Generalization of Transitive Cayley Digraphs

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Abstract

This paper proves several extended theoretical results of transitive Cayley digraphs. Several generalization of transitive Cayley digraphs also have been provided. Moreover, various graph properties have been expressed in terms of algebraic properties. This did not attract much attention in the literature.

Keywords: Cayley digraphs, transitive digraphs, quasi-transitive digraphs, k-transitive digraphs, k-quasi-transitive digraphs

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1. Introduction

A *binary relation* on a set V is a subset E of $V \times V$. A *digraph* is a pair (V, E) where V is a nonempty set (called *vertex set*) and E is a binary relation on V. The elements of E are the *edges* of the digraph. An edge of the form (x, x) is called a *loop*. Through out this paper we assume that G is a digraph without loops. A *k-chain* or a *path* of length k, is a sequence $(x_0, x_1, \ldots, x_k)$ of vertices, each adjacent to its successor, and all the internal vertices $x_1, x_2, \ldots, x_{k-1}$ are distinct. A digraph G is said to be *transitive* if $E^2 \subseteq E$, that is, $(x, y), (y, z) \in E$ implies $(x, z) \in E$. A digraph is *quasi-transitive* if $(x, y), (y, z) \in E$ implies $(x, z) \in E$ or $(z, x) \in E$ (Galeana-Sanchez & Cesar Hernandez-Cruz, 2011). Observe that every transitive digraph is quasi-transitive.

Let G be a group and let S be a subset of G. The *Cayley digraph* of G with respect to S is defined as the digraph $X = (G, E)$, where E is a subset of $G \times G$, such that $(x, y) \in E$ if and only if $x^{-1}y \in S$ (Dobson, 2006). The Cayley digraph of G with respect to S is denoted by $\mathrm{Cay}(G, S)$. The subset S is called the connection set of X. That is, Cayley digraph $\mathrm{Cay}(G, S)$ has as its vertex-set and edge-set, respectively, $V = G$ and $E = \{(x, y) : y = xz \text{ for some } z \in S\}$. Observe that transitive/quasi-transitive Cayley digraphs are special classes of Cayley digraphs and these Cayley digraphs play an import role in algebraic graph theory. In this paper, we introduce some generalization of transitive/quasi-transitive Cayley digraphs. Moreover, we express various graph properties in terms of algebraic properties.

2. k-(quasi-) Transitive Cayley Digraphs

The concepts of k-transitive and k-quasi transitive digraphs were first introduced by Galeana-Sanchez and Cesar Hernandez-Cruz (2011). Observe that the above concepts are generalizations of transitive and quasi-transitive digraphs respectively. The aim of this section is to characterize k-transitive/k-quasi-transitive Cayley digraphs. We start with the following definitions due to Galeana-Sanchez and Cesar Hernandez-Cruz (2011).

Definition 2.1 A digraph G is *k-transitive* if the existence of a directed path $(x_0, x_1, \ldots, x_k)$ of length k in G implies that $(x_0, x_k) \in E$.

Definition 2.2 A digraph G is k-quasi-transitive if, whenever $(x_0, x_1, \ldots, x_k)$ is a directed path of length k, then $(x_0, x_k) \in E$ or $(x_k, x_0) \in E$.

Here we prove that $\mathrm{Cay}(G, S)$ is k-transitive if and only if $S^k \subseteq S$ and k-quasi-transitive if and only if $S^k \subseteq S \cup S^{-1}$. We also provide examples of k-transitive/k-quasi-transitive Cayley digraphs.

Theorem 2.3 $Cay(G, S)$ *is k-transitive if and only if* $S^k \subseteq S$.

Proof. Assume that $\text{Cay}(G, S)$ is k-transitive. Let $x \in S^k$. Then there exists $x_1, x_2, \ldots, x_k \in S$ such that

$$x = x_1 x_2 \ldots x_k \tag{1}$$

Let $z_0 = 1, z_1 = x_1, z_2 = x_1 x_2, \ldots, z_{k-1} = x_1 x_2 \ldots x_{k-1}$. Consider the following ordered pairs of elements in G:

$$(z_0, z_1), (z_1, z_2), (z_2, z_3), \ldots, (z_{k-1}, x).$$

We see that

$$z_0^{-1} z_1 = 1^{-1} x_1 \in S \Rightarrow (z_0, z_1) \in E,$$
$$z_1^{-1} z_2 = x_1^{-1} x_1 x_2 = x_2 \in S \Rightarrow (z_1, z_2) \in E,$$
$$\vdots$$
$$z_{k-1}^{-1} x = (x_1 x_2 \cdots x_{k-1})^{-1} (x_1 x_2 \cdots x_k) = x_k \in S \Rightarrow (z_{k-1}, x) \in E.$$

Hence $(z_0, z_1, z_2, \ldots, z_{k-1}, x)$ is a directed path of length k from 1 to x. Since $\text{Cay}(G, S)$ is k- transitive, $(1, x)$ is an edge in $\text{Cay}(G, S)$. In other words $x \in S$. Hence $S^k \subseteq S$.

Conversely, assume that $S^k \subseteq S$. Let $(x_0, x_1, \ldots, x_k)$ be a directed path of length k in $\text{Cay}(G, S)$. This implies that $x_0^{-1} x_1, x_1^{-1} x_2, x_2^{-1} x_3, \ldots, x_{k-1}^{-1} x_k \in S$. Equivalently,

$$(x_0^{-1} x_1)(x_1^{-1} x_2)(x_2^{-1} x_3) \ldots (x_{k-1}^{-1} x_k) \in S^k.$$

In other words $x_0^{-1} x_k \in S$. This implies that (x_0, x_k) is an edge in $\text{Cay}(G, S)$. Hence $\text{Cay}(G, S)$ is k-transitive.

Corollary 2.4 *$Cay(G, S)$ is 2-transitive(that is, transitive) if and only if $S^2 \subseteq S$.*

Theorem 2.5 *$Cay(G, S)$ is k-quasi-transitive if and only if $S^k \subseteq S \cup S^{-1}$.*

Proof. Assume that $\text{Cay}(G, S)$ is k-quasi-transitive. Let $x \in S^k$. Then there exists $x_1, x_2, \ldots, x_k$ such that $x = x_1 x_2 \ldots x_k$. Note that

$$(1, x_1, x_1 x_2, \ldots, x_1 x_2 \ldots x_{k-1}, x)$$

is a directed path of length k from 1 to x. Since $\text{Cay}(G, S)$ is k- transitive, either $(1, x)$ or $(x, 1)$ is an edge in $\text{Cay}(G, S)$. In other words $x \in S \cup S^{-1}$. Hence $S^k \subseteq S \cup S^{-1}$.

Conversely, assume that $S^k \subseteq S \cup S^{-1}$. Let $(x_0, x_1, \ldots, x_k)$ be a directed path of length k in $\text{Cay}(G, S)$. Then, $x_0^{-1} x_1, x_1^{-1} x_2, x_2^{-1} x_3, \ldots, x_{k-1}^{-1} x_k \in S$. Equivalently,

$$(x_0^{-1} x_1)(x_1^{-1} x_2)(x_2^{-1} x_3) \ldots (x_{k-1}^{-1} x_k) \in S^k.$$

In other words $x_0^{-1} x_k \in S \cup S^{-1}$. This implies that either (x_0, x_k) or (x_k, x_0) is an edge in $\text{Cay}(G, S)$. Hence $\text{Cay}(G, S)$ is k-quasi-transitive.

Corollary 2.6 *$Cay(G, S)$ is 2-quasi transitive(that is, quasi-transitive) if and only if $S^2 \subseteq S \cup S^{-1}$.*

The following are some examples of k- transitive/ k-quasi transitive Cayley digraphs.

Example 2.7 Let k be any positive integer greater than or equal to 2. Let $S = \{k, k^2, 2k^2 - k, 3k^2 - 2k, \ldots\}$. Then the Cayley digraph, $\text{Cay}(\mathbb{Z}, S)$ is an infinite k-transitive Cayley digraph.

Example 2.8 Let $\mathbb{Z}_6$ denotes the cyclic group of order 6. Then $\text{Cay}(\mathbb{Z}_6, \{2, 5\})$ and $\text{Cay}(\mathbb{Z}_6, \{4, 1\})$ are 4-transitive Cayley digraphs (see Figure 1).

Example 2.9 Consider the permutation group $S_3 = \{(1), (123), (132), (23), (12), (13)\}$. Let $S = \{(23), (12), (13)\}$. Then $\text{Cay}(S_3, S)$ is a 3-transitive Cayley digraph.

Example 2.10 Let $\mathbb{Z}_8$ denotes the cyclic group of order 8. Then the Cayley digraphs, $\text{Cay}(\mathbb{Z}_8, \{1, 3\})$ and $\text{Cay}(\mathbb{Z}_8, \{1, 5\})$ are 3-quasi-transitive Cayley digraphs (see Figure 2).
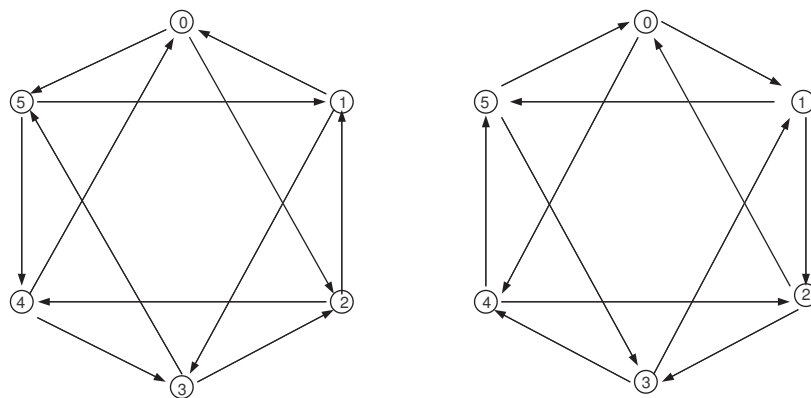
Figure 1. Cay($\mathbb{Z}_6, \{2, 5\}$) and Cay($\mathbb{Z}_6, \{4, 1\}$)
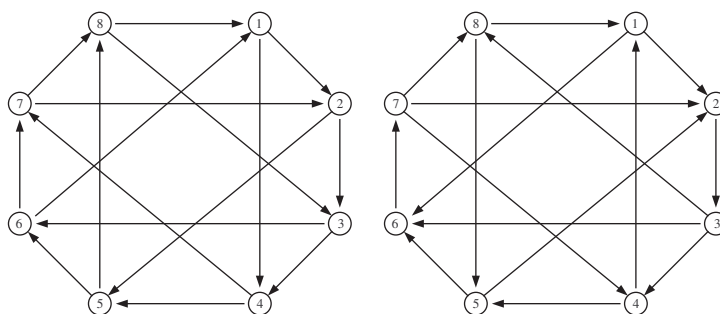
Figure 2. Cay($\mathbb{Z}_8, \{1, 3\}$) and Cay($\mathbb{Z}_8, \{1, 5\}$)
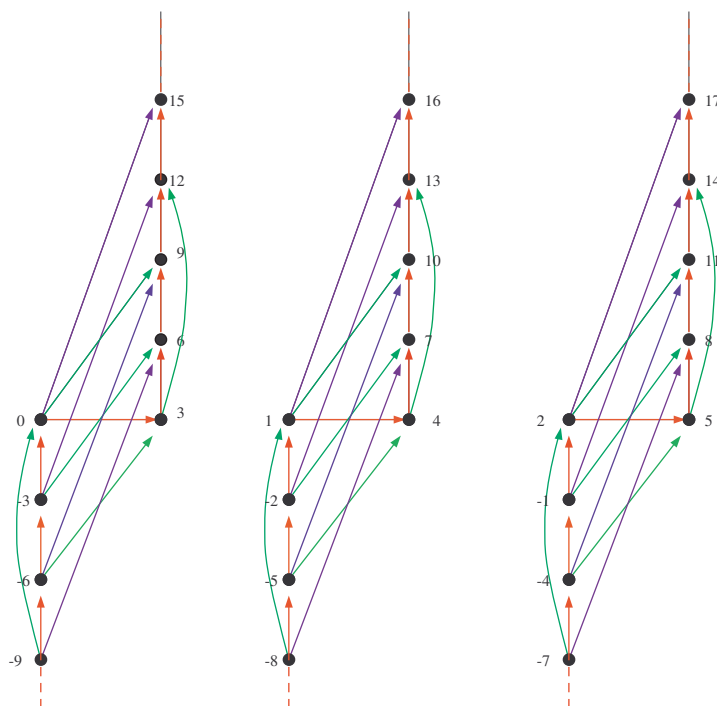
Figure 3. A portion of Cay($\mathbb{Z}, \{3, 9, 15, 21, \ldots\}$)

Remark 2.11

1) Any 2- transitive(that is, transitive) Cayley digraph is k-transitive for any positive integer k greater than equal to 3. But the converse need not be true. For example, Cay($\mathbb{Z}_6, \{2, 5\}$) is 4-transitive but not 2-transitive.

2) Cay($\mathbb{Z}, \{3, 9, 15, 21, \ldots\}$) is $2n + 1$-transitive for each $n \in \mathbb{N}$.

3) Cay($\mathbb{Z}, \{3, 9, 15, 21, \ldots\}$) is the disjoint union of three copies of the Cayley digraph: Cay($3\mathbb{Z}, \{3, 9, 15, 21, \ldots\}$) (see Figure 3).

4) Let k be any positive integer greater than or equal to 2. Let $S = \{k, k^2, 2k^2 - k, 3k^2 - 2k, \ldots\}$. Then the Cayley digraph, Cay($\mathbb{Z}, S$) is a disjoint union of k, k-transitive digraphs.

Theorem 2.12 *Any infinite k-transitive Cayley digraph Cay(G, S) is $n(k - 1) + k$-transitive for every $n \in \mathbb{N}$.*

Proof. Assume that Cay(G, S) be k-transitive. We will prove the result by induction on n. For $n = 1$, consider S^{k+k-1}. Observe that S^{k+k-1} can be written as:

$$S^{k+k-1} = S^k S^{k-1} \subseteq S S^{k-1} = S^k \subseteq S.$$

Next, assume that the theorem is true for $n = r$. Hence $S^{r(k-1)+k} \subseteq S$. We will show that the result is true for $n = r + 1$. For, it suffices to show that $S^{(r+1)(k-1)+k} \subseteq S$. Note that $S^{(r+1)(k-1)+k}$ can be written as:

$$S^{(r+1)(k-1)+k} = S^{[r(k-1)+k]+(k-1)}$$
$$= S^{[r(k-1)+k]} S^{k-1} \subseteq S S^{k-1} = S^k \subseteq S.$$

Hence the result follows by mathematical induction.

Theorem 2.13 *Cay(G, S) is k-(quasi-) transitive if and only if Cay(G, S^{-1}) is k-(quasi-) transitive.*

Proof. First, assume that Cay(G, S) is k-transitive. Let $(x_0, x_1, \ldots, x_k)$ be a directed path of length k in Cay(G, S^{-1}). This implies that

$$x_0^{-1} x_1, x_1^{-1} x_2, x_2^{-1} x_3, \ldots, x_{k-1}^{-1} x_k \in S^{-1}.$$

Equivalently, $x_1^{-1} x_0, x_2^{-1} x_1, x_3^{-1} x_2, \ldots, x_k^{-1} x_{k-1} \in S$. This tells us that

$$(x_k, x_{k-1}, \ldots, x_1, x_0)$$

is a path of length k in Cay(G, S). Since Cay(G, S) is k-transitive, $x_k^{-1} x_0 \in S$. That is $x_0^{-1} x_k \in S^{-1}$. Hence (x_0, x_k) is an edge in Cay(G, S^{-1}). The converse is straightforward.

Definition 2.14 Let G be a digraph. The least positive integer k such that G is k-transitive is called the k-transitivity number of G, denoted by $\mathscr{A}(G)$. That is,

$$\mathscr{A}(G) := \min\{k \in \mathbb{N} : G \text{ is } k \text{ - transitive}\}.$$

If no such k exists, we define $\mathscr{A}(G) = 0$. In a similar manner we can define the *k-quasi- transitivity number* of G as follows:

$$\mathscr{B}(G) = \min\{k \in \mathbb{N} : G \text{ is } k \text{ - quasi- transitive}\}.$$

We define $\mathscr{B}(G) = 0$, if no such k exists.

Theorem 2.15 *The k-transitivity number of Cay(G, S) is the least positive integer k such that $S^k \subseteq S$. That is,*

$$\mathscr{A}(Cay(G, S)) = \min\{k \in \mathbb{N} : S^k \subseteq S\}.$$

Theorem 2.16 *The k-quasi-transitivity number of Cay(G, S) is the least positive integer k such that $S^k \subseteq S \cup S^{-1}$. That is,*

$$\mathscr{B}(Cay(G, S)) = \min\{k \in \mathbb{N} : S^k \subseteq S \cup S^{-1}\}.$$

The transitivity/quasi-transitivity numbers of some Cayley digraphs are given below:

$$(i) \mathscr{A}(\text{Cay}(\mathbb{Z}_6, \{2, 5\})) = 4, \quad \mathscr{A}((\text{Cay}(\mathbb{Z}, \{3, 9, 15, 21, \ldots\})) = 3,$$
$$(ii) \mathscr{B}(\text{Cay}(\mathbb{Z}_8, \{1, 3\})) = 3, \quad \mathscr{B}(\text{Cay}(\mathbb{Z}_8, \{1, 5\})) = 3.$$

Theorem 2.17 *Let* $Cay(G, S)$ *be a cayley digraph with connection set* S. *Then we have the following:*

$$(i) \; \mathscr{A}(Cay(G, S)) = \mathscr{A}(Cay(G, S^{-1}))$$

$$(ii) \; \mathscr{B}(Cay(G, S)) = \mathscr{B}(Cay(G, S^{-1})).$$

3. k-**path-(quasi-) Transitive Cayley Digraphs**

Galeana-Sanchez and Cesar Hernandez-Cruz (2011) generalized the definition of transitive digraphs as follows:

Definition 3.18 A digraph G is called k-path transitive if whenever there is a xy directed path of length less than or equal to k and a yz directed path of length less than or equal to k, then there exists a xz-directed path of length less than or equal to k.

We define the k-path transitivity number of a digraph G as follows:

$$\mathscr{C}(G) := \min\{k \in \mathbb{N} : G \text{ is } k \text{ path transitive}\}.$$

We define $\mathscr{C}(G) = 0$, if no such k exists.

Theorem 3.19 $Cay(G, S)$ *is* k-*path-transitive if and only if* $(S \cup S^2 \cup \cdots S^k)^2 \subseteq S \cup S^2 \cdots \cup S^k$.

Proof. First assume that $Cay(G, S)$ is k-path-transitive. Let $x \in (S \cup S^2 \cup \cdots S^k)^2$. Then $x = z_1 z_2$ for some $z_1, z_2 \in (S \cup S^2 \cup \cdots S^k)$. This implies that there exists a path of length less than or equal to k from 1 to z_1 and a path of length less than or equal to k from z_1 to x. Since $Cay(G, S)$ is k-path-transitive there exists a path of length less than or equal to k from 1 to x. This implies that $x \in (S \cup S^2 \cup \cdots S^k)$. Hence $(S \cup S^2 \cup \cdots S^k)^2 \subseteq S \cup S^2 \cdots \cup S^k$.

Converse is straight forward.

Corollary 3.20 $Cay(G, S)$ *is* 1-*path-transitive (that is, transitive) if and only if* $S^2 \subseteq S$.

Theorem 3.21 *For any Cayley digraph* $Cay(G, S)$, *we have* $\mathscr{C}(Cay(G, S)) = \min\{k \in \mathbb{N} : (S \cup S^2 \cup \cdots S^k)^2 \subseteq S \cup S^2 \cdots \cup S^k\}$.

We define the following:

Definition 3.22 A digraph G is called k-path-quasi-transitive if whenever there is a xy directed path of length less than or equal to k and a yz directed path of length less than or equal to k, then there exists a xz-directed path of length less than or equal to k or there exists a zx-directed path of length less than or equal to k. We define k-path-quasi-transitivity number of G as:

$$\mathscr{D}(G) = \min\{k \in \mathbb{N} : G \text{ is } k - \text{ quasi-path- transitive}\}.$$

Theorem 3.23 $Cay(G, S)$ *is* k-*path-quasi-transitive if and only if* $(S \cup S^2 \cup \cdots S^k)^2 \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cdots \cup (S^k \cup (S^{-1})^k)$.

Corollary 3.24 $Cay(G, S)$ *is* 1-*path-quasi-transitive(that is, quasi-transitive) if and only if* $S^2 \subseteq S \cup S^{-1}$.

Theorem 3.25 $Cay(G, S)$ *is* k-*path-quasi-transitive if and only if* $Cay(G, S^{-1})$ *is* k-*path-quasi-transitive.*

Theorem 3.26 *For any Cayley digraph* $Cay(G, S)$, $\mathscr{D}(Cay(G, S))$ *is given by*

$$\mathscr{D}(Cay(G, S)) = \min\{k \in \mathbb{N} : (S \cup S^2 \cup \cdots S^k)^2 \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cdots \cup (S^k \cup (S^{-1})^k)\}.$$

Theorem 3.27 *Let* $Cay(G, S)$ *be a Cayley digraph with connection set* S. *Then we have the following:*

$$(i) \; \mathscr{C}(Cay(G, S)) = \mathscr{C}(Cay(G, S^{-1}))$$

$$(ii) \; \mathscr{D}(Cay(G, S)) = \mathscr{D}(Cay(G, S^{-1})).$$

4. (m, n)-(quasi-) Transitive Cayley Digraphs

In this section we generalize the definition of k-transitive and k-quasi-transitive digraphs as follows:

Definition 4.28 Let m and n be two positive integers such that $m > n$. A digraph G is (m, n)-transitive whenever there is a directed path of length m from x to y there is a directed path of length n from x to y. We define the

(m, n)-transitivity number of a digraph as the least positive integers m and n such that G is (m, n)-transitive. That is,

$$\mathscr{E}(G) = \min_{m,n}\{(m, n) \in \mathbb{N} \times \mathbb{N}, m > n : G \text{ is } (m, n) - \text{ transitive}\}.$$

We define, $\mathscr{E}(G) = (0, 0)$, if no such m and n exist.

Definition 4.29 Let m and n be two positive integers such that $m > n$. A digraph G is (m, n)-quasi-transitive whenever there is a directed path of length m from x to y there is a directed path of length n from x to y or a directed path of length n from y to x. We define the (m, n)-quasi-transitivity number of a digraph as the least positive integers m and n such that G is (m, n)-quasi-transitive. That is,

$$\mathscr{F}(G) = \min_{m,n}\{(m, n) \in \mathbb{N} \times \mathbb{N}, m > n : G \text{ is } (m, n) - \text{ quasi transitive}\}.$$

If no such m and n exist, we define $\mathscr{F}(G) = (0, 0)$.

Theorem 4.30 $Cay(G, S)$ is (m, n)-transitive if and only if $S^m \subseteq S^n$.

Proof. First assume that $Cay(G, S)$ is (m, n)-transitive. Let $x \in S^m$. Then

$$x = z_1 z_2 \cdots z_m \text{ for some } z_1, z_2, \ldots, z_m \in S.$$

Let $x_0 = 1, x_1 = z_1, x_2 = z_1 z_2, \ldots, x_m = x$. This implies that the sequence of vertices: $(x_0, x_1, x_2, \ldots, x_m)$ is a path of length m from x_0 to x. Since $Cay(G, S)$ is (m, n)- transitive, there is a path of length n, say, $(1, y_1, y_2, \ldots, y_{n-1}, x)$ from 1 to x. This implies that $x \in S^n$. Hence $S^m \subseteq S^n$.

Conversely, assume that $S^m \subset S^n$. We will show that $Cay(G, S)$ is (m, n)-transitive. Let $(x_0, x_1, x_2, \ldots, x_m)$ be a path of length m from x_0 to x_m. This implies that $x_0^{-1} x_1, x_1^{-1} x_2, \ldots, x_{m-1}^{-1} x_m \in S$. That is

$$(x_0^{-1} x_1)(x_1^{-1} x_2) \ldots (x_{m-1}^{-1} x_m) = x_0^{-1} x_m \in S^m \subseteq S^n$$

This implies that

$$x_0^{-1} x_m = t_1 t_2 \ldots t_n \text{ for some } t_i \in S$$

This implies that $(1, t_1, t_1 t_2, \ldots, t_1 t_2 \ldots t_n)$ is a path of length n form 1 to $t_1 t_2 \ldots t_n$. That is,

$$(x_0, x_0 t_1, x_0 t_1 t_2, \ldots, x_0 t_1 t_2 \ldots t_n)$$

is a path of length of n from x_0 to x_m. Hence $Cay(G, S)$ is (m, n)-transitive.

Corollary 4.31 $Cay(G, S)$ is $(k, 1)$-transitive (that is, k-transitive) if and only if $S^k \subseteq S$.

Corollary 4.32 $Cay(G, S)$ is $(2, 1)$-transitive (that is, transitive) if and only if $S^2 \subseteq S$.

Theorem 4.33 $Cay(G, S)$ is (m, n)-quasi-transitive if and only if $S^m \subseteq S^n \cup (S^{-1})^n$.

Corollary 4.34 $Cay(G, S)$ is $(k, 1)$-quasi-transitive (that is, k-quasi-transitive) if and only if $S^k \subseteq S \cup S^{-1}$.

Corollary 4.35 $Cay(G, S)$ is $(2, 1)$-quasi-transitive (that is, quasi-transitive) if and only if $S^2 \subseteq S \cup S^{-1}$.

Theorem 4.36 *The (m, n)-transitivity number of $Cay(G, S)$ is the least positive integers $m, n(m > n)$ such that $S^m \subseteq S^n$. That is,*
$$\mathscr{F}(Cay(G, S)) = \min_{m,n}\{(m, n) \in \mathbb{N} \times \mathbb{N}, m > n : S^m \subseteq S^n\}.$$

Theorem 4.37 *The (m, n)-quasi-transitivity number of $Cay(G, S)$ is the least positive integers $m, n(m > n)$ such that $S^m \subseteq S^n \cup (S^{-1})^n$. That is,*

$$\mathscr{F}(Cay(G, S)) = \min_{m,n}\{(m, n) \in \mathbb{N} \times \mathbb{N}, m > n : S^m \subseteq S^n \cup (S^{-1})^n\}.$$

5. (m, n)-path-(quasi-) Transitive Cayley Digraphs

Definition 5.38 Let m and n be two positive integers such that $m \geq n$. A digraph G is called (m, n)-path-transitive if whenever there is a xy directed path of length less than or equal to m and a yz directed path of length less than or equal to m, then there exists a xz-directed path of length less than or equal to n. We define the (m, n)-path-transitivity number of G as:

$$\mathscr{G}(G) = \min_{m,n}\{(m, n) \in \mathbb{N} \times \mathbb{N}, m \geq n : G \text{ is } (m, n) \text{ path transitive}\}.$$

Also, define $\mathscr{G}(G) = (0,0)$, if no such numbers m and n exist.

Theorem 5.39 $Cay(G,S)$ is (m,n)-path-transitive if and only if $(S \cup S^2 \cup \cdots S^m)^2 \subseteq (S \cup S^2 \cup \cdots S^n)$.

Corollary 5.40 $Cay(G,S)$ is (k,k)-path-transitive if and only if $(S \cup S^2 \cup \cdots S^k)^2 \subseteq (S \cup S^2 \cup \cdots S^k)$.

Corollary 5.41 $Cay(G,S)$ is $(k,1)$-path-transitive if and only if $(S \cup S^2 \cup \cdots S^k)^2 \subseteq S$.

Corollary 5.42 $Cay(G,S)$ is $(1,1)$-path-transitive if and only if $S^2 \subseteq S$.

Theorem 5.43 *The (m,n)-path-transitivity number of the (m,n)-path-transitive Cayley digraph $Cay(G,S)$ is the least positive integers $m,n(m \geq n)$ such that $(S \cup S^2 \cup \cdots S^m)^2 \subseteq (S \cup S^2 \cup \cdots S^n)$. That is,*

$$\mathscr{G}(Cay(G,S)) = \min_{m,n}\{(m,n) \in \mathbb{N} \times \mathbb{N}, m \geq n : (S \cup S^2 \cup \cdots S^m)^2 \subseteq (S \cup S^2 \cup \cdots S^n)\}.$$

Definition 5.44 Let m and n be two positive integers such that $m \geq n$. A digraph G is called (m,n)-path-quasi-transitive if whenever there is a xy directed path of length less than or equal to m and a yz directed path of length less than or equal to m, then there exists a xz-directed path of length less than or equal to n or there exists a zx-directed path of length less than or equal to n. We define the (m,n)-quasi-transitive number of G as:

$$\mathscr{H}(G) = \min_{m,n}\{(m,n) \in \mathbb{N} \times \mathbb{N}, m \geq n : G \text{ is } (m,n) \text{ -quasi-path transitive}\}.$$

Theorem 5.45 $Cay(G,S)$ *is (m,n)-path-quasi-transitive if and only if*

$$(S \cup S^2 \cup \cdots S^m)^2 \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cup \cdots (S^n \cup (S^{-1})^n).$$

Corollary 5.46 $Cay(G,S)$ *is (k,k)-path-quasi-transitive if and only if*

$$(S \cup S^2 \cup \cdots S^k)^2 \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cup \cdots (S^n \cup (S^{-1})^k).$$

Corollary 5.47 $Cay(G,S)$ *is $(k,1)$-path-quasi-transitive if and only if $(S \cup S^2 \cup \cdots S^k)^2 \subseteq S \cup S^{-1}$.*

Corollary 5.48 $Cay(G,S)$ *is $(1,1)$-path-transitive if and only if $S^2 \subseteq S \cup S^{-1}$.*

Theorem 5.49 *The (m,n)-path-quasi-transitivity number of the digraph $Cay(G,S)$ is the least positive integers $m,n(m \geq n)$ such that $(S \cup S^2 \cup \cdots S^m)^2 \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cup \cdots (S^n \cup (S^{-1})^n)$. That is, $\mathscr{I}(Cay(G,S))$ is given by*

$$\mathscr{I}(Cay(G,S)) = \min_{m,n}\{(m,n) \in \mathbb{N} \times \mathbb{N}, m > n : (S \cup \cdots \cup S^m)^2 \subseteq (S \cup S^{-1}) \cup \cdots \cup (S^n \cup (S^{-1})^n)\}.$$

6. m-n-(quasi)-transitive Transitive Cayley Digraphs

In this section we define the following:

Definition 6.50 Let m and n be positive integers such that $m > n$. A digraph G is m-n transitive whenever there is a directed path of length m from x to y there is a directed path of length almost n from x to y. We define the m-n *transitivity number* of G as:

$$\mathscr{J}(G) = \min_{m,n}\{(m,n) \in \mathbb{N} \times \mathbb{N}, m > n : G \text{ is } m - n \text{ transitive}\}.$$

Also, define $\mathscr{J}(G) = (0,0)$, if no such numbers m and n exist.

Theorem 6.51 $Cay(G,S)$ *is m-n transitive if and only if $S^m \subseteq S \cup S^2 \cdots \cup S^n$.*

Corollary 6.52 $Cay(G,S)$ *is k-1 transitive(that is, k-transitive) if and only if $S^k \subseteq S$.*

Corollary 6.53 $Cay(G,S)$ *is 2-1 transitive (that is, transitive) if and only if $S^2 \subseteq S$.*

Definition 6.54 A digraph G is m-n-quasi-transitive whenever there is a directed directed path of length m from x to y there is a directed path of length almost n from x to y or there is a directed path of length almost n from y to x. We define the m-n-quasi-transitivity number of G as:

$$\mathscr{K}(G) = \min_{m,n}\{(m,n) \in \mathbb{N} \times \mathbb{N}, m > n : G \text{ is } m - n \text{ quasi- transitive}\}.$$

Also, define $\mathscr{K}(G) = (0, 0)$, if no such numbers m and n exist.

Theorem 6.55 $Cay(G, S)$ *is m-n transitive if and only if* $S^m \subseteq (S \cup S^{-1}) \cup (S^2 \cup (S^{-1})^2) \cdots \cup S^n \cup (S^{-1})^n$.

Corollary 6.56 $Cay(G, S)$ *is k-1 quasi-transitive(that is, k-quasi-transitive) if and only if* $S^k \subseteq S \cup S^{-1}$.

Corollary 6.57 $Cay(G, S)$ *is 2-1-transitive (that is, quasi-transitive) if and only if* $S^2 \subseteq S$.

7. k-(quasi-)hasse Diagrams

We define the following:

Definition 7.58 A digraph G is a *k-hasse diagram* if the existence of a directed path $(x_0, x_1, \ldots, x_k)$ of length k in G implies that $(x_0, x_k) \notin E$.

We prove the following:

Theorem 7.59 $Cay(G, S)$ *is a k-hasse diagram if and only if* $S^k \cap S = \emptyset$.

Proof. First, assume that $S^k \cap S = \emptyset$. Let $(x_0, x_1, \ldots, x_k)$ be path of length k in G. Then we have $x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{k-1}x_k \in S$. This implies that

$$(x_0^{-1}x_1)(x_1^{-1}x_2)\ldots(x_{k-1}x_k) = x_0^{-1}x_k \in S^k$$

This implies that $x_0^{-1}x_k \notin S$. Hence (x_0, x_k) is not an edge in $Cay(G, S)$.

Conversely, suppose that $Cay(G, S)$ is a k-hasse diagram. Assume that $x \in S^k$. Then $x = z_1 z_2 \cdots z_k$ for some $z_1, z_2, \ldots, z_k \in S$. This implies that $(1, z_1, z_1 z_2, \ldots, x)$ is a path of length k in $Cay(G, S)$. Since $Cay(G, S)$ is a k-Hasse diagram, we have $(1, x) \notin S$. This implies that $x \notin S$. Hence $S^k \cap S = \emptyset$.

Definition 7.60 A digraph G is called a *hasse diagram* if and only if G is a k-hasse diagram for every $k \geq 2$.

Theorem 7.61 $Cay(G, S)$ *is a hasse diagram if and only if* $S^k \cap S = \emptyset$ *for every* $k \geq 2$.

Definition 7.62 A digraph G is a *k-strong-hasse diagram* if the existence of a directed path $(x_0, x_1, \ldots, x_k)$ of length k in G implies that $(x_0, x_k) \notin E$ and $(x_k, x_0) \notin E$.

Theorem 7.63 $Cay(G, S)$ *is a k-strong-hasse diagram if and only if* $S^k \cap S = \emptyset$ *and* $1 \notin S^k$.

Definition 7.64 A digraph G is called a *strong-hasse diagram* if and only if it is k-strong-hasse diagram for every $k \geq 2$.

Theorem 7.65 $Cay(G, S)$ *is a strong-hasse diagram if and only if* $S^k \cap S = \emptyset$ *and* $1 \notin S^k$ *for every* $k \geq 2$.

We define the following:

Definition 7.66 A digraph G is called an *anti-k-hasse diagram* if the existence of a directed path $(x_0, x_1, \ldots, x_k)$ of length k in G implies that there exists some i $(2 \leq i \leq k)$ such that $(x_0, x_i) \in E$. We define the anti-k-hasse index of G as:
$$\mathscr{L}(G) = \min\{k \in \mathbb{N} : G \text{ is an } k \text{ anti-hasse diagram}\}.$$

Sampathkumarachar et al. (2010) generalized the definition of transitive digraphs as follows:

Definition 7.67 A digraph G is said to be k- transitive if whenever $(x_0, x_1, \ldots, x_k)$ is a directed path of length k, then there exists an integer $i, 2 \leq i \leq k$ such that $(x_0, x_i) \in E$.

This definition is obviously, a generalization of transitive graphs. Observe that the definitions due to Sampathkumarachar et al. (2010) and Galeana-Sanchez and Cesar Hernandez-Cruz (2011), are entirely different even though they used the same term "k- transitive digraph". In this paper, we use the term "anti-k-hasse diagram" instead of k-transitive in the definition due to Sampathkumarachar et al. to avoid confusion.

We define the following:

Definition 7.68 A digraph G is *anti-k-quasi-hasse diagram* if the existence of a directed path $(x_0, x_1, \ldots, x_k)$ of length k in G implies that there exists some i $(2 \leq i \leq k)$ such that or $(x_0, x_i) \in E$ or $(x_i, x_0) \in E$. We define the anti-k-quasi-hasse index of G as:

$$\mathscr{M}(G) = \min\{k \in \mathbb{N} : G \text{ is an anti } k \text{ -quasi-hasse diagram}\}.$$

Theorem 7.69 $Cay(G, S)$ *is an anti-k-hasse diagram if and only if there exists an integer* i $(2 \leq i \leq k)$, *such that* $S^i \subseteq S$.

Proof. Assume that $S^i \subseteq S$ for some i. Let $(x, x_1, x_2, \ldots, y)$ be a directed path of length k from x to y. Then we have the following sets of equations:

$$
\begin{aligned}
x_1 &= xt_1, \\
x_2 &= x_1t_2, \\
x_3 &= x_2t_3, \\
&\vdots \\
y &= x_{k-1}t_k.
\end{aligned}
\tag{2}
$$

for some $t_1, t_2, \ldots, t_k \in S$. From Equation (2), x_i can be written as

$$
x_i = xt_1t_2 \ldots t_i \tag{3}
$$

Let $t = t_1t_2 \ldots t_i$. Then we have $x_i = xt$ where $t \in S^i \subseteq S$. This implies that (x, x_i) is an edge in $\mathrm{Cay}(G, S)$. Hence $\mathrm{Cay}(G, S)$ is an anti k-hasse diagram.

Conversely, assume that $\mathrm{Cay}(G, S)$ is an anti k-hasse diagram. For $i = 1, 2, \ldots k$, let x_i be an arbitrary element in S^i. Then $x_i = t_1t_2 \ldots t_i$ for some $t_i \in S$. Let $x = t_kt_{k-1} \ldots t_2t_1$. Then x is an arbitrary element in S^k. Note that $(1, x_1, x_2, \ldots, x)$ is a directed path leading from 1 to x. Then there exists an integer i such that $(1, x_i)$ is an edge in $\mathrm{Cay}(G, S)$. This implies that $x_i \in S$. Since x_i is an arbitrary element in S^i, we have $S^i \subseteq S$.

Corollary 7.70 *$\mathrm{Cay}(G, S)$ is an anti-2-hasse diagram (that is, transitive) if and only if $S^2 \subseteq S$.*

Theorem 7.71 *$\mathrm{Cay}(G, S)$ is an anti-k-quasi-hasse diagram if and only if there exists an integer i ($2 \le i \le n$), such that $S^i \subseteq S \cup S^{-1}$.*

Corollary 7.72 *$\mathrm{Cay}(G, S)$ is an anti-2-quasi-hasse diagram (that is, quasi-transitive) if and only if $S^2 \subseteq S \cup S^{-1}$.*

Theorem 7.73 *The anti-k-hasse index of $\mathrm{Cay}(G, S)$ is the least positive integer k such that there exists some i, $2 \le i \le k$ and $S^i \subseteq S$. That is,*

$$
\mathscr{L}(\mathrm{Cay}(G, S)) = \min\{k \in \mathbb{N} : S^i \subseteq S \text{ for some } i, 2 \le i \le k, \}.
$$

Theorem 7.74 *The anti-k-quasi-hasse index of $\mathrm{Cay}(G, S)$ is the least positive integer k such that there exists some i, $2 \le i \le k$ and $S^i \subseteq S \cup S^{-1}$. That is,*

$$
\mathscr{M}(\mathrm{Cay}(G, S)) = \min\{k \in \mathbb{N} : S^i \subseteq S \cup S^{-1} \text{ for some } i, 2 \le i \le k, \}.
$$

Theorem 7.75 *$\mathrm{Cay}(G, S)$ is anti-k-(quasi)-hasse diagram if and only if $\mathrm{Cay}(G, S^{-1})$ is anti-k-(quasi)-hasse diagram.*

Theorem 7.76 *Let $\mathrm{Cay}(G, S)$ be a Cayley digraph with connection set S. Then we have the following:*

$$
(i)\ \mathscr{L}(\mathrm{Cay}(G, S)) = \mathscr{L}(\mathrm{Cay}(G, S^{-1}))
$$
$$
(ii)\ \mathscr{M}(\mathrm{Cay}(G, S)) = \mathscr{M}(\mathrm{Cay}(G, S^{-1})).
$$

Acknowledgments

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