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Analysis of a Multigrid Algorithm for Mortar Element Method

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Abstract

In this paper, a multigrid algorithm is studied for mortar element method for rotated Q_1 element, the mortar condition is only dependent on the degrees of the freedom on subdomains interfaces. We prove the convergence of W-cycle multigrid and construct a variable V-cycle multigrid preconditioner which is available.

Keywords: Multigrid, Mortar element method, Rotated Q_1 element

1. Introduction

The mortar element method is a nonconforming domain decomposition method with non-overlapping subdomains. The meshes on different subdomains need not align across subdomains interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. This method offers the advantages of freely choosing highly varying mesh sizes on different subdomains. The rotat Q_1 element is an important nonconforming element. It was first proposed and analysised for numerically solving the Stokes problem, the rotated Q_1 element provides the simplest example of discretely divergence-free nonconforming element on quadrilaterals.

Let $\Omega \in R^2$ be a rectangular or L-shape bounded domain with boundary $\partial\Omega$. Partition Ω into geometrically conforming rectangular substructures, i.e..

$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$ and $\Omega_k \bigcap \Omega_l = \phi$, $k \neq l$, $\overline{\Omega}_k \bigcap \overline{\Omega}_l$ is empty set or a vertex or an edge for $k \neq l$.

Let $T_1^i = T_1^i(\Omega_i)$ be a coarsest quasi-uniform triangulation of the subdomain Ω_i, which made of elements that are rectangles whose edges are parallel to X-axis or Y-axis. Let $T_1 = \bigcup_{i=1}^{N} T_1^i$. The mesh parameter h_1 is the diameter of the largest element in T_1 the global triangulation of Ω. We refine the triangulation T_1 to produce T_2 by joining the midpoints of the edges of the rectangles in T_1. Obviously, the mesh size h_2 in T_2 satisfies $h_2 = \frac{1}{2}h_1$. Repeating this process, we get a sequel of triangulations $T_1(l = 1, 2, \cdots, L)$. Let $\Omega_{i,l}$ and $\partial\Omega_{i,l}$ be the set of vertices of the triangulation T_1^i that are in $\overline{\Omega}_i$ and $\partial\Omega_i$ respectively.

We construct the rotated Q_l element for each triangulation $T_l(\Omega_i)$ as follows.

$X_l(\Omega_i) = \{v \in L^2(\Omega_i)|v|_E = \alpha_E^1 + \alpha_E^2 x + \alpha_E^3 y + \alpha_E^4(x^2 - y^2),\ \alpha_E^2 \in R, \int_{\partial E|\partial\Omega} v|_{\partial\Omega} ds = 0,\ \forall E \in T_l(\Omega_i)$; for $E_1,\ E_2 \in T_l(\Omega_i)$, if $\partial E_1|\partial E_2 = e$, then $\int_e v|_{\partial E_1} ds = \int_e v|_{\partial E_2} ds\}$

The global discrete space is defined by

$$X_l(\Omega) = \prod_{i=1}^{N} X_l(\Omega_i)$$

The interface $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$ is broken into a set of disjoint open straight segments $\gamma_m (1 \leq m \leq M)$, i.e., $\Gamma = \bigcup_{m=1}^{M} \overline{\gamma}_m$, $\gamma_m \cap \gamma = \phi$, if $m \neq n$.

By $\gamma_{m(i)}$ we denote an edge of Ω_i called mortar and by $\delta_{m(j)}$ an edge of Ω_i that geometrically occupies the same place called nonmortar, then $\gamma_{m(i)=\delta_{m(j)}=\gamma_m}$. Since γ_m inherits two different triangulations, by $T_l(\gamma_{m(i)})$ and $T_l(\delta_{m(j)})$ denote the different triangulations across γ_m (Assume the fine side is chosen as mortar). Define $S_l(\delta_{m(j)})$ to be a subspace of $L^2(\gamma_m)$, such that its functions are piecewise constants on $T_l(\delta_{m(j)})$. The dimension of $S_l(\delta_{m(j)})$ is equal to the number of elements on the $\delta_{m(j)}$. For each nonmortar edge $\delta_{m(j)}$, define an L^2-projection operator $Q_{l,\delta} : L^2(\gamma_m) \to S_l(\delta_{m(j)})$ by

$$(Q_{l,\delta}v, \psi)_{L^2(\delta_{m(j)})} = (v, \psi)_{L^2(\delta_{m(j)})},\ \forall\psi \in S_l(\delta_{m(j)}) \tag{1}$$

The purpose of this paper is to study the multigrid method for mortar element for the rotated Q_1 element. An intergrid transfer operator is presented for nonnested mortar element spaces. On the basis of this operator, we give a multigrid algorithm. Using the theory developed by Bramble, Pasciak, Xu, we prove the W-cycle multigrid is optimal, i.e., the convergence rate is independent of mesh size and mesh level. Furthermore, a variable V-cycle multigrid preconditioner is developed, which results in a preconditioned system with uniformly bounded condition number.

The remainder of this paper is organized as follows. In section two we introduce Multigrid algorithm. Section three presents some lemmas. Last section gives our results.

2. Multigrid algorithm

We must define a suitable intergrid transfer operator for nonnested mesh space V_l. First introduce a local intergrid operator J_l^i from $X_{l-1}(\Omega_i)$ to $X_l(\Omega_i)$ by

$$\frac{1}{|e|}\int_e J_l^i v ds \begin{cases} 0 & e \subset \partial\Omega_i \cap \partial\Omega \\ \frac{1}{|e|}\int_e v ds & e \subset \partial\Omega_i \setminus \partial\Omega \\ \frac{1}{|e|}\int_e v ds & e \not\subset \partial E \qquad E \in T_{l-1}^i \\ \frac{1}{2|e|}\int_e (v\,|_{E_1} + v\,|_{E_2}) ds & e \subset \partial E_1 \cap \partial E_2 \quad E_1,\ E_2 \in T_{l-1}^i \end{cases}$$

Where $e \in \partial E,\ E \in T_l^i$.

Based on the operator J_l^i, a global intergrid transfer operator $J_l : X_{l-1}(\Omega) \to X_l(\Omega)$ introduced as follows.

$$J_l v = (J_l^1 v,\ J_l^2 v^2, \cdots, J_l^N v^N), \quad \forall v = (v^1,\ v^2, \cdots, v^N) \in X_{l-1}(\Omega)$$

To construct an intergrid operator in mortar element spaces we define an operator $\varepsilon_{l,\delta_{m(j)}}$:

$$X_l(\Omega) \to X_l(\Omega) \quad by \quad \int_e \varepsilon_{l,\delta_{m(j)}}(v) ds = \begin{cases} \int_e Q_{l,\delta}\left(I_l^\gamma Q_{l,\gamma} v\,|_{\gamma_{m(i)}} - v\,|_{\delta_{m(j)}}\right) ds & e \in T_l(\delta_{m(j)}) \\ 0 & otherwise \end{cases}$$

Then for any $v \in X_l(\Omega)$, let

$$v^* = v + \sum_{m=1}^{M} \varepsilon_{l,\delta_{m(j)}}(v) \tag{2}$$

It is easy to check that $v^* \in V_l$, since for any $\psi \in S_l(\delta_{m(j)})$, we can derive

$$\begin{aligned} \int_{\delta_{l,m(j)}} v^*|_{\delta_{m(j)}}\psi ds &= \int_{\delta_{m(j)}} v|_{\delta_{m(j)}}\psi ds + \int_{\delta_{m(j)}} \varepsilon_{l,\delta_{m(j)}}(v)|_{\delta_{m(j)}}\psi ds \\ &= \int_{\delta_{m(j)}} v|_{\delta_{m(j)}}\psi ds + \int_{\delta_{m(j)}} Q_{l,\delta}(I_l^\gamma Q_{l,\gamma} v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}})\psi ds \\ &= \int_{\delta_{m(j)}} v\,|_{\delta_{m(j)}}\ \psi ds + \int_{\delta_{m(j)}} (I_l^\gamma Q_{l,\gamma} v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}})\psi ds \\ &= \int_{\delta_{m(j)}} I_l^\gamma Q_{l,\gamma} v|_{\gamma_{m(i)}}\psi ds \\ &= \int_{\delta_{m(j)}} I_l^\gamma Q_{l,\gamma} v^*|_{\gamma_{m(i)}}\psi ds \end{aligned}$$

After above preparation, we can construct an intergrid transfer operator I_l in mortar element spaces.

$$I_l : X_{l-1}(\Omega) \to V_l \quad \text{by} \quad I_l v = J_l v + \sum_{m=1}^{M} \varepsilon_{l,\delta_{m(j)}}(J_l v),\ \forall v \in X_{l-1}(\Omega) \tag{3}$$

To present our multigrid algorithm, we describe some auxiliary operators. For $l = 1, 2, \cdots, L$, define $A_l : V_l \to V_l$, $P_{l-1} : V_l \to V_{l-1}$, and $P^0_{l-1} : V_l \to V_{l-1}$ respectively by $(A_l u, v) = \alpha_l(u, v)$, $\forall u, v \in V_l$, $(P^0_{l-1}u, v) = (u, I_l v)$, $\forall u \in V_l$, $v \in V_{l-1}$, $a_{l-1}(P_{l-1}u, v) = a_l(u, I_l v)$, $\forall u \in V_l$, $v \in V_{l-1}$,

Furthermore we must find smoothing operator R_l, including Gauss-Seidel, conjugate gradient iterations and so on, which satisfy the following condition.

(R). There exists a constant $C_R \geq 1$ independent of l such that

$$\frac{\|u\|_0^2}{\lambda_1} \leq C_R(\overline{R}_l u, v),\ \forall u \in V_l \tag{4}$$

For both $\overline{R}_l = (I - K_l^* K_l)A_l^{-1}$ or $\overline{R}_l = (I - K_l K_l^*)A_l^{-1}$, where $K_l = I - R_l A_l$, $K_l^* = I - R_l^T A_l$, R_l^T is the adjoint of R_l with respect to $(\cdot, \cdot)$ and λ_l is the maximum eigenvalue of A_l.

$$\text{Define } R_l^{(k)} = \begin{cases} R_l & k \text{ is odd} \\ R_l^T & k \text{ is even} \end{cases}$$

A general multigrid operator $B_l : V_l \to V_l$ can be defined recursively as follows.

Multigrid Algorithm. Set $B_1 = A_1^{-1}$. Let $2 \leq l \leq L$ and p be a positive integer, assume that B_{l-1} has been defined and define B_{1g} for $g \in V_l$ by

(1) Set initial value X^0 and let $q^0 = 0$.

(2) Define x^k for $k = 1, 2, \cdots, m(l)$ by $x^k = x^{k-1} + R_l^{(k+m(l))}(g - A_l x^{k-1})$.

(3) Define $y^{m(l)} = x^{m(l)} + I_l q^p y$, where q^i for $i = 1, \cdots, p$ are determined by $q^i = q^{i-1} + B_{l-1}(P^0_{l-1}(g - A_l x^{m(l)}) - A_{l-1}q^{i-1})$

(4) Define y^k for $k = m(l) + 1, \cdots, 2m(l)$ by $y^k = y^{k-1} + R_l^{(k+m(l))(g - A_l y^{k-1})}$.

(5) Set $B_{1g} = y^{2m(l)}$.

Remark. In the Multigrid Algorithm, $m(l)$ gives the number of presmoothing and postsmoothing steps, it can vary as a function of l. If $p = 1$, we have a V-cycle method, and $p = 2$ denotes a W-cycle method. A variable V-cycle algorithm is one in which the number of smoothing $m(l)$ increase exponentially as l decreases, i.e., the number of smoothing $m(l)$ satisfies $\beta_0 m(l) \leq m(l-1) \leq \beta_1 m(l)$, with $1 < \beta_0 < \beta_1$.

3. Some lemmas

To reach our conclusion, we present some auxiliary technical lemmas and prove an approximation assumption.

Define an operator $M_{l,i} : X_l(\Omega_i) \to V_l^{\frac{1}{2}}(\Omega_i)$ as follows.

Definition 1. Given $v \in X_l(\Omega_i)$, let $M_{l,i}v \in V_l^{\frac{1}{2}}(\Omega_i)$ by the values of $M_{l,i}v$ at the vertices of the partition $T_l^{\frac{1}{2}}(\Omega_i)$.

(1) If P is a central point of E, $E \in T_l(\Omega_i)$, then $(M_{l,i}v)(P) = \frac{1}{4} \sum_{e_i \in \partial E} \frac{1}{|e_i|} \int_{e_i} v ds$.

(2) If P is a midpoint of one edge $e \in \partial E$, $E \in T_l(\Omega_i)$, then $(M_{l,i}v)(P) = \frac{1}{|e_i|} \int_e v ds$.

(3) If $P \in \Omega_{i,l} \setminus \partial\Omega_{i,l}$, then $(M_{l,i}v)(P) = \frac{1}{4} \sum_{e_i} \frac{1}{|e_i|} \int_{e_i} v ds$. Where the sum is taken over all edges e_i with the common vertex P, $e_i \in \partial E_i$, $E_i \in T_l(\Omega_i)$.

(4) If $P \in \partial\Omega_{i,l} \setminus \{c_1, \cdots, c_n\}$, then $(M_{l,i}v)(P) = \frac{1}{2}(\frac{1}{|e_l|} \int_{e_l} v ds + \frac{1}{|e_\gamma|} \int_{e_\gamma} v ds)$, where $e_l \in \partial E_1 \cap \partial\Omega_i$ and $e_\gamma \in \partial E_2 \cap \partial\Omega_i$ are the left and right neighbor edges of P, $E_1, E_2 \in T_l(\Omega_i)$, $c_1, \cdots, c_n$ are the vertices of subdomain Ω_i.

(5) If $P \in \{c_1, \cdots, c_n\}$, then

$$(M_{l,i}v)(P) = \frac{|e_l|}{|e_l| + |e_\gamma|}(\frac{1}{|e_l|}\int_{e_l} vds) + \frac{|e_\gamma|}{|e_l| + |e_\gamma|}(\frac{1}{|e_\gamma|}\int_{e_\gamma} vds)$$

For the above operator $M_{l,j}$, we have the following result.

Lemma 1. For any $v \in X_l(\Omega_i)$, we have $|M_{l,i}v|_{H^1(\Omega_i)} \approx \|v\|_{l,i}$.

Lemma 2. $\|v - Q_{l,\delta}v\|_{L^2(\gamma m)} \le h_l^{\frac{1}{2}}|v|_{H^{\frac{1}{2}}(\gamma m)}\ \forall v \in H^{\frac{1}{2}}(\gamma m)$.

Lemma 3. For any $v \in X_l(\Omega_i)$, then $\|Q_{l,\delta}I_l^\gamma Q_{l,\gamma}v|_{\gamma m(i)} - I_l^\gamma Q_{l,\gamma}v|_{\gamma m(i)}\|_{L^2(\gamma m(i))} \le h_l^{\frac{1}{2}}\|v\|_{l,i}$.

$\|I_l^r Q_{l,r}v|_{\gamma m(i)} - Q_{l,r}v|_{\gamma m(i)}\|_{L^2(\gamma m(i))} \le h_l^{\frac{1}{2}}\|v\|_{l,i}$

Lemma 4. For any $v^i \in V_{l-1}(\Omega_i)$, we have $\|J_l^i v^i\|_{l,i} \le \|v^i\|_{l-1,i}$, $\|v^i - J_l^i v^i\|_{0,i} \le h_l\|v^i\|_{l-1,i}$.

Lemma 5. For any $v \in V_{l-1}$, it holds that $\|I_l v\|_l \le \|v\|_{l-1}$, $\|v - I_l v\|_0 \le h_l\|v\|_{l-1}$.

Lemma 6. For the operator Π_l, we have $\|\xi - \Pi_l\xi\|_0 + h_l\|\xi - \Pi_l\xi\|_l \le h_l^2|\xi|_2$, $\forall \xi \in H_0^1(\Omega) \cap H^2(\Omega)$.

Lemma 7. For any $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$, we have $\|\xi - I_L\Pi_{l-1}\xi\|_l \le h_l|\xi|_2$.

The proofs of the above all lemmas can be found in relevant references. Let's come to see the last two lemmas.

Lemma 8. The operator P_{l-1} has following property $\|v - P_{l-1}v\|_0 \le h_l\|v\|_l$, $\forall v \in V_l$.

Proof. Consider the auxiliary problem as follows

$$\begin{cases} -\triangle\xi = v - P_{l-1}v & in\ \Omega \\ \xi = 0 & on\ \partial\Omega \end{cases}$$

then $\|v - P_{l-1}v\|_0^2 = (-\triangle\xi, v - P_{l-1}v) = (\alpha_l(\xi, v) - \alpha_{l-1}(\xi, P_{l-1}v)) - \sum\limits_{K\in T_l}\oint\limits_{\partial K}\frac{\partial\xi}{\partial n}vds + \sum\limits_{K\in T_{l-1}}\oint\limits_{\partial K}\frac{\partial\xi}{\partial n}P_{l-1}vds$
$:= F_1 + F_2 + F_3$

Lemma 4 and Lemma 2 reveal $|F_2| \le h_l|\xi|_2\|v\|_l = h_l\|v - P_{l-1}v\|_0\|v\|_l$. Using Lemma 5, we can see $\|P_{l-1}v\|_{l-1}^2 = \alpha_{l-1}(P_{l-1}v, P_{l-1}v) = \alpha_l(v, I_lP_{l-1}v) \le \|v\|_l\|P_{l-1}v\|_l$. So $\|P_{l-1}v\|_{l-1} \le \|v\|_l$.

By Lemma 2 and above inequality, we have $|F_3| \le h_l|\xi|_2\|P_{l-1}v\|_{l-1} = h_l\|v - P_{l-1}v\|_0\|v\|_l$. Now we estimate F_1.

$$\begin{aligned} |F_1| &= |\alpha_l(\xi, v) - \alpha_{l-1}(\Pi_{l-1}\xi, P_{l-1}v) + \alpha_{l-1}(\Pi_{l-1}\xi, P_{l-1}v) - \alpha_{l-1}(\xi, P_{l-1}v)| \\ &\le |\alpha_l(\xi - I_l\Pi_{l-1}\xi, v)| + |\alpha_{l-1}(\xi - \Pi_{l-1}\xi, P_{l-1}v)| \\ &\le h_l|\xi|_2(\|v\|_l + \|P_{l-1}v\|_{l-1}) \le h_l|\xi|_2\|v\|_l \le |v - P_{l-1}v\|_0\|v\|_l \end{aligned}$$

All the above inequalities give the proof. Now, the approximation assumption theory is given as follows.

Lemma 9. $|\alpha_l((I - I_lP_{l-1})v, v)| \le \left(\frac{\|A_lv\|_0^2}{\gamma_l}\right)^{\frac{1}{2}}\alpha_l(v, v)^{\frac{1}{2}}$, $\forall v \in V_l$. Proof. By triangular inequality, Lemma 5 and Lemma 8, we derive $\|v - I_lP_{l-1}v\|_0 \le \|v - P_{l-1}v\|_0 + \|(I - I_l)P_{l-1}v\|_0 \le h_l(\|v\|_l + \|P_{l-1}v\|_{l-1}) \le h_l\|v\|_l$ On the other hand

$$\begin{aligned} \|v - I_lP_{l-1}v\|_l &= \sup_{\omega\in V_l, \|\omega\|_{l=1}} \alpha_l(v - I_lP_{l-1}v, \omega) \\ &= \sup_{\omega\in V_l, \|\omega\|_{l=1}} \alpha_l(v, \omega - I_lP_{l-1}\omega) \\ &\le \sup_{\omega\in V_l, \|\omega\|_{l=1}} \|A_lv\|_0\|\omega - I_lP_{l-1}\omega\|_0 \\ &\le h_l\|A_lv\|_0 \end{aligned}$$

Then, we can obtain

$$|\alpha_l((I - I_lP_{l-1})v, v)| \le \|(I - I_lP_{l-1})v\|_l\|v\|_l \le h_l\|A_lv\|_0\|v\|_l \le \left(\frac{\|A_lv\|_0^2}{\lambda_l}\right)^{\frac{1}{2}}\alpha_l(v, v)^{\frac{1}{2}}.$$

4. Main result

We now state the convergence results for the multigrid algorithm. The convergence rate for the multigrid algorithm on the l th level is measured by a convergence factor

$$\delta_l \text{ satisfying } |\alpha_l((I - B_l A_l)v, v)| \leq \delta_l \alpha_l(v, v), \ \forall v \in V_l \tag{5}$$

Following the above analysis, we propose two propositions:

Proposition 1. (W-cycle). Under Lemma 9, if $p = 2$ and $m(l) = m$ is large enough, then the convergence factor in (5) is $\delta_l = \frac{C}{C+m^{\frac{1}{2}}}$

Proposition 2. (variable V-cycle preconditioner) Under Lemma 9, and the number of smoothing $m(l)$ increases as decreases in such a way that $\beta_0 m(l) \leq m(l-1) \leq m(l)$, hold with $1 \leq \beta_0 \leq \beta_1$. then there exists $M > 0$ independent of L such that $C_0^{-1}\alpha_l(v, v) \leq \alpha_l(B_l A_l v, v) \leq C_0 \alpha_l(v, v)$, $\forall v \in V_l$, with $C_0 = \frac{M+m(l)^{\frac{1}{2}}}{m(l)^{\frac{1}{2}}}$.

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