Some Inequalities and Asymptotics for a Weighted Alternate Binomial Sum

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Abstract

We establish strict inequality bounds for the binomial sums $\sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{2i+1}$ and prove the asymptotic result:

$$\sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{2i+1} \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2n+1}}, \text{ as } n \to \infty.$$

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1. Introduction

We are interested in studying the sums $\sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{1}{2i+1}$ for all $n \in \mathbb{N}$.

The sums are shown to be positive and strictly decreasing with n, having limit zero as n tends to infinity. The main results are bounds for these sums as well as their asymptotic behaviour.

2. Main Results

Denote $a_n = \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{2i+1}$. We use the notations $\exp(x) = e^x$ and $x^\wedge y = x^y$ in order to make some formulas easier to be read.

Proposition 1 *For all* $n \in \mathbb{N}$, $\sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{2i+1} > 0$.

Proof. $\forall n \in \mathbb{N}, 0 < \int_0^{\pi/2} \cos^{2n+1} x \, dx = \int_0^1 (1 - u^2)^n du = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{u^{2i+1}}{2i+1} \Big|_0^1$.

Proposition 2 $\forall n \in \mathbb{N}, a_n > a_{n+1}$.

Proof. $\forall n \in \mathbb{N}, a_n = \int_0^{\pi/2} \cos^{2n+1} x \, dx > \int_0^{\pi/2} \cos^{2n+1} x \cos^2 x \, dx = a_{n+1}$.

Proposition 3 $\lim_{n \to \infty} a_n = 0$.

Proof. $\forall m > 1, \exists n \in \mathbb{N}, \left(\cos\left(\frac{1}{m}\right) \right)^{2n+1} < \frac{1}{m}$,

$$a_n = \int_0^{1/m} \cos^{2n+1} x \, dx + \int_{1/m}^{\pi/2} \cos^{2n+1} x \, dx < \frac{1}{m} + \frac{1}{m}\left(\frac{\pi}{2} - \frac{1}{m}\right) < \frac{\pi}{m}$$

and the result follows from Propositions 1 and 2.

Theorem 1 *Let* $\mu \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *such that* $0 < \mu < \sqrt{(2n+1)\ln 2}$. *Then the following inequality*

$$\frac{1}{\sqrt{2n+1}} \frac{1}{\sqrt{\ln 2}} \int_0^\mu e^{-v^2} dv + \frac{1}{2n+2}\left[\left(\frac{\pi}{2} - z\right)\left(1 - \frac{1}{2}z^2\right)^{2n+1} \right]$$

$$< \sum_{i=0}^{n} \binom{n}{i}(-1)^i \frac{1}{2i+1} < \sqrt{\frac{6\pi}{11}} \frac{1}{\sqrt{2n+1}} + \frac{(\pi/2 - 1)^{2n+2}}{2n+2},$$

is valid, where $z = \mu/\sqrt{(2n+1)\ln 2}$.

Proof. From Taylor's formula

$$\cos x = 1 - \frac{x^2}{2!} + (\cos p)\frac{x^4}{4!} \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

for some $p \in [0, x]$ and

$$\cos x \le 1 - \frac{x^2}{2!} + \frac{x^2}{4!} = 1 - \frac{11}{24}x^2$$

for all $x \in [0, 1]$ with strict inequality on $(0,1)$.

Remember that $\ln(1 - u) = -\sum_{i=1}^{\infty} \frac{u^i}{i}$ for all $u \in [0, 1)$, so that $\forall x \in (0, 1)$

$$\cos^{2n+1} x < \exp\left((2n + 1)\ln\left(1 - \frac{11}{24}x^2\right)\right) < \exp\left(-(2n + 1)\frac{11}{24}x^2\right).$$

Summing the following inequalities

$$\int_0^1 \cos^{2n+1} x\,dx < \int_0^1 e^{-(2n+1)\frac{11}{24}x^2}\,dx < \sqrt{\frac{24}{11(2n + 1)}} \int_0^{\infty} e^{-v^2}\,dv = \sqrt{\frac{6\pi}{11(2n + 1)}}$$

$$\int_1^{\pi/2} \cos^{2n+1} x\,dx < \int_1^{\pi/2} \left(\frac{\pi}{2} - x\right)^{2n+1} dx = \frac{\left(\frac{\pi}{2} - 1\right)^{2n+2}}{2n + 2},$$

we establish the upper bound. We have used the famous formula $\int_0^{\infty} e^{-v^2}\,dv = \frac{\sqrt{\pi}}{2}$ to obtain the first inequality. See Nicholas and Yates (1950).

Again, from Taylor's formula, for all $x \in (0, \pi/2)$, $\cos x > 1 - \frac{1}{2}x^2$ and we have for all $x \in (0, 1)$

$$\cos^{2n+1} x \quad > \quad \exp\left((2n + 1)\ln\left(1 - \frac{1}{2}x^2\right)\right) = \exp\left(-(2n + 1)\sum_{i=1}^{\infty} \frac{(\frac{1}{2}x^2)^i}{i}\right)$$

$$> \quad \exp\left(-(2n + 1)\left(\sum_{i=1}^{\infty}\left(\frac{(\frac{1}{2})^i}{i}\right)\right)x^2\right) = \exp\left(-(2n + 1)(\ln 2)x^2\right).$$

Since $z < 1$ we write

$$a_n \quad > \quad \int_0^z \exp\left(-(2n + 1)(\ln 2)x^2\right)dx + \int_z^{\pi/2}\left(\left(1 - \frac{1}{2}z^2\right) + \frac{-\left(1 - \frac{1}{2}z^2\right)}{\left(\frac{\pi}{2} - z\right)}(x - z)\right)^{2n+1} dx$$

$$= \quad \frac{1}{\sqrt{(2n + 1)\ln 2}} \int_0^{\mu} e^{-v^2}\,dv + \frac{1}{2n + 2}\frac{\left(\frac{\pi}{2} - z\right)}{\left(1 - \frac{1}{2}z^2\right)}\left(1 - \frac{1}{2}z^2\right)^{2n+2}$$

and the proof is complete.

For sequences x_n and y_n we write $x_n \sim y_n$ when $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$.

Denote ℓ_n and u_n the lower and upper bounds given in Theorem 1. Now, letting $n \to \infty$ we have their asymptotic behaviour:

Clearly $u_n \sim \sqrt{\frac{6\pi}{11}}\frac{1}{\sqrt{2n+1}}$ and, choosing for example $\mu = \sqrt{(2n + 1)\ln 2} - 1/n$, we have

$$\ell_n \quad = \quad \frac{1}{\sqrt{2n + 1}}\frac{1}{\sqrt{\ln 2}} \int_1^{\mu} e^{-v^2}\,dv + \frac{1}{2n + 2}\left[\left(\frac{\pi}{2} - z\right)\left(1 - \frac{1}{2}z^2\right)^{2n+1}\right]$$

$$\sim \quad \frac{1}{\sqrt{2n + 1}}\frac{1}{\sqrt{\ln 2}} \int_0^{\infty} e^{-v^2}\,dv = \frac{1}{2}\sqrt{\frac{\pi}{\ln 2}}\frac{1}{\sqrt{2n + 1}}.$$

Define b_n by $a_n = b_n\frac{1}{\sqrt{2n+1}}$. We have shown that

$$\frac{1}{2}\sqrt{\frac{\pi}{\ln 2}} \le \liminf b_n \le \limsup b_n \le \sqrt{\frac{6\pi}{11}}.$$

We can improve these bounds as it is shown by the following theorems and propositions:

Theorem 2 $\forall n \in \mathbb{N}, \forall m > 1$,

$$a_n \quad < \quad \sqrt{\frac{2}{1-(1/12m^2)}}\frac{1}{\sqrt{2n+1}}\int_0^{\frac{1}{m}\sqrt{\frac{2n+1}{2}\left(1-\frac{1}{12m^2}\right)}}e^{-v^2}dv + \frac{\pi}{2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}$$

$$< \quad \sqrt{\frac{\pi}{2-(1/6m^2)}}\frac{1}{\sqrt{2n+1}} + \frac{\pi}{2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}.$$

Proof. For $0 < x < 1/m, m > 1$,

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^2}{4!}\left(\frac{1}{m}\right)^2 = 1 - \frac{1}{2}\left(1 - \frac{1}{12m^2}\right)x^2,$$

$$a_n \quad < \quad \int_0^{1/m}\cos^{2n+1}xdx + \int_{1/m}^{\pi/2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}dx$$

$$< \quad \int_0^{1/m}\exp\left(-(2n+1)\frac{1}{2}\left(1-\frac{1}{12m^2}\right)x^2\right)dx + \frac{\pi}{2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}$$

$$= \quad \sqrt{\frac{2}{(2n+1)(1-1/12m^2)}}\int_0^{\frac{1}{m}\sqrt{\frac{(2n+1)}{2}\left(1-\frac{1}{12m^2}\right)}}e^{-v^2}dv + \frac{\pi}{2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}$$

and the result follows from $\int_0^\infty e^{-v^2}dv = \frac{\sqrt{\pi}}{2}$.

Proposition 4 $\limsup b_n \leq \sqrt{\frac{\pi}{2}}$.

Proof. Letting $m = \sqrt[3]{2n+1}$ and $c_n = \frac{\pi}{2}\left(\cos\left(\frac{1}{m}\right)\right)^{2n+1}(2n+1)$ we have

$$\lim_{n\to\infty}\frac{1}{m}\sqrt{\frac{2n+1}{2}\left(1-\frac{1}{12m^2}\right)} = +\infty$$

and

$$\lim_{n\to\infty}c_n = \lim_{y\to\infty}y^3\left(\cos\left(\frac{1}{y}\right)\right)^{y^3} = 0,$$

since, for sufficiently large y,

$$\cos\left(\frac{1}{y}\right) > 0, \quad \left(\sin\left(\frac{1}{2y}\right)\right)/(1/2y) > \frac{1}{2} \quad \text{and} \quad \left(1 - 2\sin^2\left(\frac{1}{2y}\right)\right)^\wedge\left(-\frac{1}{2\sin^2\left(\frac{1}{2y}\right)}\right) > 2$$

so that

$$0 < y^3\left(\cos\left(\frac{1}{y}\right)\right)^{y^3} = y^3\left(\left(1-2\sin^2\left(\frac{1}{2y}\right)\right)^\wedge\left(-\frac{1}{2\sin^2\left(\frac{1}{2y}\right)}\right)\right)^\wedge\left(-y^3 2\sin^2\left(\frac{1}{2y}\right)\right) < y^3 2^{-y.2.\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2} \to 0$$

as $y \to \infty$.

Now, from Theorem 2,

$$a_n < \sqrt{\frac{2}{\left(1-\frac{1}{12m^2}\right)}}\frac{1}{\sqrt{2n+1}}\frac{\sqrt{\pi}}{2} + c_n\frac{1}{2n+1},$$

from which

$$b_n < \sqrt{\frac{1}{\left(1-\frac{1}{12m^2}\right)}}\sqrt{\frac{\pi}{2}} + c_n\frac{1}{\sqrt{2n+1}} \to \sqrt{\frac{\pi}{2}} \quad \text{as} \quad n \to \infty.$$

Note that the upper bounds given in Theorem 2 or in the proof of Proposition 4 may not improve the approximation of a_n, relative to that of Theorem 1, for small n.

Theorem 3 *Let $M \in \mathbb{R}$, $M > 1$, $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $0 < \mu < \frac{1}{M}\sqrt{(2n+1)\left\{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)\right\}}$. Then we have*

$$\frac{1}{\sqrt{2n+1}\sqrt{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)}} \int_0^\mu e^{-v^2}\mathrm{d}v + \frac{1}{2n+2}\left[\left(\frac{\pi}{2} - z\right)\left(1 - \frac{1}{2}z^2\right)^{2n+1}\right] < a_n,$$

where $z = \mu / \left(\frac{1}{M}\sqrt{(2n+1)\left\{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)\right\}}\right)$.

Proof. Observe that for all x, $0 < x < \frac{1}{M}$,

$$\cos^{2n+1} x > \exp\left(-(2n+1)\sum_{i=1}^\infty \frac{\left(\frac{1}{2}x^2\right)^i}{i}\right) > \exp\left(-(2n+1)\left\{\frac{1}{2}x^2 + \frac{1}{M^2}\sum_{i=2}^\infty \frac{\left(\frac{1}{2}\right)^i}{i}x^{2i-2}\right\}\right)$$

$$> \exp\left(-(2n+1)\left\{\frac{1}{2} + \frac{1}{M^2}\left(\sum_{i=2}^\infty \frac{(1/2)^i}{i}\right)\right\}x^2\right) = \exp\left(-(2n+1)\left\{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)\right\}x^2\right).$$

Noting that $0 < z < 1$, follow the steps at the end of theorem's 1 proof.

Proposition 5 $\liminf b_n \geq \sqrt{\frac{\pi}{2}}$.

Proof. Let $M = \sqrt[4]{2n+1}$ and $\mu = \frac{1}{M}\sqrt{(2n+1)\left\{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)\right\}} - \frac{1}{n}$. Then $\mu \sim \sqrt{\frac{1}{2}}\sqrt[4]{2n+1} \to \infty$ as $n \to \infty$. Now, from Theorem 3,

$$\liminf b_n \geq \lim_{n\to\infty} \sqrt{2n+1}\left(\frac{1}{\sqrt{2n+1}\sqrt{\left\{\frac{1}{2} + \frac{1}{M^2}\left(\ln 2 - \frac{1}{2}\right)\right\}}} \int_0^\mu e^{-v^2}\mathrm{d}v + \frac{1}{2n+2}\left[\left(\frac{\pi}{2} - z\right)\left(1 - \frac{1}{2}z^2\right)^{2n+1}\right]\right)$$

$$= \sqrt{2}\int_0^\infty e^{-v^2}\mathrm{d}v = \sqrt{\frac{\pi}{2}}.$$

Propositions 4 and 5 tell us that there exists the limit $\lim_{n\to\infty} b_n$ and that it is equal to $\sqrt{\pi/2}$.

Theorem 4 (asymptotic behaviour)

$$\sum_{i=1}^n \binom{n}{i}\frac{(-1)^i}{2i+1} = \int_0^{\pi/2} \cos^{2n+1} x\,\mathrm{d}x \sim \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2n+1}} \quad as \ n \to \infty.$$

Proof. Propositions 4 and 5.

3. Final Remarks

We observe that the bounds can be improved by dividing the interval $(0, \pi/2)$ in more than two pieces as we have done here and the most direct approach seems to be piecewise linearly approximating the cosine function both from above and bellow except at a neighborhood of the origin where it is approximated by Taylor's formula. Although this procedure will improve the bounds for small n, of course it will not change the asymptotic result. We remark that from the bounds for a_n obtained in this work we can derive bounds for Catalan's numbers and from the asymptotic result we can derive Wallis product formula. See (Bustoz & Ismail, 1986; Everett, 1970; Paule & Pillwein, 2010) and references therein.

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