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Toeplitz Operators on the Super-sphere of Dimension (2 | 2)

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Abstract

In this paper we present the boundedness condition of the Toeplitz super-operators and we describe the quasi-Toeplitz operators over classical Bergman space with quasi-homogeneous symbols. Finally, we proved that every even super-operator acting over the Bergman super-space in the super-sphere can be written as a Toeplitz super-operator.

Keywords: Toeplitz operators, Bergman spaces, supermanifolds and graded manifolds

1. Introduction

Borthwick et al. (1993) introduced a general theory of the non-perturbative quantization of the super-disk. The quantization scheme is based on the notion of a Toeplitz super-operator on a suitable $\mathbb{Z}_2$-graded Hilbert space of super-holomorphic functions. The quantized supermanifold arises as the $\mathbb{C}^*$-algebra generated by such operators.

In (Sánchez-Nungaray), Sánchez-Nungaray showed the scheme of Toeplitz super-operators for the super-sphere $S^{(2|2)}$. Moreover, the author introduced the Bergman theory for the super-sphere and he also presented the form for the Bergman super-spaces and Toeplitz super-operators. In this work he characterized the invariant functions under the action of the super-circle. Moreover, he showed that the C^*-algebra of the Toeplitz operators with invariant symbols under the action of the super-circle is commutative. In (Loaiza & Sánchez-Nungaray, 2010), Loaiza and Sánchez-Nungaray studied the Toeplitz operators with radial symbols acting on the Bergman space of the super-disk and they proved that every Toeplitz super-operator with radial symbol is diagonal. This result generalizes the classical case, implying that the algebra generated by all Toeplitz super-operators with radial symbols is commutative.

Vasilevski (2001) proved that every Toeplitz operator with radial symbol acting on weigthed Bergman spaces is unitary equivalent to a multiplication operator on the unit disk. In general, the product of two Toeplitz operators not necessarily is a Toeplitz operator (Vasilevski, 2001). The situation changes in the sphere because weigthed Bergman spaces are finite-dimensional. In this way, Prieto-Sanabria (2009) showed that Toeplitz operators with radial symbols, acting on weigthed Bergman spaces of the sphere, are unitary equivalent to multiplication operators. Using this fact, he also proved that every operator acting in weigthed Bergman spaces of the sphere is a Toeplitz operator.

In the case of the super-sphere $S^{(2|2)}$, we know that Bergman spaces are finite dimensional, and also every Toeplitz super-operator is an even operator. We prove in this paper that every even operator is a Toeplitz super-operator. In particular, the product of two Toeplitz super-operators is a Toeplitz super-operator. In this way, we obtain an extension of the result presented for the sphere S^2 to the super-sphere $S^{(2|2)}$.

In Sections 2 and 3 we introduce the Bergman theory for the super-sphere, for example, by defining the notion of Bergman super-space and Toeplitz super-operator and giving the description of these operators. In Section 4 we present the boundedness condition of the Toeplitz super-operators as follows. In Proposition 3.1 we consider super-functions having no super-variables. In Proposition 3.2 we study odd super-functions. In Proposition 3.3 we study even super-functions vanishing in the part without super-variables. Finally, the Corollary 3.4 is summary of these propositions. In Section 5 we describe the quasi-Toeplitz operators over classical Bergman spaces with quasi-homogeneous symbols. In the last section we prove that every even operator over Bergman super-space on the super-sphere is a Toeplitz super-operator.

2. Bergman Super-space on the Super-sphere

The super-sphere $S^{(2|2)}$ is defined as the projective super-plane. Let $\mathcal{N}$ be the set of all nilpotent elements of $\mathbb{C}^{(2|1)}$: $\mathcal{N} = \{x \in \mathbb{C}^{(2|1)} : x^k = 0 \text{ for } k \in \mathbb{Z}\}$. Consider

$$(\mathbb{C}^{(2|1)})^{\times} = \{x \in \mathbb{C}^{(2|1)} : x \bmod \mathcal{N} \neq 0\}.$$

Let $(z_1, z_2, \theta), (z'_1, z'_2, \theta') \in (\mathbb{C}^{(2|1)})^{\times}$. We say that $(z_1, z_2, \theta) \sim (z'_1, z'_2, \theta')$ if there exists $\lambda \in (\mathbb{C}^{(1|0)})^{\times}$ such that $(z_1, z_2, \theta) = (\lambda z'_1, \lambda z'_2, \lambda\theta')$. By means of this equivalent relation we define the projective super-plane: $S^{(2|2)} = (\mathbb{C}^{(2|1)})^{\times}/\sim$.

Using the homogeneous coordinates, the local coordinates can be expressed with the two following charts of $S^{(2|2)}$:

$$(z, \theta) = \left(\frac{z_1}{z_2}, \frac{\theta}{z_2}\right), \quad (z', \theta') = \left(\frac{z_2}{z_1}, \frac{\theta}{z_1}\right).$$

Therefore, the super-sphere can be covered by two open domains, glued by

$$(z', \theta') = \left(\frac{1}{z}, \frac{\theta}{z}\right).$$

See Ninnemann (1992) for more details of the super-sphere space.

An element $f \in C^{\infty}(S^{(2|2)})$ can be written, in local coordinates, as

$$f(z, \theta, \bar{\theta}) = f_{00}(z) + f_{10}(z)\theta + f_{01}(z)\bar{\theta} + f_{11}(z)\theta\bar{\theta}, \tag{2.1}$$

where $f_{ij} \in C^{\infty}(\mathbb{C})$.

The following definitions and propositions in this section can be seen in (Sánchez-Nungaray).

Definition 2.1 A function $f \in C^{\infty}(S^{(2|2)})$ is called **super-holomorphic** if $\partial_{\bar{z}} f = 0$ and $\partial_{\bar{\theta}} f = 0$, or equivalently, if

$$f(z, \theta) = f_0(z) + f_1(z)\theta, \tag{2.2}$$

where f_0 and f_1 are holomorphic functions in $\mathbb{C}$. In the follow we use a collective notation for the coordinates, namely, $Z = (z, \theta)$.

Definition 2.2 We define the **super-spherical measure** by

$$d\mu(Z) := -\frac{1}{\pi}\left(1 + z\bar{z} - \theta\bar{\theta}\right)^{-1} dA(z) d\theta \wedge d\bar{\theta}, \tag{2.3}$$

where $dA(z) = \frac{i}{2} dz \wedge d\bar{z}$.

We consider the following perturbations of the measure (2.3):

$$d\mu_h(Z) = \left(1 + z\bar{z} - \theta\bar{\theta}\right)^{-1/h} d\mu(Z) = -\frac{1}{\pi}\left(1 + z\bar{z} - \theta\bar{\theta}\right)^{-1-N} dA(z) d\theta \wedge d\bar{\theta}, \tag{2.4}$$

where $h = 1/2, 1/3, \ldots$ and $N = 1/h \in \mathbb{N}$.

Let f, g be functions defined on $S^{(2|2)}$. We define a **semi-inner product** by

$$(f, g)_h := \int_{S^{(2|2)}} f(Z)\overline{g(Z)} d\mu_h(Z), \tag{2.5}$$

where f, g have the form (2.1), and f_{ij} are measurable functions on S^2.

Expanding (2.5) we obtain

$$(f, g)_h = \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{f_{00}(z)\,\overline{g_{00}(z)}\,dA(z)}{(1 + z\bar{z})^{N+2}} +$$
$$+ \frac{1}{\pi} \int_{\mathbb{C}} \frac{(f_{11}(z)\,\overline{g_{00}} + f_{10}(z)\,\overline{g_{10}(z)} - f_{01}(z)\,\overline{g_{01}(z)} + f_{00}(z)\,\overline{g_{11}(z)})dA(z)}{(1 + z\bar{z})^{N+1}}.$$

Note that the above semi-inner product is not positive definite. Now we consider the restriction of this semi-inner product to the set of super-holomorphic functions. Thus, this semi-inner product turns out to be positive definite and so it defines the following inner product

$$(f, g)_h = \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{f_0(z)\overline{g_0(z)}dA(z)}{(1+z\bar{z})^{N+2}} + \frac{1}{\pi} \int_{\mathbb{C}} \frac{f_1(z)\overline{g_1(z)}dA(z)}{(1+z\bar{z})^{N+1}}.$$

The completion of the set of super-holomorphic functions with respect to the norm $\|\cdot\|_h$ is a Hilbert space. This super-space is called the **weighted Bergman super-space** on the super-sphere, and is denoted by $\mathcal{A}_h^2(S^{(2|2)})$. Being finite dimensional, the space $\mathcal{A}_h^2(S^{(2|2)})$ is closed.

For $f \in L^\infty(S^{(2|2)})$, we define the **weighted Bergman projection** by

$$P_h(f(Z)) = \int_{S^{(2|2)}} f(W)K^h(Z, W)d\mu_h(W),$$

where

$$K^h(Z, W) = (1 + z\bar{w} - \theta\bar{\eta})^N.$$

An important result about the Bergman theory over the super-sphere is the following.

Theorem 2.3 *If $f \in L_h^\infty(S^{(2|2)})$, then $P_h(f) \in \mathcal{A}_h^2(S^{(2|2)})$, and $P_h(f) = f$ for $f \in \mathcal{A}_h^2(S^{(2|2)})$.*

3. Toeplitz Super-operators

We refer to (Sánchez-Nungaray) for more details about the topics of this section.

Definition 3.1 For $f \in L^\infty(S^{(2|2)})$ and $\varphi \in \mathcal{A}_h^2(S^{(2|2)})$, we define the **Toeplitz operator** on the Bergman super-space with weight parameter $1/h$ by

$$T_f^h\varphi(Z) = \int_{S^{(2|2)}} f(W)\varphi(W)K^h(Z, W)d\mu_h(W). \tag{3.1}$$

Definition 3.2 Let $h = 1/2, 1/3, ...,$ and $a \in L^\infty(S^2)$. We define the **classical weighted Bergman** space on the sphere S^2 by

$$A_N^2(S^2) = A_h^2(S^2) = \left\{f \in L^2(S^2, (N+1)d\mu_h) \ : \ f \text{ is holomorphic}\right\},$$

where $d\mu_h = (1/\pi)(1 + z\bar{z})^{-N-2}dA(z)$. This space can be denoted by $A_N^2(S^2)$, where $N = 1/h \in \mathbb{N}$.

Definition 3.3 The Toeplitz operator with symbol a acting on $A_h^2(S^2)$ is defined by

$$T_N(a)(\varphi)(z) = T_h(a)(\varphi)(z) = B_h\big(a(w)\varphi(w)\big)(z),$$

where B_h is the classical Bergman projection onto the weighted Bergman space $A_h^2(S^2)$.

Now we can represent the super-space $\mathcal{A}_h^2(S^{(2|2)})$ as a direct sum of Bergman spaces on the sphere S^2, that is,

$$\mathcal{A}_h^2(S^{(2|2)}) = A_N^2(S^2) \oplus A_{N-1}^2(S^2)\theta,$$

where $(\varphi_0, \varphi_1) \in A_N^2(S^2) \oplus A_{N-1}^2(S^2)$ whenever $\varphi = \varphi_0 + \varphi_1\theta \in \mathcal{A}_h^2(S^{(2|2)})$. Moreover, the Bergman spaces $A_N^2(S^2)$, $A_{N-1}^2(S^2)\theta$ are the even and odd parts of the Bergman super-space, respectively.

Now we describe the Toeplitz operator acting on the super-space $A_N^2(S^2) \oplus A_{N-1}^2(S^2)\theta$.

Note that if $f(W)$ has the form (2.1) and $\varphi(W) = \varphi_0(w) + \varphi_1(w)\eta$, with $\varphi(W) \in \mathcal{A}_h^2(S^{(2|2)})$, then

$$f(W)\varphi(W) = \varphi_0(w)[f_{00}(w) + f_{10}(w)\eta + f_{01}(w)\bar{\eta} + f_{11}(w)\eta\bar{\eta}] + \varphi_1(w)[f_{00}(w)\eta - f_{01}(w)\eta\bar{\eta}],$$

and

$$K^h(Z, W)d\mu_h(W) = \frac{-1}{\pi}\left[\frac{(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}\right]dA(w)d\eta\wedge d\bar{\eta} + \frac{-1}{\pi}\left[-\frac{N(1+z\bar{w})^{N-1}\theta\bar{\eta}}{(1+w\bar{w})^{N+1}} + \frac{(N+1)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}}\eta\bar{\eta}\right]dA(w)d\eta\wedge d\bar{\eta}.$$

The elements in the integrals that do not vanish are those containing the term $\eta\bar{\eta}$. Therefore

$$T_f^h\varphi(Z) = \frac{-1}{\pi}\int_{\mathbb{C}}\frac{\varphi_1(w)f_{01}(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}dA(w) + \frac{N+1}{\pi}\int_{\mathbb{C}}\left(f_{00}(w)+\frac{f_{11}(w)(1+z\bar{z})}{N+1}\right)\frac{\varphi_0(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}}dA(w)$$
$$+\left[\frac{N}{\pi}\int_{\mathbb{C}}\frac{\varphi_0(w)f_{10}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w)\right]\theta + \left[\frac{N}{\pi}\int_{\mathbb{C}}\frac{\varphi_1(w)f_{00}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w)\right]\theta.$$

Thus, the Toeplitz operator T_f^h is equivalent to the following operator on the space $A_N^2(S^2)\bigoplus A_{N-1}^2(S^2)\theta$:

$$\begin{pmatrix} T_N\left(f_{00}+\dfrac{f_{11}(1+z\bar{z})}{(N+1)}\right) & -T_N^{N-1}(f_{01}) \\ T_{N-1}^N(f_{10}) & T_{N-1}(f_{00}) \end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix} \tag{3.2}$$

where $T_N\left(f_{00}+\dfrac{f_{11}(1+z\bar{z})}{(N+1)}\right)$ and $T_{N-1}(f_{00})$ are ordinary Toeplitz operators on the weighted Bergman spaces $A_N^2(S^2)$ and $A_{N-1}^2(S^2)$, respectively. The remaining operators have the following form:

$$T_N^{N-1}(f_{01}) : A_{N-1}^2(S^2) \longrightarrow A_N^2(S^2),$$

is defined by

$$T_N^{N-1}(f_{01})(\varphi_1)(z) = \frac{1}{\pi}\int_{\mathbb{C}}\frac{\varphi_1(w)f_{01}(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}dA(w), \tag{3.3}$$

and

$$T_{N-1}^N(f_{01}) : A_N^2(S^2) \longrightarrow A_{N-1}^2(S^2),$$

is defined by

$$T_{N-1}^N(f_{10})(\varphi_0)(w) = \frac{N}{\pi}\int_{\mathbb{C}}\frac{\varphi_0(w)f_{10}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w). \tag{3.4}$$

Note that the Toeplitz super-operator (3.2) is an even operator on the Bergman super-space, because every even operator leaves invariant the parity of the super space.

4. Boundedness Conditions

In this section we present some results over boundedness conditions for Toeplitz operators on Bergman super-spaces.

Proposition 4.1 *Let $\varphi,\psi\in\mathcal{A}_h^2(S^{(2|2)})$, and $a\in L^\infty(S^{(2|2)})$ such that $a(Z)=a_{00}(z)$. Then*

$$\left|\int_{S^{(2|2)}}\varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| \le \|a\|_0\,\|\varphi\|_h\,\|\psi\|_h. \tag{4.1}$$

Proof. Since $a(Z)=a_{00}(z)$ we have

$$\int_{S^{(2|2)}}a_{00}(Z)\varphi(Z)\overline{\psi(Z)}d\mu_h(Z) = \frac{N+1}{\pi}\int_{\mathbb{C}}\frac{a_{00}(Z)\varphi_0(z)\overline{\psi_0(z)}dA(z)}{(1+z\bar{z})^{N+2}} + \frac{1}{\pi}\int_{\mathbb{C}}\frac{a_{00}(Z)\varphi_1(z)\overline{\psi_1(z)}dA(z)}{(1+z\bar{z})^{N+1}}.$$

Thus

$$\begin{aligned}
&\left|\int_{S^{(2|2)}}a_{00}(Z)\varphi(Z)\overline{\psi(Z)}d\mu_h(Z)\right| \\
\le\ & \|a_{00}(Z)\|_0\frac{N+1}{\pi}\left\{\int_{\mathbb{C}}\frac{|\varphi_0(z)|^2dA(z)}{(1+z\bar{z})^{N+2}}\right\}^{1/2}\left\{\int_{\mathbb{C}}\frac{|\psi_0(z)|^2dA(z)}{(1+z\bar{z})^{N+2}}\right\}^{1/2} \\
+\ & \|a_{00}(Z)\|_0\frac{1}{\pi}\left\{\int_{\mathbb{C}}\frac{|\varphi_1(z)|^2dA(z)}{(1+z\bar{z})^{N+1}}\right\}^{1/2}\left\{\int_{\mathbb{C}}\frac{|\psi_1(z)|^2dA(z)}{(1+z\bar{z})^{N+1}}\right\}^{1/2}.
\end{aligned}$$

By the above equation, inequality (4.1) holds.

Proposition 4.2 *For $\varphi, \psi \in \mathcal{A}_h^2(S^{(2|2)})$, and $a \in L^\infty(S^{(2|2)})$ such that $a(Z) = a_{10}\theta$ or $a(Z) = a_{01}\bar{\theta}$, we have*

$$\left|\int_{S^{(2|2)}} \varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| \leq h^{1/2}\|a_{ij}\|_0\|\varphi\|_h\|\psi\|_h, \tag{4.2}$$

where $0 \leq i, j \leq 1$ and $i + j = 1$.

Proof. We just prove the result for $a = a_{10}(z)\theta$, because the reasoning is analogous for $a = a_{01}(z)\bar{\theta}$. For the first case we have

$$\left|\int_{S^{(2|2)}} \varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| = \left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(z)\overline{\psi_1(z)}a_{10}(z)dA(z)}{(1+z\bar{z})^{N+1}}\right| \leq \|a_{10}(z)\|_0 \left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(z)\overline{\psi_1(z)}dA(z)}{(1+z\bar{z})^{N+1}}\right|. \tag{4.3}$$

We know that the functions φ_0 and φ_1 have the following form:

$$\varphi_0(z) = \sum_{n=0}^{N} a_n z^n, \quad \psi_1(z) = \sum_{n=0}^{N-1} b_n z^n.$$

Hence we replace the previous equation in (4.3), obtaining

$$\begin{aligned}
&\left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(z)\overline{\psi_1(z)}dA(z)}{(1+z\bar{z})^{N+1}}\right| \\
&\leq \left|\sum_{n=0}^{N-1} a_n\bar{b}_n \int_0^\infty \frac{u^n}{(1+u)^{N+1}}du\right| = \left|\sum_{n=0}^{N-1} a_n\bar{b}_n \left(N\binom{N-1}{n}\right)^{-1}\right| \\
&\leq \left\{\sum_{n=0}^{N-1} |a_n|^2 \left(N\binom{N-1}{n}\right)^{-1}\right\}^{1/2} \left\{\sum_{n=0}^{N-1} |b_n|^2 \left(N\binom{N-1}{n}\right)^{-1}\right\}^{1/2} \\
&\leq N^{-1/2}\|\varphi\|_h\|\psi\|_h.
\end{aligned}$$

Note that if $a(Z) = a_{11}(z)\theta\bar{\theta}$, with a_{11} bounded, then there is not guaranteed that the operator is bounded. This fact is shown in the following

Example Take $a(Z) = \theta\bar{\theta}$. Since $z^N \in \mathcal{A}_h^2(S^{(2|2)})$, we have

$$\left|\int_{S^{(2|2)}} z^N a(Z)\overline{z^N}d\mu_h(Z)\right| = \int_{\mathbb{C}} \frac{|z|^{2N}dA(z)}{(1+z\bar{z})^{N+1}},$$

with the left-side expression divergent.

Proposition 4.3 *If $\varphi, \psi \in \mathcal{A}_h^2(S^{(2|2)})$, and $a \in L^\infty(S^{(2|2)})$, such that*

$$a(Z) = \frac{a_{11}(z)\theta\bar{\theta}}{(1+z\bar{z})^\epsilon}, \quad \text{for } z \in \mathbb{C} \text{ and } \epsilon > 0, \tag{4.4}$$

then

$$\left|\int_{S^{(2|2)}} \varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| \leq \|a_{11}\|_0\, \|\varphi\|_h\, \|\psi\|_h. \tag{4.5}$$

Proof. Since $a(Z) = a_{11}(z)(1+z\bar{z})^{-\epsilon}\theta\bar{\theta}$, the Toeplitz operator with symbol a is bounded because the following bilinear form is also bounded:

$$\left|\int_{S^{(2|2)}} \varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| = \left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(z)\overline{\psi_0(z)}a_{11}(z)dA(z)}{(1+z\bar{z})^{N+1+\epsilon}}\right| \leq \|a_{11}\|_0 \left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(z)\overline{\psi_0(z)}d^2z}{(1+z\bar{z})^{N+1+\epsilon}}\right|. \tag{4.6}$$

We know that

$$\varphi_0(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad \psi_0(z) = \sum_{n=0}^{N} b_n z^n.$$

In consequence

$$\left|\frac{1}{\pi}\int_{\mathbb{C}}\frac{\varphi_0(z)\overline{\psi_0(z)}d^2z}{(1+z\bar{z})^{N+2}}\right| \leq \left|\sum_{n=0}^{N} a_n\bar{b}_n\int_0^\infty \frac{u^n}{(1+u)^{N+2}}du\right|$$

$$\leq \left\{\frac{1}{N+1}\sum_{n=0}^{N}|a_n|^2\left(N\binom{N}{n}\right)^{-1}\right\}^{1/2}\left\{\frac{1}{N+1}\sum_{n=0}^{N}|b_n|^2\left(N\binom{N}{n}\right)^{-1}\right\}^{1/2}$$

$$\leq N^{-1}||\varphi||_h||\psi||_h.$$

In summary we have the following result

Corollary 4.4 *Let $\varphi, \psi \in \mathcal{A}_h^2(S^{(2|2)})$ and $a \in L^\infty(S^{(2|2)})$ such that*

$$a(Z) = a_{00}(z)(z) + a_{10}(z)\theta + a_{01}(z)\bar{\theta} + \frac{a_{11}(z)}{(1+z\bar{z})}\theta\bar{\theta},$$

where the functions a_{ij} are bounded. Then

$$\left|\int_{S^{(2|2)}}\varphi(Z)a(Z)\overline{\psi(Z)}d\mu_h(Z)\right| \leq ||a||_0\,||\varphi||_h\,||\psi||_h. \tag{4.7}$$

5. Form of Quasi-Toeplitz Operators with Radial and Quasi-homogeneous Symbols

Remember that each Toeplitz super-operator has the form (3.2), where the components of the super-operator are both classical Toeplitz operators and quasi-Toeplitz operators. Toeplitz operators with radial and quasi-homogeneous symbols were studied in Prieto-Sanabria (2009). In this section, we analyze quasi-Toeplitz operators with radial and quasi-homogeneous symbols.

Consider the basis $\mathcal{B}_N = \{1, z, ..., z^N\}$ for the space $A_N^2(S^2)$. If a is a radial function, then we consider the following operator:

$$T_{N-1}^N(a) : A_N^2(S^2) \longrightarrow A_{N-1}^2(S^2).$$

This operator has a representation over the bases $\mathcal{B}_N$ and $\mathcal{B}_{N-1}$ given by the following expression:

$$T_{N-1}^N(a)(z^n) = \begin{cases} \hat{a}_N(n)z^n & \text{for } n = 0, 1, \ldots, N-1; \\ 0 & \text{for } n = N, \end{cases}$$

where

$$\hat{a}_N(n) = 2N\binom{N-1}{n}\int_0^\infty \frac{a(r)r^{2n+1}dr}{(1+r^2)^{N+1}}.$$

The last expression is a matrix of dimension $N \times (N+1)$ given by

$$\begin{pmatrix} \hat{a}_{N-1}(0) & 0 & \ldots & \ldots & 0 & 0 \\ 0 & \hat{a}_{N-1}(1) & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & \ldots & 0 & \hat{a}_{N-1}(N-1) & 0 \end{pmatrix} \tag{5.1}$$

Analogously, if a is a radial function, then the quasi-Toeplitz operator

$$T_N^{N-1}(a) : A_{N-1}^2(S^2) \longrightarrow A_N^2(S^2)$$

has the form

$$T_N^{N-1}(a)(z^n) = \hat{a}_{N-1}(n)z^n, \quad \text{for } n = 0, 1, \ldots, N-1,$$

where
$$\hat{a}_{N-1}(n) = 2\binom{N}{n}\int_0^\infty \frac{a(r)r^{2n+1}dr}{(1+r^2)^{N+1}}.$$
The corresponding matrix of dimension $(N+1)\times N$ is given by
$$\begin{pmatrix} \hat{a}_N(0) & 0 & \dots & \dots & 0 \\ 0 & \hat{a}_N(1) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \hat{a}_N(N-1) \\ 0 & \dots & & \cdots & 0 \end{pmatrix} \tag{5.2}$$

Definition 5.1 Let g be a function on S^2. We say that g is **quasi-homogeneous** if it has the form $e^{ik\theta}a$, where a is a radial function and $k \in \mathbb{Z}$. In this case we say that g has degree k.

Now we analyze the case when the symbol is a quasi-homogeneous function.

Given $k \in \mathbb{Z}$, we consider the quasi-Toeplitz operator with quasi-homogeneous symbol. The first case of this type of operators has the following form:
$$T^N_{N-1}(e^{ik\theta}a) : A^2_N(S^2) \longrightarrow A^2_{N-1}(S^2).$$
The representation of this operator in the bases $\mathcal{B}_N$ and $\mathcal{B}_{N-1}$ is as follows:
$$T^N_{N-1}(e^{ik\theta}a)(z^n) = \begin{cases} \hat{a}^k_N(n)z^{n+k} & \text{for } n = 0,1,\dots,N-k-1, \text{ with } 0 \le k \le N; \\ \hat{a}^k_N(n)z^{n+k} & \text{for } n = -k,\dots,N, \text{ with } -N \le k < 0; \\ 0 & \text{otherwise,} \end{cases}$$
where
$$\hat{a}^k_N(n) = 2N\binom{N-1}{n+k}\int_0^\infty \frac{a(r)r^{2n+k+1}dr}{(1+r^2)^{N+1}}.$$
For the other quasi-Toeplitz operator with quasi-homogeneous symbol we have
$$T^{N-1}_N(e^{ik\theta}a) : A^2_{N-1}(S^2) \longrightarrow A^2_N(S^2)$$
and whose expression is
$$T^{N-1}_N(e^{ik\theta}a)(z^n) = \begin{cases} \hat{a}^k_{N-1}(n)z^{n+k}, & \text{for } n = 0,1,\dots,N-k, \text{ for } 0 \le k \le N; \\ \hat{a}^k_{N-1}(n)z^{n+k}, & \text{for } n = -k,\dots,N, \text{ for } -N \le k < 0; \\ 0, & \text{otherwise,} \end{cases}$$
where
$$\hat{a}^k_{N-1}(n) = 2\binom{N}{n+k}\int_0^\infty \frac{a(r)r^{2n+k+1}dr}{(1+r^2)^{N+1}}.$$

6. Inverse Problem

In this section we prove the following fact: every even operator over Bergman super-space on the super-sphere is a Toeplitz super-operator. Prieto-Sanabria (2009) showed that every bounded operator B, acting on weighted Bergman spaces, is a Toeplitz operator, i.e., there exists $a \in L^\infty(S^2)$ such that the Toeplitz operator $T_N(a)$ is equal to B. Now we prove the following lemma which is necessary to obtain the inverse problem for the super case. In (Prieto-Sanabria, 2009) there exists an analogous result for the sphere.

Lemma 6.1 *Let $k \in \mathbb{N}\cup\{0\}$ be fixed. If $(b_0,\dots,b_{N-k}) \in \mathbb{C}^{N+1-k}$ or $(b_k,\dots,b_N) \in \mathbb{C}^{N-k}$, then there exists $a \in L^\infty(\mathbb{R}_+)$ such that the following assertions holds*
$$\int_0^\infty \frac{a(r)r^{2n+k+1}dr}{(1+r^2)^{N+1}} = b_n, \quad \textit{for } n = 0,1,\dots,N-k \tag{6.1}$$

or

$$\int_0^\infty \frac{a(r)r^{2n-k+1}dr}{(1+r^2)^{N+1}} = b_{n-k}, \quad \textit{for } n = k, \ldots, N. \tag{6.2}$$

Proof. We just prove the first assertion because the second is analogous. Let a be defined by

$$a(r) = \sum_{s=0}^{N-k} \frac{c_s r^{2s+k}}{(1+r^2)^{2(N+1)}},$$

where the c_s's are complex numbers.

If we replace the function a in (6.1), we obtain the following linear system of equations:

$$\sum_{s=0}^{N-k} c_s \int_0^\infty \frac{r^{2(n+k+s)+1}dr}{(1+r^2)^{3(N+1)}} = b_n, \quad \text{for } n = 0, 1, \ldots, N-k. \tag{6.3}$$

Note that the set of functions $\{r^{2s+k}\}_{s=0}^{N-k} \subset L^2(\mathbb{R}_+, du_N(r))$ is linearly independent, where

$$du_N(r) = \frac{rdr}{(1+r^2)^{3(N+1)}}.$$

Thus

$$\det(\langle r^{2n+k}, r^{2s+k}\rangle) \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}_+, du_N(r))$ defined by

$$\langle r^{2n+k}, r^{2s+k}\rangle = \int_0^\infty \frac{r^{2(n+k+s)+1}dr}{(1+r^2)^{3(N+1)}}.$$

In consequence, the system (6.3) of linear equations has an unique solution.

Now we present the following concept:

Definition 6.2 Let $A = [a_{ij}]$ be a matrix of dimension $p \times q$, and $k \in \mathbb{Z}$. We say that A is **quasi-diagonal** of type k if $a_{i+k,j} = 0$ for $i + k \neq j$.

Note that every matrix A can be written as a sum of quasi-diagonal matrices. Furthermore, every quasi-Toeplitz operator with quasi-homogeneous symbol is a quasi-diagonal matrix with respect to the representation.

Theorem 6.3 *Let B be a bounded operator from $A_N(S^2)$ to $A_{N-1}(S^2)$, such that the matrix representation of B with respect to the bases $\mathcal{B}_N$ and $\mathcal{B}_{N-1}$ is a quasi-diagonal matrix. Then there exists a radial function a such that*

$$T_{N-1}^N(e^{itk}a(r)) = B.$$

Analogously, let B be a bounded operator from $A_{N-1}(S^2)$ into $A_N(S^2)$, such that the matrix representation of B with respect to the bases $\mathcal{B}_N$ and $\mathcal{B}_{N-1}$ is a quasi-diagonal matrix. Then there exists a radial function b such that

$$T_N^{N-1}(e^{itk}b(r)) = B.$$

Proof. We just prove the first case, since the second is analogous. Consider a quasi-diagonal matrix B of type k. Now, take a radial function a such that $T_{N-1}^N(e^{itk}a(r)) = B$, where a is the solution function of a linear system similar to (6.1). This solution can be found by using Lemma 6.1.

As a consequence of the previous result, every arbitrary operator can be written as a quasi-Toeplitz operator where its symbol is a sum of quasi-homogeneous functions. Thus, we have the following

Corollary 6.4 *Let B be an arbitrary bounded operator from $A_N(S^2)$ into $A_{N-1}(S^2)$. Then there exists $a \in L^\infty(S^2)$ such that*

$$T_{N-1}^N(a) = B.$$

Analogously, let B be a bounded operator from $A_{N-1}(S^2)$ into $A_N(S^2)$. Then there exists $a \in L^\infty(S^2)$ such that

$$T_N^{N-1}(a) = B.$$

The main result of this paper is the following:

Theorem 6.5 *Let B be an even super-operator on the Bergman super-space. Then there exists a super-function a such that B can be written as a Toeplitz super-operator with symbol a.*

Proof. We know that every even super-operator in the Bergman space has the following matrix representation:

$$\begin{pmatrix} B_0 & B_1 \\ B_2 & B_3 \end{pmatrix},$$

where B_0 is an operator acting on $A_N(S^2)$, B_1 is an operator from $A_{N-1}(S^2)$ into $A_N(S^2)$, B_2 is an operator from $A_N(S^2)$ into $A_{N-1}(S^2)$, and B_3 is an operator on $A_{N-1}(S^2)$.

Using the fact that every operator is Toeplitz on weighted Bergman spaces, we can find a function a_{00} such that $B_3 = T_{N-1}(a_{00})$. Analogously, there exists a function a_{11} such that $T_N(a_{11}) = (N+1)\big(B_1 - T_N(a_{00})\big)$. Moreover, by Corollary 6.4, there exists two functions a_{10} and a_{01} such that $B_1 = T_N^{N-1}(-a_{01})$ and $B_1 = T_{N-1}^{N}(a_{10})$. Thus we take the Toeplitz super-operator with symbol

$$a = a_{00} + a_{10}\theta + a_{01}\bar{\theta} + \frac{a_{11}\theta\bar{\theta}}{(1+z\bar{z})},$$

and we have that $T^h(a) = B$.

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