_n, the analogues of the classical Bargmann transform and its inverse for five different types of commutative subgroups of biholomorphisms of D_n were constructed in Quiroga-Barranco and Vasilevski (2007). In particular, for the parabolic case they found an isometric isomorphisms

$$U : \mathcal{A}^2_\lambda(D_n) \to l^2(\mathbb{Z}^{n-1}_+, L^2(\mathbb{R}_+)),$$

which is the Bargmann type transform, where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$.

In this work polyanalytic function spaces are defined via the complex structure of $\mathbb{C}^n$ induced by the tangential Cauchy-Riemann equations given for the Heisenberg group Boggess (1991). Let L be $(l_1, \ldots, l_n) \in \mathbb{N}^n$. The poly-Bergman type space of D_n, denote by $\mathcal{A}^2_{\lambda L}(D_n)$ or simply by $\mathcal{A}^2_{\lambda L}$, is the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all L-analytic functions, i.e., functions that satisfy the equations

$$\begin{aligned}\left(\frac{\partial}{\partial \overline{z_k}} - 2iz_k \frac{\partial}{\partial \overline{z_n}}\right)^{l_k} f &= 0, \quad 1 \leq k \leq n-1 \\ \left(\frac{\partial}{\partial \overline{z_n}}\right)^{l_n} f &= 0,\end{aligned}$$

where, as usual, $\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \frac{1}{i}\frac{\partial}{\partial y_k}\right)$ and $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \frac{1}{i}\frac{\partial}{\partial y_k}\right)$. In particular, a function f is analytic in the Siegel domain if it satisfies

$$\begin{aligned}\frac{\partial f}{\partial \overline{z_k}} - 2iz_k \frac{\partial f}{\partial \overline{z_n}} &= 0, \quad 1 \leq k \leq n-1 \\ \frac{\partial f}{\partial \overline{z_n}} &= 0.\end{aligned}$$

Functions in $\mathcal{A}^2_{\lambda L}$ will be also called polyanalytic functions.

Anti-polyanalytic functions are just complex conjugation of polyanalytic functions, but they constitute a linearly independent space. For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the anti-poly-Bergman type space $\tilde{\mathcal{A}}^2_{\lambda L}(D_n)$ (or simply $\tilde{\mathcal{A}}^2_{\lambda L}$) as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all L-anti-analytic functions, i.e., functions satisfying the equations

$$\begin{aligned}\left(\frac{\partial}{\partial z_k} + 2i\overline{z_k} \frac{\partial}{\partial z_n}\right)^{l_k} f &= 0, \quad k = 1, ..., n-1 \\ \left(\frac{\partial}{\partial z_n}\right)^{l_n} f &= 0.\end{aligned}$$

We define the spaces of true-L-analytic and true-L-anti-analytic functions as

$$\mathcal{A}^2_{\lambda(L)} = \mathcal{A}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \mathcal{A}^2_{\lambda, L-e_j}\right),$$

$$\tilde{\mathcal{A}}^2_{\lambda(L)} = \tilde{\mathcal{A}}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \tilde{\mathcal{A}}^2_{\lambda, L-e_j}\right),$$

where $\mathcal{A}^2_{\lambda S} = \tilde{\mathcal{A}}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$, and $\{e_k\}_{k=1}^n$ stand for the canonical basis of $\mathbb{R}^n$.

The main results obtained in this work go as follows:

1) The space $L^2(D_n, d\mu_\lambda)$ admits the decomposition

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}^2_{\lambda(L)}\right) \bigoplus \left(\bigoplus_{L\in\mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)}\right).$$

2) There exists an unitary operator

$$W : L^2(D_n, d\mu_\lambda) \longrightarrow \mathcal{H} = l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, rdr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, y^\lambda dy)$$

for which

$$\mathcal{A}^2_{\lambda(L)} \cong \mathcal{K}^+_{(L)} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l_n-1}$$

and

$$\tilde{\mathcal{A}}^2_{\lambda(L)} \cong \mathcal{K}^-_{(L)} \otimes L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l_n-1},$$

where $\mathcal{L}_{l_n-1}$ is the one-dimensional space generated by the Laguerre function of degree $l_n - 1$ and order λ, and $\mathcal{K}^\pm_{(L)}$ is the subspace of $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, rdr)$ consisting of all sequences $\{c_m(r)\}_{\mathbb{Z}^{n-1}}$ such that c_m belongs to a finite dimensional space generated by Hermite type functions.

2. CR Manifolds

For a smooth submanifold M of $\mathbb{C}^n$, recall that $T_p(M)$ is the real tangent space of M at the point p. In general, $T_p(M)$ is not invariant under the complex structure map J for $T_p(\mathbb{C}^n)$. For a point $p \in M$, the complex tangent space of M at p is the vector space

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

This space is sometimes called the holomorphic tangent space. Using the Euclidian inner product on $T_p(\mathbb{R}^{2n})$, denote by $X_p(M)$ the totally real part of the tangent space of M which is the orthogonal complement of $H_p(M)$ in $T_p(M)$. We have that $T_p(M) = H_p(M) \oplus X_p(M)$ and $J(X_p(M))$ is trasversal to $T_p(M)$. A submanifold M of $\mathbb{C}^n$ is called a CR submanifold of $\mathbb{C}^n$ if $\dim_{\mathbb{R}} H_p(M)$ is independient of $p \in M$. The complexifications of $T_p(M)$, $H_p(M)$ and $X_p(M)$ are denoted by $T_p(M) \otimes \mathbb{C}$, $H_p(M) \otimes \mathbb{C}$ and $X_p(M) \otimes \mathbb{C}$, respectively. The complex structure map J on $T_p(\mathbb{R}^{2n}) \otimes \mathbb{C}$ restrict to a complex structure map on $H_p(M) \otimes \mathbb{C}$ because $H_p(M)$ is J-invariant. Moreover $H_p(M) \otimes \mathbb{C}$ is the direct sum of the $+i$ and $-i$ eigenspace of J which are denoted by $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$, respectively.

The following result establishes the form of the basis of $H_p(M)$. It also provides an expression for the generators of $H_p(M)$. We refer to Boggess (1991) for its proof.

Theorem 2.1 *Suppose* $M = \{(x + iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : y = h(x, w)\}$, *where* $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$ *is of class* C^m $(m \geq 2)$ *with* $h(0)$ *and* $Dh(0) = 0$. *A basis for* $H_p^{1,0}(M)$ *near of the origin is given by*

$$\Lambda_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^{d} \left(\sum_{m=1}^{d} \mu_{lm} \frac{\partial h_m}{\partial w_k} \frac{\partial}{\partial z_l} \right), \quad 1 \leq k \leq n - d$$

where μ_{lm} *is the* (l, m)*th element of the* $d \times d$ *matrix* $\left(I - i\frac{\partial h}{\partial x}\right)^{-1}$. *A basis for* $H_p^{0,1}$ *near the origin is given by* $\overline{\Lambda_1}, \ldots, \overline{\Lambda_{n-d}}$.

If the graphing function h of M is independient of the variable x, then the local basis of $H_p^{1,0}(M)$ has the following simple form

$$\Lambda_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^{d} \frac{\partial h_l}{\partial w_k} \frac{\partial}{\partial z_l}, \quad 1 \leq k \leq n - d. \tag{1}$$

We refer to Example 7.3-1 of Boggess (1991) for the details on the following construction of the Heisenberg group, which use the Equation (1). For the real hypersurface in $\mathbb{C}^n$ defined by

$$M = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \operatorname{Im} z_n = |z'|^2\},$$

the generators for $H^{1,0}(M)$ are given by

$$\Lambda_k = \Lambda_{k-}^{-} = \frac{\partial}{\partial z_k} + 2i\overline{z_k} \frac{\partial}{\partial z_n}, \quad 1 \leq k \leq n - 1 \tag{2}$$

and the generators for $H^{0,1}(M)$ are given by

$$\overline{\Lambda_k} = \Lambda_{k+}^{+} = \frac{\partial}{\partial \overline{z_k}} - 2iz_k \frac{\partial}{\partial \overline{z_n}}, \quad 1 \leq k \leq n - 1. \tag{3}$$

3. Cauchy-Riemann Equations for the Siegel Domain

Let $d\mu(z) = dx_1 dy_1 \cdots dx_n dy_n$ stand for the usual Lebesgue measure in $\mathbb{C}^n$, where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $z_k = x_k + iy_k$. We often rewrite z as (z', z_n), where $z' = (z_1, ..., z_{n-1})$. On the other hand, the usual norm in $\mathbb{C}^n$ is denoted by $|\cdot|$. In the Siegel domain

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \operatorname{Im} z_n - |z'|^2 > 0\}$$

we consider the weighted Lebesgue measure

$$d\mu_\lambda(z) = (\operatorname{Im} z_n - |z'|^2)^\lambda d\mu(z), \quad \lambda > -1.$$

Recall now the well known weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$, defined as the space of all holomorphic functions in $L^2(D_n, d\mu_\lambda)$. Thus, for $f \in \mathcal{A}_\lambda^2(D_n)$,

$$\frac{\partial f}{\partial \overline{z}_k} = 0, \quad k = 1, ..., n.$$

Let $\mathcal{D}$ be the subset $\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}^n$. Consider the mapping

$$\kappa : w = (z', u, v) \in \mathcal{D} \longmapsto z = (z', u + iv + i|z'|^2) \in D_n$$

and the unitary operator $U_0 : L^2(D_n, d\mu_\lambda) \to L^2(\mathcal{D}, d\eta_\lambda)$ given by

$$(U_0 f)(w) = f(\kappa(w)),$$

where

$$d\eta_\lambda(w) = v^\lambda d\mu(w).$$

Our aim is to introduce poly-Bergman type spaces in the Siegel domain, and then realize them in the space $L^2(\mathcal{D}, d\eta_\lambda)$ in order to apply Fourier transform techniques for their study. We start with the image space $\mathcal{A}_0(\mathcal{D}) = U_0(\mathcal{A}^2_\lambda)$, which consists of all functions $\varphi(z', u, v) = (U_0 f)(w)$ satisfying the equations

$$\begin{aligned} U_0 \frac{\partial}{\partial \bar{z}_k} U_0^{-1} \varphi &= \left(\frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial v} \right) \varphi = 0, \quad 0 \le k \le n-1 \\ U_0 \frac{\partial}{\partial \bar{z}_n} U_0^{-1} \varphi &= \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \varphi = 0. \end{aligned} \tag{4}$$

For functions satisfying this last equation, the first type equation in (4) can be rewritten as

$$U_0 \frac{\partial}{\partial \bar{z}_k} U_0^{-1} \varphi = \left(\frac{\partial}{\partial \bar{z}_k} - i z_k \frac{\partial}{\partial u} \right) \varphi = 0, \quad k = 1, ..., n-1. \tag{5}$$

These kind of equations were used in Quiroga-Barranco and Vasilevski (2007), and without any restriction on φ, they proved to be more usefull than the first type of equations in (4), as explained right now. At first stage, our aim was to introduce poly-Bergman type spaces such that they densely fill the space $L^2(D_n, d\mu_\lambda)$, we additionaly required that such poly-Bergman type spaces be isomorphic to tensorial products of L^2-spaces. Thus, following the techniques given in Quiroga-Barranco and Vasilevski (2007), equations (5) gave positive results for our porpuse. In this way the differential operators given in (3) were found, and they certainly satisfy

$$U_0 \overline{\Lambda}_k U_0^{-1} = \frac{\partial}{\partial \bar{z}_k} - i z_k \frac{\partial}{\partial u}, \quad k = 1, ..., n-1.$$

Obviously, a continuous function f is holomorphic in D_n if and only if

$$\begin{aligned} \overline{\Lambda}_k f &= 0, \quad k = 1, ..., n-1 \\ \frac{\partial}{\partial \bar{z}_n} f &= 0. \end{aligned}$$

We will use the operators $\overline{\Lambda}_k$'s to define the first class of poly-Bergman type spaces, i.e., a certain class of polyanalytic function spaces.

On the other hand, the differential operators $\partial/\partial z_k$ ($k = 1, ..., n-1$) are used to define anti-analytic function spaces, but they can be replaced by the operators given in (2). By the way,

$$U_0 \Lambda_k U_0^{-1} = \frac{\partial}{\partial z_k} + i \bar{z}_k \frac{\partial}{\partial u}, \quad k = 1, ..., n-1.$$

In addition we must consider

$$U_0 \frac{\partial}{\partial z_n} U_0^{-1} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right).$$

As expected, we use the operators Λ_k's to define anti-polyanalytic function spaces.

4. Orthogonal Polynomials Required

We will prove that poly-Bergman type spaces are isomorphic to tensorial products of one-dimensional spaces generated by orthogonal polynomials of two kinds. The first one is the set of Laguerre polynomials of order λ:

$$L_j^\lambda(y) := e^y \frac{y^{-\lambda}}{j!} \frac{d^j}{dy^j} (e^{-y} y^{j+\lambda}), \quad j = 0, 1, 2, ...$$

Laguerre polynomials constitute an orthogonal basis for the space $L^2(\mathbb{R}_+, y^\lambda e^{-y}dy)$, thus the set of functions

$$\ell_j^\lambda(y) = (-1)^j c_j L_j^\lambda(y) e^{-y/2}, \quad j = 0, 1, 2, ...$$

is an orthonormal basis of $L^2(\mathbb{R}_+, y^\lambda dy)$, where $c_j = \sqrt{j!/\Gamma(j+\lambda+1)}$ and Γ is the gamma function. Consider the one-dimensional space

$$\mathcal{L}_j = gen\{\ell_j^\lambda(y)\} \subset L^2(\mathbb{R}_+, y^\lambda dy).$$

On the other hand, for each $\nu \geq -1/2$, the second kind of polynomials consists of an orthonormal family of Hermite type polynomials in the space $L^2(\mathbb{R}_+, \tau^{2\nu+1}e^{-\tau^2}d\tau)$. These polynomials are denoted by $Q_j^\nu(\tau)$, $j = 0, 1, 2, ...$, and they are defined via the Gram-Schmidt procedure using the linearly independent set $\{1, \tau, \tau^2, ...\}$. Thus, deg $Q_j^\nu(\tau) = j$ and

$$\int_0^\infty Q_j^\nu(\tau) Q_k^\nu(\tau) \tau^{2\nu+1} e^{-\tau^2} d\tau = \delta_{jk}.$$

Actually $\{Q_j^\nu(\tau)\}_{j=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}_+, \tau^{2\nu+1}e^{-\tau^2}d\tau)$. Let's prove it. Let f in $\{1, \tau, \tau^2, ...\}^\perp \subset L^2(\mathbb{R}_+, \tau^{2\nu+1}e^{-\tau^2})$, that is,

$$\int_0^\infty f(\tau)\tau^j \, \tau^{2\nu+1} e^{-\tau^2} d\tau = 0, \quad \forall j \geq 0$$

or

$$\int_0^\infty g(\tau)h(\tau)\tau^j e^{-\tau/2} = 0, \quad \forall j \geq 0$$

where $g(\tau) = f(\tau)\tau^{\nu+1/2}e^{-\tau^2/2}$ belongs to $L^2(\mathbb{R}_+)$, and $h(\tau) = \tau^{\nu+1/2}e^{-(\tau^2-\tau)/2}$ is bounded. Therefore $gh \in L^2(\mathbb{R}_+)$ and is orthogonal to the orthonormal basis $\{\ell_j^\lambda(y)\}$. Thus $gh = 0$, i.e., $f = 0$.

We have proved that the Hermite type functions

$$H_j^\nu(\tau) = Q_j^\nu(\tau)\tau^\nu e^{-\tau^2/2}, \quad j = 0, 1, ...$$

form an orthonormal basis for $L^2(\mathbb{R}_+, \tau d\tau)$. We will refer to τ^ν as the potential weight of both the polynomials and Hermite type functions.

All the polynomials $Q_j^\nu(\tau)$ come out in our computations but we can work instead with the polynomials $Q_j^0(\tau)$ via the unitary operator $T_\nu : L^2(\mathbb{R}_+, \tau d\tau) \to L^2(\mathbb{R}_+, \tau d\tau)$ defined by

$$T_\nu : Q_j^\nu(\tau)\tau^\nu e^{-\tau^2/2} \longmapsto Q_j^0(\tau)e^{-\tau^2/2}, \quad \nu \geq -1/2. \tag{6}$$

Let rdr denote the product measure $\prod_{k=1}^{n-1} r_k dr_k$ on $\mathbb{R}_+^{n-1}$, so that

$$L^2(\mathbb{R}_+^{n-1}, rdr) = L^2(\mathbb{R}_+, r_1 dr_1) \otimes \cdots \otimes L^2(\mathbb{R}_+, r_{n-1}dr_{n-1}).$$

For $m = (m_1, ..., m_{n-1})$, $J' = (j_1, ..., j_{n-1}) \in \mathbb{Z}_+^{n-1}$, we introduce the following Hermite type functions of several variables:

$$\begin{aligned} H_{J'}^m(r) &= H_{j_1}^{m_1}(r_1) \cdots H_{j_{n-1}}^{m_{n-1}}(r_{n-1}) \\ &= Q_{j_1}^{m_1}(r_1) \cdots Q_{j_{n-1}}^{m_{n-1}}(r_{n-1}) r^m e^{-r^2/2}, \end{aligned}$$

where $r = (r_1, ..., r_{n-1})$, $r^2 = r_1^2 + \cdots + r_{n-1}^2$, and $r^m = r_1^{m_1} \cdots r_{n-1}^{m_{n-1}}$. Introduce the one-dimensional space

$$\mathcal{H}_{J'}^m = gen\{H_{J'}^m(r)\} \subset L^2(\mathbb{R}_+^{n-1}, rdr).$$

For each $m \in \mathbb{Z}_+^{n-1}$, the set $\{H_{J'}^m(r)\}_{J' \in \mathbb{Z}_+^{n-1}}$ is an orthonormal basis for $L^2(\mathbb{R}_+^{n-1}, rdr)$. We can now define an unitary operator

$$T_m : L^2(\mathbb{R}_+^{n-1}, rdr) \to L^2(\mathbb{R}_+^{n-1}, rdr)$$

by

$$T_m = T_{m_1} \otimes \cdots \otimes T_{m_{n-1}} : H_{J'}^m(r) \longmapsto H_{J'}^0(r). \tag{7}$$

We need a partial order in $\mathbb{Z}^N$. We say that $0 \leq J \leq L$ if $0 \leq j_k \leq l_k$ for $k = 1, ..., N$, where $J = (j_1, ..., j_N)$, $L = (l_1, ..., l_N)$.

5. Poly-Bergman Type Spaces

For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the poly-Bergman type space $\mathcal{A}^2_{\lambda L}$ as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all functions f satisfying the equations

$$\begin{aligned} \left(\frac{\partial}{\partial \overline{z}_k} - 2iz_k \frac{\partial}{\partial \overline{z}_n}\right)^{l_k} f &= 0, \quad k = 1, ..., n-1 \\ \left(\frac{\partial}{\partial \overline{z}_n}\right)^{l_n} f &= 0. \end{aligned}$$

Let $\{e_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$. We define the space of true-L-analytic functions as

$$\mathcal{A}^2_{\lambda(L)} = \mathcal{A}^2_{\lambda L} \ominus \left(\sum_{j=1}^n \mathcal{A}^2_{\lambda, L-e_j}\right),$$

where $\mathcal{A}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$.

It is much more convenient to deal with $\mathcal{A}_{0,\lambda L}(\mathcal{D}) = U_0(\mathcal{A}^2_{\lambda L}) \subset L^2(\mathcal{D}, d\eta_\lambda)$ in order to apply Fourier techniques in the study of the poly-Bergman type space. For $\varphi = U_0 f \in \mathcal{A}_{0,\lambda L}(\mathcal{D})$ we have then

$$\begin{aligned} U_0 \left(\overline{\Lambda}_k\right)^{l_k} U_0^{-1} \varphi &= \left(\frac{\partial}{\partial \overline{z}_k} - iz_k \frac{\partial}{\partial u}\right)^{l_k} \varphi = 0, \quad k = 1, ..., n-1 \\ U_0 \left(\frac{\partial}{\partial \overline{z}_n}\right)^{l_n} U_0^{-1} \varphi &= \frac{1}{2^{l_n}} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}\right)^{l_n} \varphi = 0. \end{aligned}$$

Once and for all we introduce all the operators to be considered. Fourier transforms on $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ play a very important role in this work, where $\mathbb{T} = S^1$ is the unit circumference. We begin with the tensorial decomposition

$$L^2(\mathcal{D}, d\eta_\lambda) = L^2(\mathbb{C}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v^\lambda dv).$$

We use now polar coordinates for the first tensorial factor space. For $z' = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$, we write $z_k = r_k t_k$ with $r_k \geq 0$ and $t_k \in \mathbb{T}$. For $t = (t_1, ..., t_{n-1})$ and $r = (r_1, ..., r_{n-1})$, we often write rt to mean z', and we identify z' with (t, r). Then

$$L^2(\mathbb{C}^{n-1}) = L^2(\mathbb{T}^{n-1}, d\Theta) \otimes L^2(\mathbb{R}^{n-1}_+, rdr),$$

where

$$d\Theta = d\Theta_{n-1} = \frac{1}{(2\pi)^{(n-1)/2}} \prod_{k=1}^{n-1} \frac{dt_k}{it_k}.$$

Obviously

$$L^2(\mathcal{D}, d\eta_\lambda) = L^2(\mathbb{T}^{n-1}, d\Theta) \otimes L^2(\mathbb{R}^{n-1}_+, rdr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v^\lambda dv). \tag{8}$$

Let F denote the Fourier transform on $L^2(\mathbb{R})$, and let $\mathcal{F}$ be the discrete Fourier transform on $L^2(\mathbb{T}, dt/(it))$:

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-i\xi u} du,$$

$$(\mathcal{F} g)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} g(t) \bar{t}^k \frac{dt}{it}.$$

Let $\mathcal{F}_{(n-1)}$ be the tensorial product of $\mathcal{F}$ with itself taken $n-1$ times. Now, according to the decomposition (8) we introduce the unitary operators

$$U_1 = I \otimes I \otimes F \otimes I,$$

$$U_2 = \mathcal{F}_{(n-1)} \otimes I \otimes I \otimes I.$$

Of course, the operator U_2 acts from $L^2(\mathcal{D}, d\eta_\lambda)$ onto the Hilbert space

$$\mathcal{H} = l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, rdr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v^\lambda dv). \tag{9}$$

Consider now the decomposition

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where

$$\mathcal{H}^\pm = l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, rdr) \otimes L^2(\mathbb{R}_\pm) \otimes L^2(\mathbb{R}_+, v^\lambda dv).$$

We introduce the unitary operator

$$U_3 = [T^+ \otimes I \otimes I] \oplus [T^- \otimes I \otimes I] : \ \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $T^\pm$ is the operator on $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, rdr)$ given by

$$T^\pm : \{c_m(r)\}_{m\in\mathbb{Z}^{n-1}} \mapsto \{T_{m^\pm}(c_m(r))\}_{m\in\mathbb{Z}^{n-1}}$$

with T_m given by (7), $m^\pm = (m_1^\pm, ..., m_{n-1}^\pm)$, $m_j^+ = \max\{m_j, 0\}$ and $m_j^- = m_j^+ - m_j$.

Finally, according to the tensorial product (8), we consider the following unitary operators on $L^2(\mathcal{D}, d\eta_\lambda)$:

$$V_1 : \phi(z', \xi, v) \longmapsto \psi(z', x, y) = \frac{1}{(2|x|)^{(\lambda+1)/2}} \phi(z', x, \frac{y}{2|x|}),$$

$$V_2 : \psi(t, r, x, y) \longmapsto \Psi(t, \rho, x, y) = \frac{1}{(\sqrt{2|x|})^{n-1}} \psi(t, \frac{1}{\sqrt{2|x|}}\rho, x, y), \quad \rho = \sqrt{2|x|}r.$$

Let $\mathcal{K}_L^+$ be the subspace of $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, \rho d\rho)$ consisting of all sequences

$$\{c_m(\rho)\}_{m\in\mathbb{Z}^{n-1}}$$

such that

$$\begin{array}{ll} c_m = 0 & \text{for } L' + m - e \notin \mathbb{Z}_+^{n-1} \\ c_m \in \displaystyle\bigoplus_{0\le J'\le L'-m^- -e} \mathcal{H}_{J'}^0 & \text{for } L' + m - e \in \mathbb{Z}_+^{n-1} \end{array}$$

where $e = (1, ..., 1) \in \mathbb{Z}^{n-1}$.

Theorem 5.1 *The unitary operator* $W = U_3U_2V_2V_1U_1U_0$ *maps* $L^2(D_n, d\mu_\lambda)$ *onto*

$$\mathcal{H} = l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, rdr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, y^\lambda dy).$$

The poly-Bergman type space $\mathcal{A}_{\lambda L}^2$ *is isomorphic to the subspace*

$$\mathcal{H}_L^+ = \mathcal{K}_L^+ \otimes L^2(\mathbb{R}_+) \otimes \left(\bigoplus_{j_n=0}^{l_n-1} \mathcal{L}_{j_n} \right).$$

Let $\mathcal{K}_{(L)}^+$ be the subspace of $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, \rho d\rho)$ consisting of all sequences

$$\{c_m(\rho)\}_{m\in\mathbb{Z}^{n-1}}$$

such that

$$\begin{array}{ll} c_m = 0 & \text{for } L' + m - e \notin \mathbb{Z}_+^{n-1} \\ c_m \in \mathcal{H}_{L'-m^- -e}^0 & \text{for } L' + m - e \in \mathbb{Z}_+^{n-1}. \end{array}$$

Corollary 5.2 *The restriction of W to the space* $\mathcal{A}_{\lambda(L)}^2$ *given by*

$$W : \mathcal{A}_{\lambda(L)}^2 \longrightarrow \mathcal{H}_{(L)}^+ = \mathcal{K}_{(L)}^+ \otimes L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l_n-1}$$

is an isomorphisms. Furthermore

$$\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}_{\lambda(L)}^2 \cong \mathcal{H}^+.$$

Proof of Theorem 5.1. If $\mathcal{A}_{1,\lambda L} = U_1(\mathcal{A}_{0,\lambda L}(\mathcal{D}))$, then $\phi = U_1\varphi$ belongs to $\mathcal{A}_{1,\lambda L}$ if and only if

$$\begin{aligned}\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k\right)^{l_k} \phi &= 0, \quad (k = 1, ..., n-1)\\ \frac{i^{l_n}}{2^{l_n}}\left(\xi + \frac{\partial}{\partial v}\right)^{l_n} \phi &= 0.\end{aligned}$$

Let $\mathcal{A}'_{1,\lambda L}$ denote the image space $V_1(\mathcal{A}_{1,\lambda L})$. Then $\psi = V_1\phi$ belongs to $\mathcal{A}'_{1,\lambda L}$ if and only if

$$\begin{aligned}V_1\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k\right)^{l_k} V_1^{-1}\psi &= \left(\frac{\partial}{\partial \overline{z}_k} + x z_k\right)^{l_k} \psi &= 0, \quad k = 1, ..., n-1\\ \frac{i^{l_n}}{2^{l_n}}V_1\left(\xi + \frac{\partial}{\partial v}\right)^{l_n} V_1^{-1}\psi &= \frac{i^{l_n}|x|^{l_n}}{2^{l_n}}\left(sign(x) + 2\frac{\partial}{\partial y}\right)^{l_n} \psi &= 0.\end{aligned} \tag{10}$$

The last equation in (10) separates the variable y from the rest of variables; this means that certain independent solutions for it can be expressed in the form $f(x, z')g(y)$ as shown below. But we must do the corresponding part for the first kind of equation in (10). In polar coordinates, the first kind of equation in (10) takes the form

$$\left[\frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k} + 2x r_k\right)\right]^{l_k} \psi = 0.$$

Define now $\mathcal{A}'_{2,\lambda L} = V_2(\mathcal{A}'_{1,\lambda L})$. Then $\Psi = V_2\psi$ belongs to $\mathcal{A}'_{2,\lambda L}$ if and only if

$$\begin{aligned}\left[\sqrt{2|x|}\frac{t_k}{2}\left(\frac{\partial}{\partial \rho_k} - \frac{t_k}{\rho_k}\frac{\partial}{\partial t_k} + sign(x)\rho_k\right)\right]^{l_k} \Psi &= 0, \quad k = 1, ..., n-1\\ \frac{i^{l_n}|x|^{l_n}}{2^{l_n}}\left(sign(x) + 2\frac{\partial}{\partial y}\right)^{l_n} \Psi &= 0.\end{aligned} \tag{11}$$

The general solution of the last equation in (11) is given by

$$\Psi(t, \rho, x, y) = \sum_{j_n=0}^{l_n-1} \psi_{0 j_n}(t, \rho, x) y^{j_n} e^{-(sgn\ x)y/2}.$$

Since $\Psi(t, \rho, x, y)$ has to be in $L^2(\mathcal{D}, d\eta_\lambda)$, we must take only positive values of x. Morever, by rearranging polynomial terms we can express $\Psi(t, \rho, x, y)$ as

$$\Psi(t, \rho, x, y) = \chi_+(x) \sum_{j_n=0}^{l_n-1} \psi_{j_n}(t, \rho, x)\ell^{\lambda}_{j_n}(y). \tag{12}$$

Let $\mathcal{A}_{2,\lambda L}$ denote the space $U_2(\mathcal{A}'_{2,\lambda L})$. In order to simplify our computations let's consider the function

$$\Psi_{j_n} = \chi_+(x)\psi_{j_n}(t, \rho, x)\ell^{\lambda}_{j_n}(y)$$

instead of the whole function Ψ given in (12). Then

$$\{d_{m j_n}\}_{m\in\mathbb{Z}^{n-1}} := U_2\Psi_{j_n} = \chi_+(x)\ell^{\lambda}_{j_n}(y)\{c_{m j_n}(\rho, x)\}_{m\in\mathbb{Z}^{n-1}}, \tag{13}$$

where $c_{m j_n} \in L^2(\mathbb{R}^{n-1}_+, \rho d\rho) \otimes L^2(\mathbb{R}_+)$ is given by

$$c_{m j_n}(\rho, x) = \int_{\mathbb{T}^{n-1}} \psi_{j_n}(t, \rho, x)t^{-m} d\Theta. \tag{14}$$

Obviously

$$\Psi_{j_n} = U_2^*\{d_{m j_n}\}_{m\in\mathbb{Z}^{n-1}} = \chi_+(x)\ell^{\lambda}_{j_n}(y) \sum_{m\in\mathbb{Z}^{n-1}} c_{m j_n}(\rho, x)t^m.$$

Thus $\{d_{mj_n}\}_{m\in\mathbb{Z}^{n-1}}$, as in (13), belongs to $\mathcal{A}_{2,\lambda L}$ if and only if

$$U_2\left[\sqrt{2|x|}\frac{t_k}{2}\left(\frac{\partial}{\partial\rho_k}-\frac{t_k}{\rho_k}\frac{\partial}{\partial t_k}+sign(x)\rho_k\right)\right]^{l_k}U_2^{-1}\{d_{mj_n}\}=0.$$

Let R denote the left hand side of this equation for the particular case $l_k = 1$, and let $G(x,y)$ be the function $\chi_+(x)\ell^{\lambda}_{j_n}(y)$. We have

$$\begin{aligned}P &:= U_2^{-1}R\\ &= \sqrt{2|x|}\frac{t_k}{2}\left(\frac{\partial}{\partial\rho_k}-\frac{t_k}{\rho_k}\frac{\partial}{\partial t_k}+sign(x)\rho_k\right)\sum_{m\in\mathbb{Z}^{n-1}}G(x,y)c_{mj_n}(\rho,x)t^m\\ &= \sqrt{2|x|}G(x,y)\sum\frac{t_k}{2}\left(t^m\frac{\partial c_{mj_n}}{\partial\rho_k}-\frac{m_k}{\rho_k}c_{mj_n}t^m+sign(x)\rho_k c_{mj_n}t^m\right)\\ &= \sqrt{2|x|}G(x,y)\sum t^m\frac{t_k}{2}\left(\frac{\partial}{\partial\rho_k}-\frac{m_k}{\rho_k}+sign(x)\rho_k\right)c_{mj_n},\end{aligned}$$

that is,

$$R=\chi_+(x)\sqrt{2|x|}\ell^{\lambda}_{j_n}(y)\left\{\frac{1}{2}\left(\frac{\partial}{\partial\rho_k}-\frac{m_k-1}{\rho_k}+sign(x)\rho_k\right)c_{m-e_k,j_n}\right\}_{m\in\mathbb{Z}^{n-1}}.$$

Thus, the function $\{d_{mj_n}\}_{m\in\mathbb{Z}^{n-1}}=U_2\Psi_{j_n}$ belongs to $\mathcal{A}_{2,\lambda L}$ if and only if for each m and $k=1,...,n-1$:

$$\left(\frac{\partial}{\partial\rho_k}-\frac{m_k}{\rho_k}+sign(x)\rho_k\right)^{l_k}c_{mj_n}=0,\quad \text{with } c_{mj_n}\in L^2. \tag{15}$$

Fixed $m\in\mathbb{Z}^{n-1}_+$, the general solution of this system of equations has the form

$$c_{mj_n}=\sum_{0\le J'\le L'-e}g_{mJ}(x)\rho^{J'}\rho^m e^{-sign(x)\rho^2/2},\quad (x>0) \tag{16}$$

where $J'=(j_1,...,j_{n-1})$ and $J=(J',j_n)$. Alternately, the general solution is given by

$$c_{mj_n}=\sum_{0\le J'\le L'-e}\chi_+(x)f_{mJ}(x)H^m_{J'}(\rho),\quad m\in\mathbb{Z}^{n-1}_+. \tag{17}$$

For arbitrary $m\in\mathbb{Z}^{n-1}$, the general solution of the system of differential equations (15) can also be written as

$$c_{mj_n}=\chi_+(x)p_1(\rho_1)\cdots p_{n-1}(\rho_{n-1})\rho^m e^{-\rho^2/2}, \tag{18}$$

where $p_k(\rho_k)$ is a polynomial of degree at most l_k-1 and whose coeficients are functions in x. Suppose that $m=(m_1,...,m_{n-1})\notin\mathbb{Z}^{n-1}_+$. Take $m_k<0$. Since c_{mj_n} must be in $L^2(\mathbb{R}^{n-1}_+,\rho d\rho)$, the polynomial $p_k(\rho_k)$ is necessarily divisible by $\rho_k^{|m_k|}$. Thus, if $l_k\le|m_k|$, then $p_k(\rho_k)=0$; but if $|m_k|\le l_k-1$ then $p_k(\rho_k)\rho^{m_k}$ is a polynomial of degree at most $l_k-1-|m_k|$. Thus, the potential weight $\rho_k^{m_k}$ is canceled in (18), and the set of solutions is reduced by the L^2-condition. We have non-trivial solutions for $L'+m-e\ge 0$, they are given by

$$c_{mj_n}=\sum_{0\le J'\le L'-m^--e}\chi_+(x)f_{mJ}(x)H^{m^+}_{J'}(\rho). \tag{19}$$

Then the function $U_2\Psi_{j_n}$ belongs to $\mathcal{A}_{2,\lambda L}$ if and only if

$$U_2\Psi_{j_n}=\chi_+(x)\ell^{\lambda}_{j_n}(y)\left\{\sum_{0\le J'\le L'-m^--e}H^{m^+}_{J'}(\rho)f_{mJ}(x)\right\}_{m\in\mathbb{Z}^{n-1}},$$

where $f_{mJ}=0$ for $L'+m-e\notin\mathbb{Z}^{n-1}_+$. Therefore

$$U_3U_2\Psi_{j_n}=\ell^{\lambda}_{j_n}(y)\left\{\sum_{0\le J'\le L'-m^--e}H^0_{J'}(\rho)\chi_+(x)f_{mJ}(x)\right\}_{m\in\mathbb{Z}^{n-1}_+}.$$

Finally $U_3U_2\Psi = \sum_{j_n=0}^{l_n-1} U_3U_2\Psi_{j_n}$ belongs to $\mathcal{H}_L^+$, and it is easy to see that W maps $\mathcal{A}^2_{\lambda L}(D_n)$ onto $\mathcal{H}_L^+$.

6. Anti-poly-Bergman Type Spaces

Anti-polyanalytic functions are just complex conjugation of polyanalytic functions, but they constitute a linearly independent space. For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the anti-poly-Bergman type space $\tilde{\mathcal{A}}^2_{\lambda L}$ as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all functions f satisfying the equations

$$\begin{aligned}\left(\frac{\partial}{\partial z_k} + 2i\overline{z_k}\frac{\partial}{\partial z_n}\right)^{l_k} f &= 0, \quad k = 1, ..., n-1\\ \left(\frac{\partial}{\partial z_n}\right)^{l_n} f &= 0.\end{aligned}$$

We define the space of true-L-anti-analytic functions as

$$\tilde{\mathcal{A}}^2_{\lambda(L)} = \tilde{\mathcal{A}}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \tilde{\mathcal{A}}^2_{\lambda, L-e_j}\right),$$

where $\tilde{\mathcal{A}}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$.

The following theorem is the main result of this work.

Theorem 6.1 *The Hilbert space $L^2(D_n, d\mu_\lambda)$ admits the decomposition*

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}^2_{\lambda(L)}\right) \bigoplus \left(\bigoplus_{L\in\mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)}\right).$$

Proof. Follows from Corollary 5.2 and Corollary 6.3 below.

Let $\mathcal{K}_L^-$ be the subspace of $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, \rho d\rho)$ consisting of all sequences

$$\{c_m(\rho)\}_{m\in\mathbb{Z}^{n-1}}$$

such that

$$\begin{aligned}c_m &= 0 && \text{for } L' - m - e \notin \mathbb{Z}_+^{n-1}\\ c_m &\in \bigoplus_{0\le J'\le L'-m^+-e} \mathcal{H}^0_{J'} && \text{for } L' - m - e \in \mathbb{Z}_+^{n-1}.\end{aligned}$$

Let $\mathcal{K}_{(L)}^-$ be the subspace of $l^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, \rho d\rho)$ consisting of all sequences

$$\{c_m(\rho)\}_{m\in\mathbb{Z}^{n-1}}$$

such that

$$\begin{aligned}c_m &= 0 && \text{for } L' - m - e \notin \mathbb{Z}_+^{n-1}\\ c_m &\in \mathcal{H}^0_{L'-m^+-e} && \text{for } L' - m - e \in \mathbb{Z}_+^{n-1}\end{aligned}$$

Theorem 6.2 *Under the unitary operator $W = U_3U_2V_2V_1U_1U_0$ acting on $L^2(D_n, d\mu_\lambda)$, the anti-poly-Bergman type space $\tilde{\mathcal{A}}^2_{\lambda L}$ is isomorphic to the subspace*

$$\mathcal{H}_L^- = \mathcal{K}_L^- \otimes L^2(\mathbb{R}_-) \otimes \left(\bigoplus_{j_n=0}^{l_n-1} \mathcal{L}_{j_n}\right).$$

Corollary 6.3 *The restriction of W to the space $\tilde{\mathcal{A}}^2_{\lambda(L)}$ given by*

$$W : \tilde{\mathcal{A}}^2_{\lambda(L)} \longrightarrow \mathcal{H}^-_{(L)} = \mathcal{K}^-_{(L)} \otimes L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l_n-1}$$

is an isomorphisms. Furthermore

$$\bigoplus_{L\in\mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)} \cong \mathcal{H}^-.$$

Proof of Theorem 6.2. The image space $\tilde{\mathcal{A}}_{0,\lambda L}(\mathcal{D}) = U_0(\tilde{\mathcal{A}}^2_{\lambda L}(D_n)) \subset L^2(\mathcal{D}, d\eta_\lambda)$ consists of all functions $\varphi = U_0 f$ satisfying the equations

$$U_0 (\Lambda_k)^{l_k} U_0^{-1}\varphi = \left(\frac{\partial}{\partial z_k} + i\bar{z}_k \frac{\partial}{\partial u}\right)^{l_k} \varphi = 0, \quad (k = 1, ..., n-1)$$

$$U_0 \left(\frac{\partial}{\partial z_n}\right)^{l_n} U_0^{-1}\varphi = \frac{1}{2^{l_n}} \left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)^{l_n} \varphi = 0.$$

Now if $\tilde{\mathcal{A}}_{1,\lambda L} = U_1(\tilde{\mathcal{A}}_{0,\lambda L}(\mathcal{D}))$, then $\phi = U_1\varphi$ belongs to $\tilde{\mathcal{A}}_{1,\lambda L}$ if and only if

$$\begin{aligned} \left(\frac{\partial}{\partial z_k} - \xi\bar{z}_k\right)^{l_k} \phi &= 0, \quad (k = 1, ..., n-1) \\ \frac{i^{l_n}}{2^{l_n}} \left(\xi - \frac{\partial}{\partial v}\right)^{l_n} \phi &= 0. \end{aligned} \tag{20}$$

In polar coordinates, the first type equation in (20) takes the form

$$\left[\frac{\bar{t}_k}{2}\left(\frac{\partial}{\partial r_k} + \frac{t_k}{r_k}\frac{\partial}{\partial t_k} - 2\xi r_k\right)\right]^{l_k} \phi = 0. \tag{21}$$

Under the transformation $\Psi = V_2 V_1 \phi$, the system of equations (20) is now equivalent to

$$\begin{aligned} \left[\sqrt{2|x|}\frac{\bar{t}_k}{2}\left(\frac{\partial}{\partial \rho_k} + \frac{t_k}{\rho_k}\frac{\partial}{\partial t_k} - sign(x)\rho_k\right)\right]^{l_k} \Psi &= 0, \\ \frac{i^{l_n}|x|^{l_n}}{2^{l_n}}\left(sign(x) - 2\frac{\partial}{\partial y}\right)^{l_n} \Psi &= 0. \end{aligned} \tag{22}$$

Thus the general solution of the this last equation has the form

$$\Psi(t, \rho, x, y) = \sum_{j_n=0}^{l_n-1} \psi_{0j_n}(t, \rho, x) y^{j_n} e^{(sgn\ x)y/2}.$$

Since $\Psi(t, \rho, x, y)$ has to be in $L^2(\mathcal{D}, d\eta_\lambda)$, we must take only negative values of x. Morever, by rearranging polynomial terms we can take

$$\Psi(t, \rho, x, y) = \chi_-(x) \sum_{j_n=0}^{l_n-1} \psi_{j_n}(t, \rho, x)\ell^\lambda_{j_n}(y). \tag{23}$$

For the function $\Psi_{j_n} = \chi_-(x)\psi_{j_n}(t, \rho, x)\ell^\lambda_{j_n}(y)$ we have

$$\{d_{mj_n}\}_{m\in\mathbb{Z}^{n-1}} := U_2\Psi_{j_n} = \chi_-(x)\ell^\lambda_{j_n}(y)\{c_{mj_n}(\rho, x)\}_{m\in\mathbb{Z}^{n-1}}, \tag{24}$$

where $c_{mj_n}(\rho, x) \in L^2(\mathbb{R}^{n-1}_+, \rho d\rho)$ is given by formula (14).

Define $\tilde{\mathcal{A}}_{2,\lambda L} = U_2 V_2 V_1(\tilde{\mathcal{A}}_{1,\lambda L})$. Thus $\{d_{mj_n}\}_{m\in\mathbb{Z}^{n-1}}$, as in (24), belongs to $\tilde{\mathcal{A}}_{2,\lambda L}$ if and only if

$$U_2\left[\sqrt{2|x|}\frac{\bar{t}_k}{2}\left(\frac{\partial}{\partial \rho_k} + \frac{t_k}{\rho_k}\frac{\partial}{\partial t_k} - sign(x)\rho_k\right)\right]^{l_k} U_2^{-1}\{d_{mj_n}\} = 0, \quad x < 0.$$

Again, let P denote the left hand side of this equation for $l_k = 1$, and let $G(x, y)$ be the function $\chi_-(x)\ell^\lambda_{j_n}(y)$. We have

$$\begin{aligned} R &:= U_2^{-1}P \\ &= \sqrt{2|x|}\frac{\bar{t}_k}{2}\left(\frac{\partial}{\partial \rho_k} + \frac{t_k}{\rho_k}\frac{\partial}{\partial t_k} - sign(x)\rho_k\right)\sum_{m\in\mathbb{Z}^{n-1}} G(x, y)c_{mj_n}(\rho, x)t^m \\ &= \sqrt{2|x|}G(x, y)\sum \frac{\bar{t}_k}{2}\left(t^m\frac{\partial c_{mj_n}}{\partial \rho_k} + \frac{m_k}{\rho_k}c_{mj_n}t^m - sign(x)\rho_k c_{mj_n}t^m\right) \\ &= \sqrt{2|x|}G(x, y)\sum t^m\frac{\bar{t}_k}{2}\left(\frac{\partial}{\partial \rho_k} + \frac{m_k}{\rho_k} - sign(x)\rho_k\right)c_{mj_n}, \end{aligned}$$

that is,

$$P = \chi_-(x)\sqrt{2|x|}\ell^{\lambda}_{j_n}(y)\left\{\frac{1}{2}\left(\frac{\partial}{\partial\rho_k} + \frac{m_k+1}{\rho_k} - sign(x)\rho_k\right)c_{m+e_k,j_n}\right\}_{m\in\mathbb{Z}^{n-1}}.$$

The function $\{d_{mj_n}\} = U_2\Psi_{j_n}$ belongs to $\tilde{\mathcal{A}}_{2,\lambda L}$ if and only if for each m and $k = 1, ..., n-1$

$$\left(\frac{\partial}{\partial\rho_k} + \frac{m_k}{\rho_k} - sign(x)\rho_k\right)^{l_k} c_{ms_n} = 0, \quad (c_{ms_n} \in L^2).$$

Fixed m, the general solution of this system of differential equations has the form

$$c_{mj_n} = \sum_{0\le J'\le L'-e} g_{mJ}(x)\rho^{J'}\rho^{-m}e^{sign(x)\rho^2/2}, \quad (x<0).$$

Adding the L^2-condition we get non-trivial solutions for $L' - m - e \ge 0$, they are given by

$$c_{mj_n} = \sum_{0\le J'\le L'-m^+-e} \chi_-(x)f_{mJ}(x)H^{m^-}_{J'}(\rho). \tag{25}$$

Then the function $U_2\Psi_{j_n}$ belongs to $\mathcal{A}_{2,\lambda L}$ if and only if

$$U_2\Psi_{j_n} = \chi_-(x)\ell^{\lambda}_{j_n}(y)\left\{\sum_{0\le J'\le L'-m^+-e} H^{m^-}_{J'}(\rho)f_{mJ}(x)\right\}_{m\in\mathbb{Z}^{n-1}},$$

where $f_{mJ} = 0$ for $L' - m - e \notin \mathbb{Z}^{n-1}_+$. Therefore

$$U_3U_2\Psi_{j_n} = \ell^{\lambda}_{j_n}(y)\left\{\sum_{0\le J'\le L'-m^+-e} H^0_{J'}(\rho)\chi_-(x)f_{mJ}(x)\right\}_{m\in\mathbb{Z}^{n-1}_+}.$$

Finally $U_3U_2\Psi = \sum_{j_n=0}^{l_n-1} U_3U_2\Psi_{j_n}$ belongs to $\mathcal{H}^-_L$, and it is easy to see that W maps $\tilde{\mathcal{A}}^2_{\lambda L}(D_n)$ onto $\mathcal{H}^-_L$.

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