Poly-Bergman Type Spaces on the Siegel Domain: Quasi-parabolic Case

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Abstract

We introduce poly-Bergman type spaces on the Siegel domain $D_n \subset \mathbb{C}^n$, and we prove that they are isomorphic to tensorial products of one-dimensional spaces generated by orthogonal polynomials of two kinds: Laguerre polynomials and Hermite type polynomials. The linear span of all poly-Bergman type spaces is dense in the Hilbert space $L^2(D_n, d\mu)$, where $d\mu = (\text{Im } z_n - |z|^2 - \cdots - |z_{n-1}|^2)^l dx_1 dy_1 \cdots dx_n dy_n$, with $\lambda > -1$.

Keywords: Siegel Domain, Poly-Bergman Space, Laguerre polynomials, Hermite polynomials

1. Introduction

In this paper we generalized the concept of polyanalytic function on the Siegel domain $D_n \subset \mathbb{C}^n$, which is the unbounded realisation of the unit ball $B^n \subset \mathbb{C}^n$.

The spaces of polyanalytic functions on the unit disc $D$, or the upper half-plane as its unbounded realisation, were introduced and studied in Balk (1997), Balk and Zuev (1970), Dzhuraev (1985) and Dzhuraev (1992). Recall some preliminaries known facts. Let $\Pi \subset \mathbb{C}$ be the upper half-plane and let $l \in \mathbb{N}$. We denote by $\mathcal{A}_l^2(\Pi)$ the subspace of $L^2(\Pi)$ consisting of all $l$-analytic functions [$l$-anti-analytic functions], i.e., the functions satisfying the equation $(\partial/\partial \bar{z})^l \varphi = 0$ [$\partial/\partial \bar{z}]^l \varphi = 0$. The function space $\mathcal{A}_l^2(\Pi)$ is called poly-Bergman space of $\Pi$. Let $\mathcal{A}_0^2(\Pi) = \mathcal{A}_2^2(\Pi) \ominus \mathcal{A}_1^2(\Pi)$ and $\mathcal{A}_l^2(\Pi) = \mathcal{A}_{l-1}^2(\Pi) \ominus \mathcal{A}_{l-2}^2(\Pi)$ be the spaces of true-$l$-analytic functions and true-$l$-anti-analytic functions, respectively. Let $\chi_s$ stand for the characteristic function of $\mathbb{R}_s = \mathbb{R}_{s+1} = \{x \in \mathbb{R} : \pm x \geq 0\}$. The main result of Vasilevski (1999) says that the space $L^2(\Pi)$ admits the decomposition

$$L^2(\Pi) = \bigoplus_{l=1}^{\infty} \mathcal{A}_0^2(\Pi) \oplus \bigoplus_{l=1}^{\infty} \mathcal{A}_l^2(\Pi),$$

and that there exists an unitary operator $W : L^2(\Pi) \rightarrow L^2(\Pi)$ such that the restriction mappings

$$W : \mathcal{A}_0^2(\Pi) \rightarrow L^2(\mathbb{R}_+ \ominus \mathcal{L}_{-1}),$$

$$W : \mathcal{A}_l^2(\Pi) \rightarrow L^2(\mathbb{R}_- \ominus \mathcal{L}_{-1}),$$

are isometric isomorphisms, where $\mathcal{L}_l$ is the one-dimensional space generated by $\ell^l_1(y) = (-1)^l c_l L^l_1(y)e^{-y/2}\chi_s(y)$, with $L^l_1(y)$ the Laguerre polynomial of order $l$ and degree $l$. Note that the above restriction mappings from poly-Bergman spaces and anti-poly-Bergman spaces are the analogue of the Bargmann type transform.

For the Bergman space $\mathcal{A}_l^2(D_n)$ of the Siegel domain $D_n$, the analogues of the classical Bargmann transform and its inverse for five different types of commutative subgroups of biholomorphisms of $D_n$ were constructed in Quiroga-Barranco and Vasilevski (2007). In particular, for the parabolic case they found an isometric isomorphisms

$$U : \mathcal{A}_l^2(D_n) \rightarrow \mathcal{L}(\mathbb{Z}_+^{n-1}, L^2(\mathbb{R}_+)), $$

which is the Bargmann type transform, where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$. 


In this work polyanalytic function spaces are defined via the complex structure of \( \mathbb{C}^n \) induced by the tangential Cauchy-Riemann equations given for the Heisenberg group Boggess (1991). Let \( L \) be \( (l_1, \ldots, l_n) \in \mathbb{N}^n \). The poly-Bergman type space of \( D_n \), denote by \( \mathcal{A}^2_{\lambda}(D_n) \) or simply \( \mathcal{A}^2_{\lambda} \), is the subspace of \( L^2(D_n, d\mu_\lambda) \) consisting of all \( L \)-analytic functions, i.e., functions that satisfy the equations

\[
\left( \frac{\partial}{\partial z_k} - 2i \frac{\partial}{\partial \zeta_n} \right)^n f = 0, \quad 1 \leq k \leq n - 1
\]

\[
\left( \frac{\partial}{\partial z_k} + 2i \frac{\partial}{\partial \zeta_n} \right)^n f = 0, \quad k = 1, \ldots, n - 1
\]

where, as usual, \( \frac{\partial}{\partial z_k} = \frac{1}{2} \left( \frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) \) and \( \frac{\partial}{\partial \zeta_n} = \frac{1}{2} \left( \frac{\partial}{\partial x_n} + i \frac{\partial}{\partial y_n} \right) \). In particular, a function \( f \) is analytic in the Siegel domain if it satisfies

\[
\frac{\partial f}{\partial z_k} = \frac{\partial f}{\partial \zeta_n} = 0, \quad 1 \leq k \leq n - 1
\]

Functions in \( \mathcal{A}^2_{\lambda} \) will be also called polyanalytic functions.

Anti-polyanalytic functions are just complex conjugation of polyanalytic functions, but they constitute a linearly independent space. For \( L = (l_1, \ldots, l_n) \in \mathbb{N}^n \), we define the anti-poly-Bergman type space \( \tilde{\mathcal{A}}^2_{\lambda}(D_n) \) (or simply \( \tilde{\mathcal{A}}^2_{\lambda} \)) as the subspace of \( L^2(D_n, d\mu_\lambda) \) consisting of all \( L \)-anti-analytic functions, i.e., functions satisfying the equations

\[
\left( \frac{\partial}{\partial z_k} - 2i \frac{\partial}{\partial \zeta_n} \right)^n f = 0, \quad 1 \leq k \leq n - 1
\]

\[
\left( \frac{\partial}{\partial z_k} + 2i \frac{\partial}{\partial \zeta_n} \right)^n f = 0.
\]

We define the spaces of true-\( L \)-analytic and true-\( L \)-anti-analytic functions as

\[
\mathcal{A}^2_{\lambda}(L) = \mathcal{A}^2_{\lambda} \oplus \sum_{j=1}^{n} \mathcal{A}^2_{\lambda, L-e_j},
\]

\[
\tilde{\mathcal{A}}^2_{\lambda}(L) = \tilde{\mathcal{A}}^2_{\lambda} \oplus \sum_{j=1}^{n} \tilde{\mathcal{A}}^2_{\lambda, L-e_j},
\]

where \( \mathcal{A}^2_{\lambda} = \tilde{\mathcal{A}}^2_{\lambda} = \{0\} \) if \( S \notin \mathbb{N}^n \), and \( \{e_k\}_{k=1}^{n} \) stand for the canonical basis of \( \mathbb{R}^n \).

The main results obtained in this work go as follows:

1) The space \( L^2(D_n, d\mu_\lambda) \) admits the decomposition

\[
L^2(D_n, d\mu_\lambda) = \left( \bigoplus_{L \in \mathbb{N}^n} \mathcal{A}^2_{\lambda}(L) \right) \oplus \left( \bigoplus_{L \in \mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda}(L) \right).
\]

2) There exists an unitary operator

\[
W : L^2(D_n, d\mu_\lambda) \longrightarrow \mathcal{H} = \overset{\cdot}{L}^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}, rdr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}, y^1 dy)
\]

for which

\[
\mathcal{A}^2_{\lambda}(L) \cong \mathcal{K}^L_{(\lambda)} \otimes L^2(\mathbb{R}^+) \otimes L_{l_n-1}
\]

and

\[
\tilde{\mathcal{A}}^2_{\lambda}(L) \cong \mathcal{K}^L_{(\lambda)} \otimes L^2(\mathbb{R}^-) \otimes L_{l_n-1},
\]

where \( L_{l_n-1} \) is the one-dimensional space generated by the Laguerre function of degree \( l_n - 1 \) and order \( \lambda \), and \( \mathcal{K}^L_{(\lambda)} \) is the subspace of \( \overset{\cdot}{L}^2(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}, rdr) \) consisting of all sequences \( \{c_m(r)\}_{\mathbb{Z}^{n-1}} \) such that \( c_m \) belongs to a finite dimensional space generated by Hermite type functions.
2. CR Manifolds

For a smooth submanifold $M$ of $\mathbb{C}^n$, recall that $T_p(M)$ is the real tangent space of $M$ at the point $p$. In general, $T_p(M)$ is not invariant under the complex structure map $J$ for $T_p(\mathbb{C}^n)$. For a point $p \in M$, the complex tangent space of $M$ at $p$ is the vector space

$$H_p(M) = T_p(M) \cap J(T_p(M)).$$

This space is sometimes called the holomorphic tangent space. Using the Euclidean inner product on $T_p(\mathbb{R}^{2n})$, denote by $X_p(M)$ the totally real part of the tangent space of $M$ which is the orthogonal complement of $H_p(M)$ in $T_p(M)$. We have that $T_p(M) = H_p(M) \oplus X_p(M)$ and $J(X_p(M))$ is transversal to $T_p(M)$. A submanifold $M$ of $\mathbb{C}^n$ is called a CR submanifold of $\mathbb{C}^n$ if $\dim_\mathbb{R} H_p(M)$ is independent of $p \in M$. The complexifications of $T_p(M)$, $H_p(M)$ and $X_p(M)$ are denoted by $T_p(M) \otimes \mathbb{C}$, $H_p(M) \otimes \mathbb{C}$ and $X_p(M) \otimes \mathbb{C}$, respectively. The complex structure map $J$ on $T_p(\mathbb{R}^{2n}) \otimes \mathbb{C}$ restrict to a complex structure map on $H_p(M) \otimes \mathbb{C}$ because $H_p(M)$ is $J$-invariant. Moreover $H_p(M) \otimes \mathbb{C}$ is the direct sum of the $+i$ and $-i$ eigenspace of $J$ which are denoted by $H_p^{(0)}(M)$ and $H_p^{(1)}(M)$, respectively.

The following result establishes the form of $H_p(M)$. It also provides an expression for the generators of $H_p(M)$. We refer to Boggess (1991) for its proof.

**Theorem 2.1** Suppose $M = \{(x + iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : y = h(x, w)\}$, where $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$ is of class $C^m$ ($m \geq 2$) with $h(0)$ and $Dh(0) = 0$. A basis for $H_p^{(0)}(M)$ near of the origin is given by

$$\Lambda_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^d \mu_{lm} \frac{\partial}{\partial w_k} \frac{\partial}{\partial z_l}, \quad 1 \leq k \leq n - d$$

where $\mu_{lm}$ is the $(l,m)$th element of the $d \times d$ matrix $(1 - i \frac{\partial h}{\partial x})^{-1}$. A basis for $H_p^{(1)}(M)$ near of the origin is given by $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_{n-d}$.

If the graphing function $h$ of $M$ is independent of the variable $x$, then the local basis of $H_p^{(1)}(M)$ has the following simple form

$$\tilde{\Lambda}_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^d \frac{\partial h_l}{\partial z_k} \frac{\partial}{\partial z_l}, \quad 1 \leq k \leq n - d.$$  \hspace{1cm} (1)

We refer to Example 7.3-1 of Boggess (1991) for the details on the following construction of the Heisenberg group, which use the Equation (1). For the real hypersurface in $\mathbb{C}^n$ defined by

$$M = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im } z_n = |z'|^2\},$$

the generators for $H^{(1)}(M)$ are given by

$$\Lambda_k = \Lambda_k^- = \frac{\partial}{\partial z_k} + 2i z_k \frac{\partial}{\partial z_n}, \quad 1 \leq k \leq n - 1$$ \hspace{1cm} (2)

and the generators for $H^{(0)}(M)$ are given by

$$\tilde{\Lambda}_k = \Lambda_k^+ = \frac{\partial}{\partial z_k} - 2i z_k \frac{\partial}{\partial z_n}, \quad 1 \leq k \leq n - 1.$$ \hspace{1cm} (3)

3. Cauchy-Riemann Equations for the Siegel Domain

Let $d\mu(z) = dx_1 dy_1 \cdots dx_n dy_n$ stand for the usual Lebesgue measure in $\mathbb{C}^n$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z_k = x_k + iy_k$. We often rewrite $z$ as $(z', z_n)$, where $z' = (z_1, \ldots, z_{n-1})$. On the other hand, the usual norm in $\mathbb{C}^n$ is denoted by $| \cdot |$. In the Siegel domain

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im } z_n - |z'|^2 > 0\}$$

we consider the weighted Lebesgue measure

$$d\mu_\lambda(z) = (\text{Im } z_n - |z'|^2)^\lambda d\mu(z), \quad \lambda > -1.$$  

Recall now the well known weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$, defined as the space of all holomorphic functions in $L^2(D_n, d\mu_\lambda)$. Thus, for $f \in \mathcal{A}^2_\lambda(D_n)$,

$$\frac{\partial f}{\partial \overline{z}_k} = 0, \quad k = 1, \ldots, n.$$
Let $\mathcal{D}$ be the subset $\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}^n$. Consider the mapping

$$\kappa : w = (z', u, v) \in \mathcal{D} \mapsto z = (z', u + iv + i|z'|^2) \in \mathcal{D}_n$$

and the unitary operator $U_0 : L^2(D_n, d\mu_{\lambda}) \to L^2(\mathcal{D}, d\eta_{\lambda})$ given by

$$(U_0f)(w) = f(\kappa(w)),$$

where

$$d\eta_{\lambda}(w) = v^1 d\mu(w).$$

Our aim is to introduce poly-Bergman type spaces in the Siegel domain, and then realize them in the space $L^2(\mathcal{D}, d\eta_{\lambda})$ in order to apply Fourier transform techniques for their study. We start with the image space $\mathcal{A}_n(\mathcal{D}) = U_0(\mathcal{A}_1^n)$, which consists of all functions $\varphi(z', u, v) = (U_0f)(w)$ satisfying the equations

$$U_0 \frac{\partial}{\partial z_k} U_0^{-1} \varphi = \left( \frac{\partial}{\partial z_k} - zk \frac{\partial}{\partial v} \right) \varphi = 0, \quad 0 \leq k \leq n - 1$$

$$(4)$$

$$U_0 \frac{\partial}{\partial \varepsilon_{\lambda}} U_0^{-1} \varphi = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial \varepsilon_{\lambda}} \right) \varphi = 0.$$  

For functions satisfying this last equation, the first type equation in (4) can be rewritten as

$$U_0 \frac{\partial}{\partial z_k} U_0^{-1} \varphi = \left( \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial u} \right) \varphi = 0, \quad k = 1, ..., n - 1.$$  

(5)

These kind of equations were used in Quiroga-Barranco and Vasilevski (2007), and without any restriction on $\varphi$, they proved to be more useful than the first type of equations in (4), as explained right now. At first stage, our aim was to introduce poly-Bergman type spaces such that they densely fill the space $L^2(D_n, d\mu_{\lambda})$, we additionally required that such poly-Bergman type spaces be isomorphic to tensorial products of $L^2$-spaces. Thus, following the techniques given in Quiroga-Barranco and Vasilevski (2007), equations (5) gave positive results for our purpose. In this way the differential operators given in (3) were found, and they certainly satisfy

$$U_0 \overline{\Lambda_k} U_0^{-1} = \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial u}, \quad k = 1, ..., n - 1.$$  

Obviously, a continuous function $f$ is holomorphic in $D_n$ if and only if

$$\overline{\Lambda_k} f = 0, \quad k = 1, ..., n - 1$$

$$\frac{\partial}{\partial \varepsilon_{\lambda}} f = 0.$$  

We will use the operators $\overline{\Lambda_k}$’s to define the first class of poly-Bergman type spaces, i.e., a certain class of polyanalytic function spaces.

On the other hand, the differential operators $\partial/\partial z_k$ ($k = 1, ..., n - 1$) are used to define anti-analytic function spaces, but they can be replaced by the operators given in (2). By the way,

$$U_0 \Lambda_k U_0^{-1} = \frac{\partial}{\partial z_k} + iz_k \frac{\partial}{\partial u}, \quad k = 1, ..., n - 1.$$  

In addition we must consider

$$U_0 \frac{\partial}{\partial \varepsilon_{\lambda}} U_0^{-1} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial \varepsilon_{\lambda}} \right).$$

As expected, we use the operators $\Lambda_k$’s to define anti-polyanalytic function spaces.

4. Orthogonal Polynomials Required

We will prove that poly-Bergman type spaces are isomorphic to tensorial products of one-dimensional spaces generated by orthogonal polynomials of two kinds. The first one is the set of Laguerre polynomials of order $\lambda$:

$$L^1_j(y) := e^y \frac{y^{-\lambda}}{j!} \frac{d^j}{dy^j} (e^{-y} y^j), \quad j = 0, 1, 2, ...$$
Laguerre polynomials constitute an orthogonal basis for the space \( L^2(\mathbb{R}_+, y^e\tau^2 dy) \), thus the set of functions
\[
\ell_j(y) = (-1)^j c_j L_j(y) e^{-y/2}, \quad j = 0, 1, 2, ...
\]
is an orthonormal basis of \( L^2(\mathbb{R}_+, y^4 dy) \), where \( c_j = \sqrt{j!/(j + \lambda + 1)} \) and \( \Gamma \) is the gamma function. Consider the one-dimensional space
\[
L_j = \text{gen}(\ell_j(y)) \subset L^2(\mathbb{R}_+, y^4 dy).
\]
On the other hand, for each \( n \geq -1/2 \), the second kind of polynomials consists of an orthonormal family of Hermite type polynomials in the space \( L^2(\mathbb{R}_+, \tau^{2n+1} e^{-\tau^2} d\tau) \). These polynomials are denoted by \( Q_j^r(\tau), j = 0, 1, 2, \ldots \), and they are defined via the Gram-Schmidt procedure using the linearly independent set \{1, \tau, \tau^2, \ldots\}. Thus, \( \deg Q_j^r(\tau) = j \) and
\[
\int_0^\infty Q_j^r(\tau)Q_k^s(\tau)\tau^{2n+1} e^{-\tau^2} d\tau = \delta_{jk}.
\]
Actually \( \{Q_j^r(\tau)\}_{j=0}^\infty \) is an orthonormal basis of \( L^2(\mathbb{R}_+, \tau^{2n+1} e^{-\tau^2} d\tau) \). Let's prove it. Let \( f \in \{1, \tau, \tau^2, \ldots\} \subset L^2(\mathbb{R}_+, \tau^{2n+1} e^{-\tau^2}) \), that is,
\[
\int_0^\infty f(\tau)\tau^j \tau^{2n+1} e^{-\tau^2} d\tau = 0, \quad \forall j \geq 0
\]
or
\[
\int_0^\infty g(\tau)h(\tau)\tau^j e^{-\tau^2} = 0, \quad \forall j \geq 0
\]
where \( g(\tau) = f(\tau)\tau^{n+1/2} e^{-\tau^2/2} \) belongs to \( L^2(\mathbb{R}_+) \), and \( h(\tau) = \tau^{n+1/2} e^{-(\tau^2-\tau)/2} \) is bounded. Therefore \( gh \in L^2(\mathbb{R}_+) \) and is orthogonal to the orthonormal basis \{\( \ell_j(y) \)\}. Thus \( gh = 0 \), i.e., \( f = 0 \).

We have proved that the Hermite type functions
\[
H_j^r(\tau) = Q_j^r(\tau)\tau^r e^{-\tau^2/2}, \quad j = 0, 1, ...
\]
form an orthonormal basis for \( L^2(\mathbb{R}_+, \tau^r d\tau) \). We will refer to \( \tau^r \) as the potential weight of both the polynomials and Hermite type functions.

All the polynomials \( Q_j^r(\tau) \) come out in our computations but we can work instead with the polynomials \( Q_j^0(\tau) \) via the unitary operator \( T_v : L^2(\mathbb{R}_+, \tau d\tau) \to L^2(\mathbb{R}_+, \tau^2 d\tau) \) defined by
\[
T_v : Q_j^r(\tau)\tau^r e^{-\tau^2/2} \mapsto Q_j^0(\tau)e^{-\tau^2/2}, \quad \nu \geq -1/2.
\]
Let \( rdr \) denote the product measure \( \prod_{k=1}^{n-1} r_k dr_k \) on \( \mathbb{R}^{n-1}_+ \), so that
\[
L^2(\mathbb{R}^{n-1}_+, rdr) = L^2(\mathbb{R}_+, r_1 dr_1) \otimes \cdots \otimes L^2(\mathbb{R}_+, r_{n-1} dr_{n-1}).
\]
For \( m = (m_1, \ldots, m_{n-1}) \), \( J' = (j_1, \ldots, j_{n-1}) \in \mathbb{Z}^{n-1}_+ \), we introduce the following Hermite type functions of several variables:
\[
H_m^p(r) = H_{j_1}^{m_1}(r_1) \cdots H_{j_{n-1}}^{m_{n-1}}(r_{n-1}) = Q_{j_1}^{m_1}(r_1) \cdots Q_{j_{n-1}}^{m_{n-1}}(r_{n-1}) r^m e^{-r^2/2},
\]
where \( r = (r_1, \ldots, r_{n-1}) \), \( r^2 = r_1^2 + \cdots + r_{n-1}^2 \), and \( r^m = r_1^{m_1} \cdots r_{n-1}^{m_{n-1}} \). Introduce the one-dimensional space
\[
H_m^p = \text{gen}[H_m^p(r)] \subset L^2(\mathbb{R}^{n-1}_+, rdr).
\]
For each \( m \in \mathbb{Z}^{n-1}_+ \), the set \( \{H_m^p(r)\}_{J' \in \mathbb{Z}^{n-1}_+} \) is an orthonormal basis for \( L^2(\mathbb{R}^{n-1}_+, rdr) \). We can now define an unitary operator
\[
T_m : L^2(\mathbb{R}^{n-1}_+, rdr) \to L^2(\mathbb{R}^{n-1}_+, rdr)
\]
by
\[
T_m = T_{m_1} \otimes \cdots \otimes T_{m_{n-1}} : H_m^p(r) \mapsto H_0^p(r).
\]
We need a partial order in $\mathbb{Z}^N$. We say that $0 \leq J \leq L$ if $0 \leq j_k \leq l_k$ for $k = 1, ..., N$, where $J = (j_1, ..., j_N)$, $L = (l_1, ..., l_N)$.

5. Poly-Bergman Type Spaces

For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the poly-Bergman type space $\mathcal{A}_{2L}^2$ as the subspace of $L^2(D_n, d\mu_i)$ consisting of all functions $f$ satisfying the equations

$$\left(\frac{\partial}{\partial z_k} - 2iz_k \frac{\partial}{\partial u} \right)^{l_k} f = 0, \quad k = 1, ..., n$$

$$\left(\frac{\partial}{\partial u} \right)^{l_k} f = 0.$$ 

Let $\{e_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$. We define the space of true-$L$-analytic functions as

$$\mathcal{A}_{2L}^2 = \mathcal{A}_{2L}^2(\mathbb{R}^n) \Theta \left( \bigoplus_{j=1}^n \mathcal{A}_{2L-\varepsilon_j}^2 \right),$$

where $\mathcal{A}_{2L}^2 = \{0\}$ if $S \notin \mathbb{N}^n$.

It is much more convenient to deal with $\mathcal{A}_{0,2L}(\mathcal{D}) = U_0(\mathcal{A}_{2L}^2) \subset L^2(\mathcal{D}, d\eta_1)$ in order to apply Fourier techniques in the study of the poly-Bergman type space. For $\varphi = U_0 f \in \mathcal{A}_{0,2L}(\mathcal{D})$ we have then

$$U_0 \left( \frac{\partial}{\partial z_k} \right)^{l_k} U_0^{-1} \varphi = \left( \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial u} \right)^{l_k} \varphi = 0, \quad k = 1, ..., n$$

$$U_0 \left( \frac{\partial}{\partial u} \right)^{l_k} U_0^{-1} \varphi = \frac{1}{2n} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)^{l_k} \varphi = 0.$$ 

Once and for all we introduce all the operators to be considered. Fourier transforms on $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ play a very important role in this work, where $\mathbb{T} = S^1$ is the unit circumference. We begin with the tensorial decomposition

$$L^2(\mathcal{D}, d\eta_1) = L^2(\mathbb{C}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v^4 dv).$$

We use now polar coordinates for the first tensorial factor space. For $z' = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$, we write $z_k = r_k t_k$ with $r_k \geq 0$ and $t_k \in \mathbb{T}$. For $t = (t_1, ..., t_{n-1})$ and $r = (r_1, ..., r_{n-1})$, we often write $rt$ to mean $z'$, and we identify $z'$ with $(t, r)$. Then

$$L^2(\mathbb{C}^{n-1}) = L^2(\mathbb{T}^{n-1}, d\Theta) \otimes L^2(\mathbb{R}_+, r^4 dr),$$

where

$$d\Theta = d\Theta_{n-1} = \frac{1}{(2\pi)^{(n-1)/2}} \prod_{k=1}^{n-1} \frac{dt_k}{dt_k}.$$ 

Obviously

$$L^2(\mathcal{D}, d\eta_1) = L^2(\mathbb{T}^{n-1}, d\Theta) \otimes L^2(\mathbb{R}_+, r^4 dr) \otimes L^2(\mathbb{R}_+, v^4 dv).$$

(8)

Let $F$ denote the Fourier transform on $L^2(\mathbb{R})$, and let $\mathcal{F}$ be the discrete Fourier transform on $L^2(\mathbb{T}, dt/(it))$:

$$(F f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-iu} du,$$

$$(\mathcal{F} g)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} g(t)t^k dt.$$ 

Let $\mathcal{F}_{(n-1)}$ be the tensorial product of $\mathcal{F}$ with itself taken $n - 1$ times. Now, according to the decomposition (8) we introduce the unitary operators

$$U_1 = I \otimes I \otimes F \otimes I,$$

$$U_2 = \mathcal{F}_{(n-1)} \otimes I \otimes I \otimes I.$$ 

Of course, the operator $U_2$ acts from $L^2(\mathcal{D}, d\eta_1)$ onto the Hilbert space

$$\mathcal{H} = \mathcal{F}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}_+^{n-1}, r^4 dr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v^4 dv).$$

(9)
Consider now the decomposition
\[ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \]
where
\[ \mathcal{H}^+ = \tilde{L}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, r dr) \otimes L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+, v^i dv). \]

We introduce the unitary operator
\[ U_3 = [T^+ \otimes I \otimes I] \oplus [T^- \otimes I \otimes I]: \mathcal{H}^+ \oplus \mathcal{H}^- \rightarrow \mathcal{H}^+ \oplus \mathcal{H}^-, \]
where \( T^\pm \) is the operator on \( \tilde{L}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, r dr) \) given by
\[ T^\pm : \{c_m(r)\}_{m \in \mathbb{Z}^{n-1}} \mapsto \{T_m^\pm(c_m(r))\}_{m \in \mathbb{Z}^{n-1}} \]
with \( T_m^\pm \) given by (7), \( m^\pm = (m_1^\pm, \ldots, m_{n-1}^\pm) \), \( m_j^+ = \max\{m_j, 0\} \) and \( m_j^- = m_j^+ - m_j \).

Finally, according to the tensorial product (8), we consider the following unitary operators on \( L^2(D, d\eta_1): \)
\[ V_1 : \psi(z', \xi, v) \mapsto \psi(z', x, y) = \frac{1}{(2|x|)^{1/(2\nu-1)}} \phi(z', x, \frac{y}{2|x|}), \]
\[ V_2 : \psi(t, r, x, y) \mapsto \Psi(t, \rho, x, y) = \frac{1}{(\sqrt{2|x|})^{2-\nu}} \psi(t, \frac{1}{\sqrt{2|x|}} \rho, x, y), \quad \rho = \sqrt{2|x|r}. \]

Let \( K_2 \) be the subspace of \( \tilde{L}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, \rho dp) \) consisting of all sequences
\[ \{c_m(\rho)\}_{m \in \mathbb{Z}^{n-1}} \]
such that
\[ c_m = 0 \quad \text{for } L' + m - e \in \mathbb{Z}^{n-1} \]
\[ c_m \in \bigoplus_{0 \leq j < \mathbb{Z}^{n-1}} H^0_{j}, \quad \text{for } L' + m - e \in \mathbb{Z}^{n-1} \]
where \( e = (1, \ldots, 1) \in \mathbb{Z}^{n-1}. \)

**Theorem 5.1** The unitary operator \( W = U_3 U_2 V_2 V_1 U_1 U_0 \) maps \( L^2(D, d\mu_1) \) onto
\[ \mathcal{H} = \tilde{L}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, r dr) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, y^i dy). \]

The poly-Bergman type space \( \mathcal{A}_{lL}^{2+} \) is isomorphic to the subspace
\[ \mathcal{H}^+_L = K^+_1 \otimes L^2(\mathbb{R}_+) \otimes \left( \bigoplus_{j \geq 0} L_{j+1}. \right) \]

Let \( K^+_1 \) be the subspace of \( \tilde{L}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}_+, \rho dp) \) consisting of all sequences
\[ \{c_m(\rho)\}_{m \in \mathbb{Z}^{n-1}} \]
such that
\[ c_m = 0 \quad \text{for } L' + m - e \in \mathbb{Z}^{n-1} \]
\[ c_m \in \mathcal{H}^0_{l, m - e}, \quad \text{for } L' + m - e \in \mathbb{Z}^{n-1}. \]

**Corollary 5.2** The restriction of \( W \) to the space \( \mathcal{A}_{l(L)}^{2+} \) given by
\[ W: \mathcal{A}_{l(L)}^{2+} \rightarrow \mathcal{H}_L^+ = K^+_1 \otimes L^2(\mathbb{R}_+) \otimes L_{l+1} \]
is an isomorphisms. Furthermore
\[ \bigoplus_{l \in \mathbb{N}} \mathcal{A}_{l(L)}^{2+} \cong \mathcal{H}^+. \]
Proof of Theorem 5.1. If $A_{1,\mathcal{L}} = U_1(\mathcal{A}_{0,\mathcal{L}}(\mathcal{D}))$, then $\phi = U_1 \varphi$ belongs to $A_{1,\mathcal{L}}$ if and only if

$$
\left( \frac{\partial}{\partial \xi_k} + \xi \zeta_k \right)^{\lambda} \frac{\partial}{\partial \xi_k} \phi = 0, \quad (k = 1, \ldots, n - 1)
$$

$$
\frac{\partial}{\partial \xi_k} \phi = 0.
$$

Let $\mathcal{A}_{1,\mathcal{L}}$ denote the image space $V_1(\mathcal{A}_{1,\mathcal{L}})$. Then $\psi = V_1 \phi$ belongs to $\mathcal{A}_{1,\mathcal{L}}$ if and only if

$$
V_1 \left( \frac{\partial}{\partial \xi_k} + \xi \zeta_k \right)^{\lambda} V_1^{-1} \psi = \left( \frac{\partial}{\partial \xi_k} + \xi \zeta_k \right)^{\lambda} \psi = 0, \quad k = 1, \ldots, n - 1
$$

The last equation in (10) separates the variable $y$ from the rest of variables; this means that certain independent solutions for it can be expressed in the form $f(x, z')g(y)$ as shown below. But we must do the corresponding part for the first kind of equation in (10). In polar coordinates, the first kind of equation in (10) takes the form

$$
\frac{\partial}{\partial \xi_k} \phi = 0.
$$

Define now $\mathcal{A}_{2,\mathcal{L}} = V_2(\mathcal{A}_{1,\mathcal{L}})$. Then $\Psi = V_2 \psi$ belongs to $\mathcal{A}_{1,\mathcal{L}}$ if and only if

$$
\left[ \sqrt{2|\lambda|} \frac{\partial}{\partial \xi_k} - \frac{\partial}{\partial \rho_k} \zeta_k + \text{sign}(\rho) \lambda_k \right]^{\lambda} \Psi = 0, \quad k = 1, \ldots, n - 1
$$

The general solution of the last equation in (11) is given by

$$
\Psi(t, \rho, x, y) = \sum_{j=0}^{l-1} \psi_{0j}(t, \rho, x) y^j e^{-(\text{sign } x)y/2}.
$$

Since $\Psi(t, \rho, x, y)$ has to be in $L^2(\mathcal{D}, d\eta_k)$, we must take only positive values of $x$. Moreover, by rearranging polynomial terms we can express $\Psi(t, \rho, x, y)$ as

$$
\Psi(t, \rho, x, y) = \chi_+(x) \sum_{j=0}^{l-1} \psi_{j}(t, \rho, x) f^j_{j+}(y).
$$

Let $\mathcal{A}_{2,\mathcal{L}}$ denote the space $U_2(\mathcal{A}_{2,\mathcal{L}})$. In order to simplify our computations let’s consider the function

$$
\Psi_{j+} = \chi_+(x) \psi_{j+}(t, \rho, x) f^j_{j+}(y)
$$

instead of the whole function $\Psi$ given in (12). Then

$$
\{d_{m_{j+}}\}_{m \in \mathbb{Z}^{n-1}} := U_2 \Psi_{j+} = \chi_+(x) f^j_{j+}(y) \{c_{m_{j+}}(\rho, x)\}_{m \in \mathbb{Z}^{n-1}},
$$

where $c_{m_{j+}} \in L^2(\mathbb{R}_+, \rho d\rho) \otimes L^2(\mathbb{R}_+)$ is given by

$$
c_{m_{j+}}(\rho, x) = \int_{\mathbb{R}_{=1}} \psi_{j+}(t, \rho, x) r^{-m} d\Theta.
$$

Obviously

$$
\Psi_{j+} = U_2^* \{d_{m_{j+}}\}_{m \in \mathbb{Z}^{n-1}} = \chi_+(x) f^j_{j+}(y) \sum_{m \in \mathbb{Z}^{n-1}} c_{m_{j+}}(\rho, x) r^m.
$$
Thus \( \{ d_{mj} \}_{m \in \mathbb{Z}^{n-1}} \), as in (13), belongs to \( \mathcal{A}_{2, L} \) if and only if
\[
U_2 \left[ \sqrt{2|x|} \frac{I_k}{2} \left( \frac{\partial}{\partial \rho_k} - \frac{t_k}{\rho_k} \frac{\partial}{\partial t_k} + \text{sign}(x) \rho_k \right) \right]^l U_2^{-1} [d_{mj}] = 0.
\]

Let \( R \) denote the left hand side of this equation for the particular case \( l_k = 1 \), and let \( G(x, y) \) be the function \( \chi_+(x)\ell_1^k(y) \). We have
\[
P := U_2^{-1} R = \sqrt{2|x|} \left( \frac{\partial}{\partial \rho_k} - \frac{t_k}{\rho_k} \frac{\partial}{\partial t_k} + \text{sign}(x) \rho_k \right) \sum_{m \in \mathbb{Z}^{n-1}} G(x, y) c_{mj, \rho, x} t^m
\]
\[
= \sqrt{2|x|} G(x, y) \sum_{m \in \mathbb{Z}^{n-1}} \left( \frac{\partial}{\partial \rho_k} - \frac{t_k}{\rho_k} \frac{\partial}{\partial t_k} + \text{sign}(x) \rho_k \right) c_{mj, \rho, x} t^m
\]
that is,
\[
R = \chi_+(x) \sqrt{2|x|} \ell_1^k(y) \left\{ \frac{1}{2} \left( \frac{\partial}{\partial \rho_k} - \frac{t_k}{\rho_k} + \text{sign}(x) \rho_k \right) c_{m, \rho, x} \right\} \quad \text{for } m \in \mathbb{Z}^{n-1}.
\]

Thus, the function \( \{ d_{mj} \}_{m \in \mathbb{Z}^{n-1}} = U_2 \Psi_{j_m} \) belongs to \( \mathcal{A}_{2, L} \) if and only if for each \( m \) and \( k = 1, ..., n - 1 \):
\[
\left( \frac{\partial}{\partial \rho_k} - \frac{m_k - 1}{\rho_k} + \text{sign}(x) \rho_k \right) c_{m, \rho, x} = 0, \quad \text{with } c_{mj, \rho, x} \in L^2.
\]

Fixed \( m \in \mathbb{Z}_{+}^{n-1} \), the general solution of this system of equations has the form
\[
c_{mj} = \sum_{0 \leq J' \leq L - e} g_{mJ}(x) \rho^J \rho^m e^{-\text{sign}(x) \rho^2/2}, \quad (x > 0)
\]
where \( J' = (j_1, ..., j_{n-1}) \) and \( J = (J', f_m) \). Alternately, the general solution is given by
\[
c_{mj} = \sum_{0 \leq J' \leq L - e} \chi_+(x) f_{mJ}(x) H^m_m(\rho), \quad m \in \mathbb{Z}_{+}^{n-1}.
\]

For arbitrary \( m \in \mathbb{Z}^{n-1} \), the general solution of the system of differential equations (15) can also be written as
\[
c_{mj} = \chi_+(x) p_1(\rho_1) \cdots p_{n-1}(\rho_{n-1}) \rho^m e^{-\rho^2/2},
\]
where \( p_k(\rho_k) \) is a polynomial of degree at most \( l_k - 1 \) and whose coefficients are functions in \( x \). Suppose that \( m = (m_1, ..., m_{n-1}) \notin \mathbb{Z}_{+}^{n-1} \). Take \( m_k < 0 \). Since \( c_{mj} \) must be in \( L^2(\mathbb{R}^{n-1}, \rho d\rho) \), the polynomial \( p_k(\rho_k) \) is necessarily divisible by \( \rho_k^{m_k} \). Thus, if \( l_k \leq |m_k| \), then \( p_k(\rho_k) = 0 \); but if \( |m_k| \leq l_k - 1 \) then \( p_k(\rho_k) \rho^m_k \) is a polynomial of degree at most \( l_k - 1 - |m_k| \). Thus, the potential weight \( \rho^m_k \) is canceled in (18), and the set of solutions is reduced by the \( L^2 \)-condition. We have non-trivial solutions for \( L' + m - e \geq 0 \), they are given by
\[
c_{mj} = \sum_{0 \leq J' \leq L - m - e} \chi_+(x) f_{mJ}(x) H^m_m(\rho).
\]

Then the function \( U_2 \Psi_{j_m} \) belongs to \( \mathcal{A}_{2, L} \) if and only if
\[
U_2 \Psi_{j_m} = \chi_+(x) \ell_1^k(y) \left\{ \sum_{0 \leq J' \leq L - m - e} H^m_m(\rho) f_{mJ}(x) \right\} \quad \text{for } m \in \mathbb{Z}_{+}^{n-1},
\]
where \( f_{mJ} = 0 \) for \( L' + m - e \notin \mathbb{Z}_{+}^{n-1} \). Therefore
\[
U_3 U_2 \Psi_{j_m} = \ell_1^k(y) \left\{ \sum_{0 \leq J' \leq L - m - e} H^m_m(\rho) \chi_+(x) f_{mJ}(x) \right\} \quad \text{for } m \in \mathbb{Z}_{+}^{n-1}.
\]
Finally \( U_3 U_2 \psi = \sum_{j=0}^{l-1} U_3 U_2 \psi_{j,0} \), belongs to \( \mathcal{H}_L^- \), and it is easy to see that \( W \) maps \( \mathcal{R}_{xL}^2(D_n) \) onto \( \mathcal{H}_L^- \).

### 6. Anti-poly-Bergman Type Spaces

Anti-polyanalytic functions are just complex conjugation of polyanalytic functions, but they constitute a linearly independent space. For \( L = (l_1, \ldots, l_d) \in \mathbb{N}^d \), we define the anti-poly-Bergman type space \( \widehat{\mathcal{R}}_{xL}^2 \) as the subspace of \( L^2(D_n, d\mu) \) consisting of all functions \( f \) satisfying the equations

\[
\left( \frac{\partial}{\partial z_k} + 2\overline{z}_k \frac{\partial}{\partial \overline{z}_k} \right)^l f = 0, \quad k = 1, \ldots, n-1
\]

\[
\left( \frac{\partial}{\partial z_n} \right)^l f = 0.
\]

We define the space of true-\( L \)-anti-analytic functions as

\[
\widehat{\mathcal{R}}_{xL}^2 = \mathcal{R}_{xL}^2 \bigoplus \left( \bigoplus_{j=1}^n \mathcal{R}_{xL}^2 \right),
\]

where \( \mathcal{R}_{xL}^2 = \{0\} \) if \( S \notin \mathbb{N}^d \).

The following theorem is the main result of this work.

**Theorem 6.1** The Hilbert space \( L^2(D_n, d\mu) \) admits the decomposition

\[
L^2(D_n, d\mu) = \left( \bigoplus_{L \in \mathbb{N}^d} \mathcal{R}_{xL}^2 \right) \bigoplus \left( \bigoplus_{L \in \mathbb{N}^d} \mathcal{R}_{xL}^2 \right).
\]

**Proof.** Follows from Corollary 5.2 and Corollary 6.3 below.

Let \( \mathcal{K}_L^- \) be the subspace of \( \mathcal{F}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^n_+, \rho d\rho) \) consisting of all sequences

\[c_m(\rho)\] such that

\[c_m = 0 \quad \text{for} \quad L' - m - e \notin \mathbb{Z}^{n-1};
\]

\[c_m \in \bigoplus_{0 \leq j \leq L' - m - e} \mathcal{H}_{L'_j}^0 \quad \text{for} \quad L' - m - e \in \mathbb{Z}^{n-1}.
\]

Let \( \mathcal{K}_{(L)}^- \) be the subspace of \( \mathcal{F}(\mathbb{Z}^{n-1}) \otimes L^2(\mathbb{R}^n_+, \rho d\rho) \) consisting of all sequences

\[c_m'(\rho)\] such that

\[c_m' = 0 \quad \text{for} \quad L' - m - e \notin \mathbb{Z}^{n-1};
\]

\[c_m' \in \mathcal{H}_{L'_j}^0 \quad \text{for} \quad L' - m - e \in \mathbb{Z}^{n-1}.
\]

**Theorem 6.2** Under the unitary operator \( W = U_3 U_2 V_2 V_1 U_1 U_0 \) acting on \( L^2(D_n, d\mu) \), the anti-poly-Bergman type space \( \widehat{\mathcal{R}}_{xL}^2 \) is isomorphic to the subspace

\[
\mathcal{H}_L^- = \mathcal{K}_L^- \otimes L^2(\mathbb{R}_-) \bigoplus \left( \bigoplus_{j=0}^{l-1} \left( \sum_{L \in \mathbb{N}^d} \mathcal{R}_{xL}^2 \right) \right).
\]

**Corollary 6.3** The restriction of \( W \) to the space \( \widehat{\mathcal{R}}_{xL}^2 \) given by

\[
W : \widehat{\mathcal{R}}_{xL}^2 \to \mathcal{H}_{(L)}^- = \mathcal{K}_{(L)}^- \otimes L^2(\mathbb{R}_-) \bigoplus \mathcal{L}_{L'-1}
\]

is an isomorphisms. Furthermore

\[
\bigoplus_{L \in \mathbb{N}^d} \mathcal{R}_{xL}^2 \cong \mathcal{H}^-.
\]
Proof of Theorem 6.2. The image space \( \mathcal{A}_{0,IL}(\mathcal{D}) = U_0(\mathcal{A}_{IL}^1(\mathcal{D}_n)) \subset L^2(\mathcal{D}, d\eta) \) consists of all functions \( \varphi = U_0 \psi \) satisfying the equations

\[
U_0(A_k)^l \varphi = \left( \frac{\partial}{\partial z_k} + i \xi_k \frac{\partial}{\partial u} \right)^l \varphi = 0, \quad (k = 1, \ldots, n - 1)
\]

\[
U_0 \left( \frac{\partial}{\partial z_n} \right)^l \varphi = \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right)^l \varphi = 0.
\]

Now if \( \mathcal{A}_{1,IL} = U_1(\mathcal{A}_{0,IL}(\mathcal{D})) \), then \( \psi = U_1 \varphi \) belongs to \( \mathcal{A}_{1,IL} \) if and only if

\[
\left( \frac{\partial}{\partial z_k} - \xi_k \right)^l \psi = 0, \quad (k = 1, \ldots, n - 1)
\]

\[
\left( \frac{\partial}{\partial v} \right)^l \psi = 0.
\]

In polar coordinates, the first type equation in (20) takes the form

\[
\left( \frac{\partial}{\partial r} \right)^l \varphi = 0.
\]

Under the transformation \( \Psi = V_2 V_1 \phi \), the system of equations (20) is now equivalent to

\[
\left( \frac{\partial}{\partial r} \right)^l \Psi = 0,
\]

\[
\left( \frac{\partial}{\partial y} \right)^l \Psi = 0.
\]

Thus the general solution of the this last equation has the form

\[
\Psi(t, \rho, x, y) = \sum_{j=0}^{l-1} \psi_{0j}(t, \rho, x) x^j e^{(\text{sign} \, x)y/2}.
\]

Since \( \Psi(t, \rho, x, y) \) has to be in \( L^2(\mathcal{D}, d\eta) \), we must take only negative values of \( x \). Moreover, by rearranging polynomial terms we can take

\[
\Psi(t, \rho, x, y) = \chi_{-}(x) \sum_{j=0}^{l-1} \psi_{j}(t, \rho, x) x^j (x < 0).
\]

For the function \( \Psi_{j} = \chi_{-}(x) \psi_{j}(t, \rho, x) \)

\[
\{d_{m_{j}}\}_{m \in \mathbb{Z}^{+}} := U_2 \Psi_{j} = \chi_{-}(x) \int_{- \infty}^{\infty} \int_{- \infty}^{\infty} G(x, y) c_{m_{j}}(\rho, x) \, d\rho \, dy,
\]

where \( c_{m_{j}}(\rho, x) \in L^2(\mathbb{R}^{+}, d\rho) \) is given by formula (14).

Define \( \mathcal{A}_{2,IL} = U_2 V_2 V_1(\mathcal{A}_{1,IL}) \). Thus \( \{d_{m_{j}}\}_{m \in \mathbb{Z}^{+}} \), as in (24), belongs to \( \mathcal{A}_{2,IL} \) if and only if

\[
U_2 \left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} - \frac{\partial}{\partial y} \right)^l \psi \equiv 0, \quad x < 0.
\]

Again, let \( P \) denote the left hand side of this equation for \( l_k = 1 \), and let \( G(x, y) \) be the function \( \chi_{-}(x) x^j (x < 0) \). We have

\[
R := U_2^{-1} P
\]

\[
= \sqrt{2|x|} \frac{\partial}{\partial \rho_k} + \frac{m_k}{\rho_k} - \text{sign}(x) \rho_k \int_{\mathbb{Z}^{+}} G(x, y) c_{m_{j}}(\rho, x) \, d\rho
\]

\[
= - \sqrt{2|x|} |G(x, y)| \int_{\mathbb{Z}^{+}} \frac{\partial}{\partial \rho_k} + \frac{m_k}{\rho_k} - \text{sign}(x) \rho_k c_{m_{j}}(\rho, x) \, d\rho
\]

\[
= - \sqrt{2|x|} |G(x, y)| \int_{\mathbb{Z}^{+}} \frac{\partial}{\partial \rho_k} + \frac{m_k}{\rho_k} - \text{sign}(x) \rho_k c_{m_{j}}(\rho, x) \, d\rho.
\]
that is,

\[ P = \chi_-(x) \sqrt{2|x|} \ell_{j_k}^1(y) \left\{ \frac{1}{2} \left( \frac{\partial}{\partial \rho_k} + \frac{m_k + 1}{\rho_k} - \text{sign}(x) \rho_k \right) c_{m_{j_k}j_k} \right\}_{m \in \mathbb{Z}^{n-1}}. \]

The function \( \{d_{m_{j_k}}\} = U_2 \Psi_{j_k} \) belongs to \( \mathcal{A}_{2,IL} \) if and only if for each \( m \) and \( k = 1, ..., n - 1 \)

\[ \left( \frac{\partial}{\partial \rho_k} + \frac{m_k}{\rho_k} - \text{sign}(x) \rho_k \right) c_{m_{j_k}j_k} = 0, \quad (c_{m_{j_k}j_k} \in L^2). \]

Fixed \( m \), the general solution of this system of differential equations has the form

\[ c_{m_{j_k}} = \sum_{0 \leq j' \leq L - m - e} g_{mJ}(x) \rho^{j'} e^{\text{sign}(x) \rho^{2}/2}, \quad (x < 0). \]

Adding the \( L^2 \)-condition we get non-trivial solutions for \( L' - m - e \geq 0 \), they are given by

\[ c_{m_{j_k}} = \sum_{0 \leq j' \leq L - m - e} \chi_-(x) f_{mJ}(x) H_{mJ}^{n-2}(\rho). \quad (25) \]

Then the function \( U_2 \Psi_{j_k} \) belongs to \( \mathcal{A}_{2,IL} \) if and only if

\[ U_2 \Psi_{j_k} = \chi_-(x) \ell_{j_k}^1(y) \left\{ \sum_{0 \leq j' \leq L - m - e} H_{mJ}^{n-2}(\rho)f_{mJ}(x) \right\}_{m \in \mathbb{Z}^{n-1}}. \]

where \( f_{mJ} = 0 \) for \( L' - m - e \notin \mathbb{Z}^{n-1} \). Therefore

\[ U_3 U_2 \Psi_{j_k} = \ell_{j_k}^1(y) \left\{ \sum_{0 \leq j' \leq L - m - e} H_{mJ}^{n-2}(\rho) \chi_-(x)f_{mJ}(x) \right\}_{m \in \mathbb{Z}^{n-1}}. \]

Finally \( U_3 U_2 \Psi = \sum_{j_k=0}^{b-1} U_3 U_2 \Psi_{j_k} \) belongs to \( \mathcal{H}_L^2 \), and it is easy to see that \( W \) maps \( \mathcal{A}_{2,IL}(D_n) \) onto \( \mathcal{H}_L^2 \)

References


