Certain Flowers of Continued Fractions
In the Garden of Generalized Lambert Series

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Abstract
In this paper, an attempt has been made to establish certain results involving Lambert series and Continued fractions.

Keywords: Lambert series, Continued fractions, Rogers-Ramanujan identity

1. Introduction

2. Notations and Definitions
For $a$ and $q$ real or complex and $|q| < 1$, we shall have

$$(a; q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n = 1, 2, 3, \ldots \end{cases}$$ (1)

We also define

$$(a)_\infty = (a; q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \text{ for } |q| < 1.$$ (2)

The infinite product diverges when $a \neq 0$.

Also

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n(a_2; q)_n\ldots(a_r; q)_n.$$ (3)
\[ \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{|q| q^n} = \frac{[q; q^3]_n [q^4; q^3]_n}{[q^2; q^3]_n [q^3; q^3]_n}, \quad |q| < 1 \]  

An expression of the form \( \frac{a_n}{b_n} \) is said to be a continued fraction. The values of \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) can be either real or complex values. A finite simple continued fraction is a simple continued fraction with only a finite number of terms. An infinite simple continued fraction is a simple continued fraction with an infinite number of terms.

The series, \( \sum_{n=1}^{\infty} a_n x^n \), considered by Lambert in connection with the convergence of power series is called Lambert series. If the series \( \sum_{n=1}^{\infty} a_n x^n \) converges then Lambert series \( \sum_{n=1}^{\infty} a_n x^n \) converges for all values of \( x \) except for \( x = \pm 1 \), otherwise it converges for those values of \( x \), for which the series \( \sum_{n=1}^{\infty} a_n x^n \) converges. A series of the form \( \sum_{n=-\infty}^{\infty} (-1)^n q^{-n^2} R(q^n) \), where \( \epsilon = 0 \) or \( 1 \), \( \lambda > 0 \), and \( R(q^n) \) is rational function of \( q^n \) is called generalized Lambert series.

We use following continued fractions for analysis in the present paper.

Rogers - Ramanujan continued fraction, \( c(q) \) is given by

\[ \frac{1}{c(q)} = \frac{1}{1 + \frac{q + q^2 + q^3 + q^4 + \ldots}{1 + \frac{q + q^2 + q^3 + q^4 + \ldots}{1 + \frac{q + q^2 + q^3 + q^4 + \ldots}{1 + \ldots}}} = \sum_{n=0}^{\infty} \frac{q^n}{|q| q^n} = \frac{[q; q^3]_n [q^4; q^3]_n}{[q^2; q^3]_n [q^3; q^3]_n}, \quad |q| < 1 \]  

\[ \frac{[q; q^3]_n [q^4; q^3]_n}{[q^2; q^3]_n [q^3; q^3]_n} = \frac{1 + q + q^2 + q^3 + q^4}{1 + 1 + 1 + 1 + \ldots}, \quad (Andrews, G. E. & Berndt, B. C., 2005, 6.2.37, p. 154). \]  

\[ \frac{[q; q^3]_n [q^4; q^3]_n}{[q^2; q^3]_n [q^3; q^3]_n} = \frac{1 + q + q^2 + q^3 + q^4}{1 + q + q^2 + q^3 + q^4 + \ldots}, \quad (Andrews, G. E. & Berndt, B. C., 2005, 6.2.38, p. 154). \]  

\[ \frac{[q; q^3]_n [q^4; q^3]_n}{[q^2; q^3]_n [q^3; q^3]_n} = \frac{1 + q + q^2 + q^3 + q^4}{1 + 1 + 1 + 1 + 1 + \ldots}, \quad (Andrews, G. E. & Berndt, B. C., 2005, 6.2.22, p. 150). \]  

The first remarkable result of Rogers - Ramanujan continued fraction involving Lambert series is given by

\[ \frac{1}{c^3(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^n} = \sum_{n=0}^{\infty} \frac{q^{5n+1} + q^{5n+2} + q^{5n+3}}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{5n+2} + q^{5n+3}}{1 - q^{5n+3}} \]  

\[ \frac{1}{c^3(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^n} = \sum_{n=0}^{\infty} \frac{q^{5n+1} + q^{5n+2} + q^{5n+3}}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{5n+1} + q^{5n+2} + q^{5n+3}}{1 - q^{5n+3}} \]  

Here $C(q)$ is called Ramanujan continued fraction with its value equal to $G(q)/H(q)$, and $G(q), H(q)$ are called Rogers - Ramanujan identities, which are as follows

\[
G(q) = \sum_{n=0}^{\infty} \frac{q^n}{[q; q]_n} = \frac{1}{[q; q^3]_\infty[q^4; q^3]_\infty},
\]

(12)

\[
H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{[q; q]_n} = \frac{1}{[q^2; q^3]_\infty[q^3; q^3]_\infty},
\]

(13)


We also use the following identities for establishing our main results.

The Rogers-Fine identity is as follows

\[
\sum_{n=0}^{\infty} \frac{[\alpha; q]_n \beta^n}{[\beta; q]_n} = \sum_{n=0}^{\infty} \frac{[\alpha; q\beta]_n [\alpha q; q\beta]_n [\beta q; q\beta]_n [\alpha q^2; q]_\infty [\beta q^2; q]_\infty}{[\beta; q]_n [\alpha q; q]_n+1} (1-\alpha q^{2n}).
\]

(14)


\[
\sum_{n=0}^{\infty} \frac{z^n}{(1-\alpha q^n)} = \sum_{n=0}^{\infty} \frac{(\alpha z^n)q^n (1-\alpha q^{2n})}{(1-\alpha q^n) (1-zq^n)}.
\]

(15)

\[
\sum_{n=0}^{\infty} \frac{q^{nj}}{(1-q^{kn+i+j})} = \sum_{n=0}^{\infty} \frac{q^{kn+2n}q^{nj}(1-q^{kn+i+j})}{(1-q^{kn+i+j}) (1-q^{kn+i+j})}.
\]

(16)

\[
\sum_{n=0}^{\infty} \frac{q^{mi}}{(1-q^{kn+i})} = \sum_{n=0}^{\infty} \frac{q^{kn+2m}(1+q^{kn+i})}{(1-q^{kn+i})},
\]

(17)


\[
\sum_{n=-\infty}^{\infty} \frac{q^{nj}}{(1-q^{kn+i})} = \frac{[q^k; q^k]_\infty [q^{k+j}; q^k]_\infty [q^{k-j}; q^k]_\infty}{[q^j; q^j]_\infty [q^i; q^i]_\infty [q^{i-j}; q^i]_\infty [q^{i+j}; q^i]_\infty}.
\]

(18)


\[
\sum_{n=0}^{\infty} \frac{(n+1)[q/\alpha; q]_n \alpha^n}{[\beta; q]_n+1} + \sum_{n=0}^{\infty} \frac{n[q/\beta; q]_n \beta^n}{[\alpha; q]_n+1} = \frac{[q; q]_\infty [\alpha q; q]_\infty}{[\alpha q]_\infty [\beta; q]_\infty}.
\]

(19)


\[
\sum_{n=0}^{\infty} \frac{(n+1)[q^{k-i}; q^k]_n q^{in}}{[q^i; q^i]_n+1} + \sum_{n=0}^{\infty} \frac{n[q^{k-j}; q^k]_n q^{jn}}{[q^j; q^j]_n+1} = \frac{[q^k; q^k]_\infty [q^{i+j}; q^k]_\infty}{[q^j; q^j]_\infty [q^i; q^i]_\infty [q^i]_\infty},
\]

(20)


\[
[q^5; q^5]_\infty G(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}},
\]

(21)

\[
[q^5; q^5]_\infty H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}},
\]

(22)

\[
[q^5; q^5]_\infty G^2(q)/H(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}},
\]

(23)

\[
[q^5; q^5]_\infty H^2(q)/G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}},
\]

(24)
Main Results

We established the following main results

\[
[q^5; q^5]^2_G(q) = \sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{5n}+1},
\]
(25)

\[
[q^5; q^5]^2_H(q) = \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+3}},
\]
(26)

\[
[q^5; q^5]^2_G(q) = \sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1 + q^{5n+1})}{1 - q^{5n+1}},
\]
(27)

\[
[q^5; q^5]^2_H(q) = \sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1 + q^{5n+2})}{1 - q^{5n+2}},
\]
(28)

\[
[q^5; q^5]^2_G(q) = \sum_{n=0}^{\infty} \frac{q^{4n}}{1 - q^{10n+1}},
\]
(29)

\[
[q^5; q^5]^2_H(q) = \sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{10n+3}},
\]
(30)


\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1 - q^{10n+4})}{(1 - q^{5n+1})(1 - q^{5n+3})} = \frac{1}{1+1+1+1+1+\ldots}.
\]
(31)

\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1 - q^{10n+3})}{(1 - q^{5n+2})(1 - q^{5n+1})} = \frac{1}{1+1+1+1+1+\ldots}.
\]
(32)

\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1 + q^{5n+1})}{(1 - q^{5n+1})} = \left(\frac{1}{1+1+1+1+1+\ldots}\right)^2.
\]
(33)

\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1 + q^{5n+2})}{(1 - q^{5n+2})} = \left(\frac{1}{1+1+1+1+1+\ldots}\right)^2.
\]
(34)

\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1 + q^{5n+2})}{(1 - q^{5n+2})} = \frac{1}{1+1+1+1+1+\ldots}.
\]
(35)

\[
\sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1 + q^{5n+3})}{(1 - q^{5n+3})} = \frac{1}{1+1+1+1+1+\ldots}.
\]
(36)
\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1 - q^{10n+4})}{(1 - q^{5n+1})(1 - q^{5n+3})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1 - q^{10n+6})}{(1 - q^{5n+4})(1 - q^{5n+2})} &= \frac{1}{1+1+1+1+1+\ldots}. \\
\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1 - q^{10n+3})}{(1 - q^{5n+1})(1 - q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1 - q^{10n+7})}{(1 - q^{5n+4})(1 - q^{5n+3})} &= \frac{1}{1+1+1+1+1+\ldots}. \\
\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1 - q^{10n+4})}{(1 - q^{5n+2})(1 - q^{5n+1})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1 - q^{10n+6})}{(1 - q^{5n+4})(1 - q^{5n+3})} &= \frac{1}{1+1+1+1+1+\ldots}. \\
\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1 - q^{10n+3})}{(1 - q^{5n+2})(1 - q^{5n+1})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1 - q^{10n+7})}{(1 - q^{5n+4})(1 - q^{5n+3})} &= \frac{1}{1+1+1+1+1+\ldots}. \\
\end{align*}
\]
4. Proof of Main Results

Proof (31)-(44):

As an illustration, we shall prove (31).
From (21) and (22) we have

\[ H(q) = \frac{\sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{3n+1}}}{G(q)} = \frac{\sum_{n=0}^{\infty} q^{3n}}{1 - q^{3n+1}} - \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{3n+4}}. \tag{55} \]

Making use of (5) and (16) in (55) and after simplification we obtain (31).

Proceeding in the same way and using the results (5), (16), (17) and (21)-(30), one can establish the results (32)-(44).

Proof of (45)-(54):

As an illustration, we shall prove (47) and (50).

Proof of (47) is as follows Taking \(i = 3, j = 4, k = 6\) in (18) we get

\[ \sum_{n=0}^{\infty} \frac{q^{4n}}{1 - q^{6n+3}} = \frac{[q; q^6]_\infty[q; q^4]_\infty[1; q^1]_\infty}{[q^3; q^6]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{56} \]

Again setting \(i = 2, j = 1, k = 6\) in (18) we get

\[ \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{6n+2}} = \frac{[q^2; q^6]_\infty[1; q^2]_\infty[3; q^4]_\infty[3; q^4]_\infty}{[q^2; q^6]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{57} \]

Now taking the ratio of (56) and (57) and making use of (16) for the assigned values of \(i, j\) and \(k\), we get

\begin{align*}
\sum_{n=0}^{\infty} \frac{q^{6n+7n}(1 - q^{12n+7})}{(1 - q^{6n+4})(1 - q^{6n+3})} & - \sum_{n=0}^{\infty} \frac{q^{6n+5n-1}(1 - q^{12n+5})}{(1 - q^{6n+3})(1 - q^{6n+2})} \\
\sum_{n=0}^{\infty} \frac{q^{6n+8n}(1 - q^{12n+9})}{(1 - q^{6n+1})(1 - q^{6n+2})} & - \sum_{n=0}^{\infty} \frac{q^{6n+9n-1}(1 - q^{12n+5})}{(1 - q^{6n+5})(1 - q^{6n+4})} = \frac{[q^7; q^6]_\infty[1; q^1]_\infty}{[q^3; q^6]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{58} 
\end{align*}

Finally making use of (6) in (58) and after some calculation, we get (47).

In order to prove (50) we proceed as follows consider \(i = 2, j = 7, k = 8\) in (18) we get

\[ \sum_{n=0}^{\infty} \frac{q^{7n}}{1 - q^{8n+2}} = \frac{[q^3; q^8]_\infty[1; q^1]_\infty[3; q^4]_\infty}{[q^3; q^8]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{59} \]

Further taking \(i = 1, j = 4, k = 8\) in (18) we obtain

\[ \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{8n+1}} = \frac{[q^3; q^8]_\infty[1; q^1]_\infty[3; q^4]_\infty}{[q^3; q^8]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{60} \]

Now taking the ratio of (59) and (60) and making use of (16) for the assigned values of \(i, j\) and \(k\), we get

\begin{align*}
\sum_{n=0}^{\infty} \frac{q^{8n+9n}(1 - q^{16n+9})}{(1 - q^{8n+7})(1 - q^{8n+6})} & - \sum_{n=0}^{\infty} \frac{q^{8n+8n-1}(1 - q^{16n+7})}{(1 - q^{8n+5})(1 - q^{8n+4})} \\
\sum_{n=0}^{\infty} \frac{q^{8n+11n}(1 - q^{16n+11})}{(1 - q^{8n+9})(1 - q^{8n+8})} & - \sum_{n=0}^{\infty} \frac{q^{8n+10n-1}(1 - q^{16n+7})}{(1 - q^{8n+7})(1 - q^{8n+6})} = \frac{[1 - q^{-1}]_\infty[3; q^4]_\infty[3; q^4]_\infty[3; q^4]_\infty}{[1 - q][q^3; q^8]_\infty[3; q^4]_\infty[3; q^4]_\infty[3; q^4]_\infty}. \tag{61} 
\end{align*}

Now making use of (7) and (8) in (61), and after some calculation we obtain (50).

For suitable selection of values for \(i, j\) and \(k\) and using the same process for the results (5) to (10) and (16) to (20), one can establish the results (45), (46), (48), (49) and (51) to (54).
5. Conclusion
The object of this article has been to show the effective applications of generalized Lambert series in obtaining a diverse variety of continued fractions.

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References


