



Bioeconomic Modelling: An Application to the North-East- Atlantic Cod Fishery

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Abstract

This paper describes a single species model, which followed by two stages, a mature and an immature stage. Conditions for existence of equilibrium points and their stability are discussed. The model in this paper has been developed on the concept of optimal management of resources based on the criterion of maximization of present values of net economic revenues. Using the data from the North-East Atlantic cod fishery, the results of the optimal stock, harvest and effort level are derived. Our simulation results show that optimal harvesting policy is much superior than the MSY policy and optimal paths always take less time than the suboptimal path to reach the optimal steady state. Our analysis also shows that, if it is insisted that a closure is take place for one of the two sub-stocks, it will be optimal to reduce fishing on the immature sub-stock rather than the mature sub-stock.

Keywords: Harvesting, Stage-structured, Global stability, MSY, Economic rent

1. Introduction

In the natural world, most of the species go through two or more life stages as they proceed from birth to death. Immature and mature are such two important stages that are to be taken into consideration. Most population models in the literature ignore such reality and assume that all individuals are identical and do not take into account any stage structure. It has been recognized that mortality, fertility and feeding depend on the size of the individuals. Some biological factors and some important rates (like rates of survival, reproduction) of individuals in a population almost always depend on stage-structure.

A stage-structured model of population growth consisting of immature and mature individuals was analyzed by Freedman and Gopalsamy (1986), where the stage-structured was modeled by introduction of a constant time delay. A prey-predator model with stage-structure for prey was established by Zhang et al. (2000) and obtained the necessary and sufficient condition of the permanence of the system. An overview on stage-structured models can be found in the recent book by Murdoch et al (2003). Recent papers of Kar (2003), Kar and Matsuda (2006), and Wiken (2004) study the stage-structure of species with or without time delays.

Harvesting has generally a strong impact on the population dynamics of a harvested species. The study of population dynamics with harvesting is a subject of mathematical bio-economics, and it is related to optimal management of renewable resources (Clark, 1990). Clark has also considered harvesting of a single species in a two fish ecologically competing population model. Chaudhuri (1986, 1988) studied combined harvesting and considered the perspectives of bio-economics

and dynamic optimization of a two species fishery. Dai and Tang (1998) considered a prey-predator system with constant rate of harvesting. They showed how to approximate the region of asymptotic stability in biological terms in the initial states that ultimately lead to co-existence of the two species.

In recent years, the optimal management of renewable resources, which has direct relationship to sustainable development has been studied extensively by some authors. Generally, speaking, the exploitation of a species should be determined by the economic and biological values of the population. In this paper we aim to find an optimal harvesting policy which ensures a lasting exploitation of the biological resource and maximize the benefit resulting from the harvesting.

To develop a mathematical model for describing the interaction among immature and mature species we make the following assumptions:

- (i) The recruitment of the mature species depends on the size of the immature species.
- (ii) Two substocks interact via cannibalism.
- (iii) Immature fish, which are product of mature fish, also become mature.

Cod fishery is a good example of the above model. An ecological interpretation of the cod cannibalism is that the cod has adopted to the cyclic recruitment pattern of capelin by eating its own offspring when the capelin stock is depleted and is in a state of rebuilding. It seems obvious if the nearly one and half a million tonnes of young cods eaten by older brothers and sisters in 1995-1997 had survived this could have prevented rebuilding of the capelin stocks in the subsequent years and thus threatened the food supply of the coming generations of cod. According to Eide (1993), following model is closely describing the North-East Atlantic cod stock.

In our model we consider the stage-structure of immature and mature of a single species. To establish a constructive management of commercial exploitation of biological resources we shall consider a special type linearly dependent harvesting efforts on both stages of the species. Let x_1, x_2 be the densities of immature, mature species respectively at any time t . Then our proposed model is of the form

$$\frac{dx_1}{dt} = r_1 x_1 \left(1 - \frac{x_1}{a_1 x_2}\right) - b_1 x_1 x_2, \quad (1)$$

$$\frac{dx_2}{dt} = r_2 x_2 \left(1 - \frac{x_2}{a_2 x_1}\right), \quad (2)$$

where r_1, r_2, a_1, a_2 and b_1 are all positive constants. Now considering the harvesting of both the species our model takes the form,

$$\frac{dx_1}{dt} = r_1 x_1 \left(1 - \frac{x_1}{a_1 x_2}\right) - b_1 x_1 x_2 - h_1, \quad (3)$$

$$\frac{dx_2}{dt} = r_2 x_2 \left(1 - \frac{x_2}{a_2 x_1}\right) - h_2. \quad (4)$$

We take the two forms of harvesting as follows:

$$h_1 = q_1 (1 - \alpha) E x_1, h_2 = q_2 \alpha E x_2, \quad (5)$$

$$h_1 = q_1 E_1 \alpha_1 x_1, h_2 = q_2 E_2 \alpha_2 x_2. \quad (6)$$

In (5), we consider the harvesting efforts $(1 - \alpha)E$ and αE to the immature and mature species (where $0 < \alpha < 1$) respectively i.e. some part of harvesting effort is applied for mature species and rest of the harvesting effort is for immature species and q_1, q_2 are respective catchability coefficient. In (6), the harvest $h_i = q_i E_i \alpha_i x_i$ where q_i is the catchability coefficient, E_i is the effort, and α_i is the fraction of the stock available to harvesting, with $0 < \alpha_i \leq 1$.

2. Results and discussion of the model taking $h_i (i = 1, 2)$ given in (5)

2.1 Equilibrium analysis

We can analytically derive the equilibrium values of populations. Recalling equations (3) and (4), the equilibria are given by the intersection of the isoclines $\dot{x}_1 = 0 = \dot{x}_2$.

It can be easily verified that the shape of the isocline $\dot{x}_1 = 0$ is a parabola but the $\dot{x}_2 = 0$ is a straight line and both depends on the effort E . The slope of the isocline $\dot{x}_2 = 0$ depends upon sign of the term $r_2 - q_2 \alpha E$. To allow the population to grow under such conditions, $r_2 - q_2 \alpha E$ is assumed to be positive, so that the isocline $\dot{x}_2 = 0$ has a positive slope. $r_2 - q_2 \alpha E > 0$

implies that $\alpha E < r_2/q_2 = BTP_{x_2}$ (biotechnical productivity of x_2). It follows from these considerations that there will be one strictly positive equilibrium (x_1^*, x_2^*) at most, where

$$x_1^* = \frac{r_2}{a_2(r_2 - q_2\alpha E)} \left[\frac{r_1\{a_1a_2(r_2 - q_2\alpha E) - r_2\}}{b_1a_1a_2(r_2 - q_2\alpha E)} - \frac{q_1(1 - \alpha)E}{b_1} \right] \quad (7)$$

$$x_2^* = \frac{r_1[a_1a_2(r_2 - q_2\alpha E) - r_2]}{b_1a_1a_2(r_2 - q_2\alpha E)} - \frac{q_1(1 - \alpha)E}{b_1} \quad (8)$$

<Table 1>

<Figure 1>

<Figure 2>

Parameters r_i & k_i ($i = 1, 2$) and b_1 are taken from Eide (1993) and other parameters are taken from Armstrong and Sumalia (2000).

This equilibrium (x_1^*, x_2^*) lies in R_+^2 if

$$\frac{r_1[a_1a_2(r_2 - q_2\alpha E) - r_2]}{b_1a_1a_2(r_2 - q_2\alpha E)} > \frac{q_1(1 - \alpha)E}{b_1} \quad (9)$$

and

$$E < r_2/q_2\alpha. \quad (10)$$

The condition (9) is equivalent to

$$f(E) = q_1q_2\alpha(1 - \alpha)E^2 - [q_1r_2(1 - \alpha) + r_1q_2\alpha]E + \left(r_1r_2 - \frac{r_1r_2}{a_1a_2}\right) > 0.$$

<Figure 3>

<Figure 4>

That is, E lies outside the interval $[\lambda, \mu]$ where

$$\lambda, \mu = \frac{[q_1r_2(1 - \alpha) + q_2r_1\alpha] \mp \sqrt{[r_2q_1(1 - \alpha) - r_1q_2\alpha]^2 + 4r_1r_2q_1q_2\alpha(1 - \alpha)/a_1a_2}}{2q_1q_2\alpha(1 - \alpha)} \quad (11)$$

$\mu > 0$ and we consider $a_1a_2 > 1$ to make $\lambda > 0$. it is obvious that

$$r_2/q_2\alpha < \mu \quad (12)$$

and

$$r_2/q_2\alpha > \lambda \quad (13)$$

$$\text{otherwise } \frac{4r_1r_2q_1q_2\alpha(1 - \alpha)}{a_1a_2} < 0$$

which is not true, because $r_1, r_2, q_1, q_2, a_1, a_2, \alpha$ are all positive and $\alpha < 1$.

Thus from (9) & (10) the equilibrium point (x_1^*, x_2^*) given in (7) and (8) lies in the interior of R_+^2 if E lies in $[0, \lambda]$. Let $\lambda = E_0$, then the sufficient condition that the system has a positive equilibrium is that

$$E < E_0 \quad (14)$$

For $E = 0$, the equilibrium (x_1', x_2') is given by

$$x_1' = \frac{r_1(a_1a_2 - 1)}{b_1a_1a_2^2}, \quad x_2' = \frac{r_1(a_1a_2 - 1)}{b_1a_1a_2} \text{ where } a_1a_2 > 1.$$

The equilibrium curve for $E = 0$ is then $x_2 = a_2x_1$. Thus for $E = 0$, the equilibrium curve originates from natural equilibrium (x_1', x_2') . As we increase harvesting effort the equilibrium point moves downwards and both the species become more extensively exploited. Both species will persist up to an effort level $E < E_0$.

2.2 Stability analysis

We now inspect the local nature of equilibrium point (x_1^*, x_2^*) in terms of stability. The Jacobian matrix of the system is

$$J[x_1, x_2] = \begin{bmatrix} r_1 \left(1 - \frac{2x_1}{a_1 x_2}\right) - b_1 x_2 - q_1(1 - \alpha)E & \frac{r_1 x_1^2}{a_1 x_2^2} - b_1 x_1 \\ \frac{r_2 x_2^2}{a_2 x_1^2} & r_2 \left(1 - \frac{2x_2}{a_2 x_1}\right) - q_2 E \alpha \end{bmatrix} \quad (15)$$

At $P(x_1^*, x_2^*)$ its characteristic equation is the quadratic in ζ , given by

$$\zeta^2 + \left(\frac{r_1 x_1^*}{a_1 x_2^*} + \frac{r_2 x_2^*}{a_2 x_1^*}\right) \zeta + \frac{r_2 b_1}{a_2} \frac{x_2^{*2}}{x_1^*} = 0 \quad (16)$$

By Hurwitz - Routh's condition the roots of this quadratic equation are real negative or complex with negative real parts if

$$\left\{ \left(\frac{r_1 x_1^*}{a_1 x_2^*} + \frac{r_2 x_2^*}{a_2 x_1^*}\right) > 0 \text{ and } \frac{r_2 b_1}{a_2} \frac{x_2^{*2}}{x_1^*} > 0 \right\} \quad (17)$$

These conditions are true if $P(x_1^*, x_2^*)$ exists in R_+^2 . Thus the interior equilibrium $P(x_1^*, x_2^*)$ is locally stable if it exists. In otherwords we can say that the equilibrium remains stable up to a maximum effort level E_0 beyond which the system ceases to have a locally stable equilibrium.

<Figure 5>

<Figure 6>

To establish the non-existence of periodic orbit encircling (x_1^*, x_2^*) , we use Bendixon-Dulac criterion.

Let $h(x_1, x_2) = 1/x_1 x_2$. Obviously $h(x_1, x_2) > 0$ for all $x_1, x_2 > 0$. We define

$$f_1(x_1, x_2) = r_1 x_1 \left(1 - \frac{x_1}{a_1 x_2}\right) - b_1 x_1 x_2 - q_1(1 - \alpha) E x_1$$

$$f_2(x_1, x_2) = r_2 x_2 \left(1 - \frac{x_2}{a_2 x_1}\right) - q_2 \alpha E x_2$$

Now we find that

$$\frac{\partial}{\partial x_1}(f_1 h) + \frac{\partial}{\partial x_2}(f_2 h) = -\frac{r_1}{a_1 x_2^2} - \frac{r_2}{a_2 x_1^2} < 0$$

for all $x_1, x_2 > 0$, since all other parameters are strictly positive. Therefore, by Bendixon-Dulac criterion, there will be no periodic orbit within the interior of the first quadrant of state of space around (x_1, x_2) . So local stability of the equilibrium point (x_1^*, x_2^*) implies the global asymptotic stability. From the point of view of ecological managers, it may be desirable to have an unique positive equilibrium which is globally asymptotic stable, in order to plan harvesting and keep sustainable development of ecosystem.

2.3 Bionomic equilibrium

The term bionomic equilibrium is the combined concept of biological equilibrium as well as economic equilibrium. For biological equilibrium $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, which implies, $x_1 = 0$ or, $E = \frac{r_1}{q_1(1 - \alpha)} \left(1 - \frac{x_1}{a_1 x_2}\right) - \frac{b_1}{q_1(1 - \alpha)} x_2$,

and $x_2 = 0$ or, $E = \frac{r_2}{q_2 \alpha} \left(1 - \frac{x_2}{a_2 x_1}\right)$.

Therefore, for non-trivial biological equilibrium the solution lies on the curve,

$$\left[\frac{r_1}{q_1(1 - \alpha)} - \frac{r_2}{q_2 \alpha} \right] - \frac{r_1}{q_1(1 - \alpha)} \frac{x_1}{a_1 x_2} + \frac{r_2}{q_2 \alpha} \frac{x_2}{a_2 x_1} - \frac{b_1}{q_1(1 - \alpha)} x_2 = 0 \quad (18)$$

At an economic equilibrium the total revenue (TR) obtained by selling the harvested biomass is equal to the total cost (TC) for effort devoted to harvest.

If c_1 = unit effort cost of immature species, c_2 = unit effort of mature species, p_1 = constant price per unit of biomass of immature species, p_2 = constant price per unit of biomass of mature species, then the economic rent *i.e.* net revenue (TR-TC) at any time t is given by

$$\pi(x_1, x_2, E) = [p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c] E$$

Thus at economic equilibrium

$$[p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c] E = 0 \quad (19)$$

where $c = c_1(1 - \alpha) + c_2\alpha$ and the bionomic equilibrium is the intersection of (18) and (19).

From (19) we get

$$x_1 = \frac{c - p_2 q_2 \alpha x_2}{p_1 q_1 (1 - \alpha)} \quad (20)$$

which exists in R_+^2 if

$$x_2 < c / p_2 q_2 \alpha \quad (21)$$

At the intersection of (20) and (18)

$$\left[\frac{r_1}{q_1(1 - \alpha)} - \frac{r_2}{q_2 \alpha} \right] - \frac{r_1}{q_1(1 - \alpha)a_1} \frac{c - p_2 q_2 \alpha x_2}{p_1 q_1 x_2 (1 - \alpha)} + \frac{r_2}{q_2 a_2 \alpha} \frac{x_2 p_1 q_1 (1 - \alpha)}{c - p_2 q_2 \alpha x_2} - \frac{b_1 x_2}{q_1(1 - \alpha)} = 0. \quad (22)$$

Simplifying this we get the cubic equation in x_2 ,

$$Ax_2^3 + Bx_2^2 + Cx_2 + D = 0 \quad (23)$$

where

$$A = a_1 a_2 A_3 A_4 A_5$$

$$B = -[a_1 a_2 A_3(A_1 - A_2)A_4 + A_1 A_4^2 a_2 - a_1 A_2 A_3^2 + a_1 a_2 A_3 A_5 c]$$

$$C = a_1 a_2 A_3(A_1 - A_2)c + 2A_1 A_4 a_2 c$$

$$D = -A_1 a_2 c$$

$$\text{and } A_1 = \frac{r_1}{q_1(1 - \alpha)}, A_2 = \frac{r_2}{q_2 \alpha}, A_3 = p_1 q_1 (1 - \alpha), A_4 = p_2 q_2 \alpha, A_5 = \frac{b_1}{q_1(1 - \alpha)}.$$

Since $a_1, a_2, q_1, q_2, r_1, r_2, \alpha, p_1, p_2, c$ are all positive we have $A > 0$ and $D < 0$. The co-efficient B and C may be positive, negative or zero. So the following cases may appear:

Case I. When both B and C are positive then all the co-efficients of (23) are positive except the last co-efficient D. Hence by Descarte's rule of signs (23) has exactly one positive root. That is in this case unique positive bionomic equilibrium exists if (21) is satisfied.

Case II. When both B and C are negative then all the co-efficients of (23) are negative except the leading coefficient which is essentially positive. Hence (23) has only one positive root and so unique interior bionomic equilibrium exists if (21) is satisfied.

Case III. When $B > 0$ and $C < 0$, first two co-efficients of (23) are positive and last two are negative. Hence by Descarte's rule of signs (23) has exactly one positive root that is in turn unique positive bionomic equilibrium exists if (21) is true.

<Figure 7>

<Figure 8>

Case IV. When $B < 0, C > 0$ then equation (23) may have more than one positive roots and system may have non-unique interior bionomic equilibrium if (21) is also true.

Case V. In all the cases when $B = 0$ & either $C > 0$ or $C < 0$ and $C = 0, B > 0$ or $C < 0$ and $B = 0, C = 0$ the equation (23) has exactly one positive root *i.e.* the system has unique interior bionomic equilibrium if (21) is also true.

From all the cases studied above it is clear that equation (23) has at least one positive root. If this positive root be less than $c/p_2 q_2 \alpha$ we denote it by $x_{2\infty}$ and from (20) we get a value of x_1 denoted by $x_{1\infty}$ and the bionomic equilibrium point $R(x_{1\infty}, x_{2\infty})$ is obtained. Now (23) will have a positive root less than $c/p_2 q_2 \alpha$ i.e. a root in $(0, c/p_2 q_2 \alpha)$ if $(Ax_2^3 + Bx_2^2 + Cx_2 + D)$ has opposite signs for $x_2 = 0$ and $x_2 = c/p_2 q_2 \alpha$ that is if

$$(a_1 a_2 p_1 p_2 q_2 b_1 \alpha)(c/p_2 q_2 \alpha)^3$$

$$-\left[a_1 a_2 p_1 p_2 \{r_1 q_2 \alpha - r_2 q_1 (1 - \alpha)\} + \frac{r_1 a_2 p_2^2 q_2^2 \alpha^2}{q_1} - \frac{r_2}{q_2 \alpha} a_1 p_1^2 q_1^2 (1 - \alpha)^2 + a_1 a_2 p_1 b_1 c \right] \\ \left(\frac{c}{p_2 q_2 \alpha} \right)^2 + \left[\frac{a_1 a_2 p_1 c}{q_2 \alpha} \{r_1 q_2 \alpha - r_2 q_1 (1 - \alpha)\} + \frac{2r_1 a_2 p_2 q_2 \alpha c}{q_1 (1 - \alpha)} \left(\frac{c}{p_2 q_2 \alpha} \right) - \frac{a_2 r_1 c}{q_1 (1 - \alpha)} \right] > 0. \quad (24)$$

Thus the system has a bionomic equilibrium if (24) is satisfied by the parameters of the model.

Then for 40% and 60% of the effort level devoted to mature species i.e. for $\alpha = 0.4$ and $\alpha = 0.6$ the inequation (23) is satisfied. For $\alpha = 0.4$ the bionomic equilibrium point is (0.33, 0.37) and for $\alpha = 0.6$ the bionomic equilibrium is attained at the point (0.31, 0.32).

2.4 Optimal harvest policy

In this section we shall discuss the optimal harvest policy. For that we consider the present value J of a continuous time-stream of revenues as

$$J = \int_0^{\infty} e^{-\delta t} \left[p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c \right] E dt \quad (25)$$

where, δ denotes the instantaneous annual rate of discount. Our problem is to maximize J subject to the state equation (3) and (4) by invoking Pontryagin et al. (1962). The control variable $E(t)$ is subjected to the constraint set $0 \leq E \leq E_{\max}$. We consider the current value of Hamiltonian as

$$H = \left[p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c \right] E + \mu_1 \left\{ r_1 \left(1 - \frac{x_1}{a_1 x_2} \right) - b_1 x_2 - q_1 (1 - \alpha) E \right\} + \mu_2 \left\{ r_2 \left(1 - \frac{x_2}{a_2 x_1} \right) - q_2 \alpha E \right\} \quad (26)$$

where $\mu_1(t)$, $\mu_2(t)$ are adjoint variables. The adjoint equations are

$$\frac{d\mu_1}{dt} = \delta \mu_1 - \frac{\partial H}{\partial x_1} \\ \frac{d\mu_2}{dt} = \delta \mu_2 - \frac{\partial H}{\partial x_2}. \quad (27)$$

Here we use the steady state solution as we are concerned with optimal equilibrium and we consider x_1, x_2 as constants in the subsequent steps.

Then using the steady state solution the adjoint equations (27) become,

$$\frac{d\mu_1}{dt} = \left(\delta + \frac{r_1 x_1}{a_1 x_2} \right) \mu_1 - \mu_2 \frac{r_2 x_2^2}{a_2 x_1^2} - p_1 q_1 (1 - \alpha) E, \\ \frac{d\mu_2}{dt} = \left(\delta + \frac{r_2 x_2}{a_2 x_1} \right) \mu_2 - \mu_1 \left(\frac{r_1 x_1^2}{a_1 x_2^2} - b_1 x_1 \right) - p_2 q_2 \alpha E. \quad (28)$$

We write these equations as

$$(D - m_1) \mu_1 + m_2 \mu_2 = -m_3 \\ (D - m_4) \mu_2 + m_5 \mu_1 = -m_6 \quad (29)$$

where $D \equiv d/dt$, and

$$m_1 = \delta + \frac{r_1 x_1}{a_1 x_2}, \quad m_2 = \frac{r_2 x_2^2}{a_2 x_1^2}, \quad m_3 = p_1 q_1 (1 - \alpha) E,$$

$$m_4 = \delta + \frac{r_2 x_2}{a_2 x_1}, \quad m_5 = \frac{r_1 x_1^2}{a_1 x_2^2} - b_1 x_1, \quad m_6 = p_2 q_2 \alpha E.$$

Eliminating μ_1 from the equation (29) we get the differential equation

$$[D^2 - (m_1 + m_4)D + m_1 m_4 - m_2 m_5] \mu_2 = m_1 m_6 + m_3 m_5$$

$$\therefore \mu_2 = A e^{\lambda_1 t} + B e^{\lambda_2 t} + \frac{m_1 m_6 + m_3 m_5}{m_1 m_4 - m_2 m_5} \quad (30)$$

where A,B are arbitrary constants and λ_1, λ_2 are the roots of

$$\lambda^2 - (m_1 + m_4)\lambda + m_1 m_4 - m_2 m_5 = 0.$$

It is clear that μ is bounded if A,B, are zero or $\lambda_1, \lambda_2 < 0$. We consider, for simplicity, that A,B are zero.

$$\therefore \mu_2 = \frac{m_1 m_6 + m_3 m_5}{m_1 m_4 - m_2 m_5}$$

. Similarly we can derive that $\mu_1 = \frac{m_3 m_4 + m_2 m_6}{m_1 m_4 - m_2 m_5}$.

Now $\partial H / \partial E = 0$ implies

$$p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c = \mu_1 q_1 (1 - \alpha) x_1 + \mu_2 q_2 \alpha x_2.$$

Using the values of μ_1, μ_2 we get,

$$\begin{aligned} & (p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c) \left(\left(\delta + \frac{r_1 x_1}{a_1 x_2} \right) \left(\delta + \frac{r_2 x_2}{a_2 x_1} \right) - \frac{r_2 x_2^2}{a_2 x_1^2} \left(\frac{r_1 x_1^2}{a_1 x_2^2} - b_1 x_1 \right) \right) \\ &= \left[\left(\delta + \frac{r_2 x_2}{a_2 x_1} \right) p_1 q_1 (1 - \alpha) + \frac{r_2 x_2^2}{a_2 x_1^2} p_2 q_2 \alpha \right] E q_1 (1 - \alpha) x_1 \\ &+ \left[\left(\delta + \frac{r_1 x_1}{a_1 x_2} \right) p_2 q_2 \alpha + p q (1 - \alpha) \left(\frac{r_1 x_1^2}{a_2 x_2^2} - b_1 x_1 \right) \right] E q_2 \alpha x_2. \end{aligned} \quad (31)$$

Solving (31), (10) & (11) we can get a value of E, the optimal harvesting effort.

Since the optimal control problem is linear of its control variable E, the optimal harvesting policy will be a combination of bang-bang and singular controls. Let T be the time taken to reach the optimal state by using the bang-bang control. Then the optimal control policy will be

$$E(t) = \begin{cases} \bar{E} & \text{for } 0 \leq t \leq T \\ E^* & \text{for } t > T, \end{cases}$$

where

$$\bar{E}(t) = \begin{cases} E_{max} & \text{for } \sigma(t) > 0 \\ E_{max} & \text{for } \sigma(t) < 0, \end{cases}$$

here

$$\sigma(t) = p_1 q_1 (1 - \alpha) x_1 + p_2 q_2 \alpha x_2 - c - \mu_1 q_1 (1 - \alpha) x_1 - \mu_2 q_2 \alpha x_2$$

is the switching function.

<Figure 9-11>

Phase diagram showing the changes in the equilibrium biomass as a response to changes in the effort allocation. Both the equilibrium biomass tends to zero with the efforts increases. Here $\alpha = 0.4$ for (a) and $\alpha = 0.6$ for (b).

<Figure 12> <Figure 13>

Then the following table shows the interior equilibrium (x_1^*, x_2^*) & optimal harvest E^* , E_{max} , E_{min} for 40% and 60% of the effort devoted to the mature species.

<Table 2>

The following table shows the time taken by different approach paths to reach an equilibrium point.

<Table 3>

Management of renewable resources has generally been relied on the concept of Maximum Sustainable Yield (MSY). Recently, several objections have been raised against the use of MSY on both biological and socioeconomic grounds (Matsuda and Abrams, 2006). One of the most serious objection is obviously the non-recognition of the cost factor. The inadequacy of MSY concept has resulted in a trend to replace it with a concept of optimal resource management based on criterion of maximization of present values of net economic revenues. In this paper we have attempted to find optimal steady state solution (i.e. optimal stock, optimal harvest and effort level) that ensures the long run sustainability of the resource and excludes the possibility of depletion due to over exploitation and give maximum benefit.

Approach paths of optimal and suboptimal harvesting policy are depicted in figures 12 and 13. Optimal paths always take less time than the suboptimal path to reach the optimal steady state.

In both the cases for $\alpha = 0.4$ and $\alpha = 0.6$, it is observed that optimal harvesting effort $E^* < E_{MSY} < E_{max}$. The MSY are 0.084 (for $\alpha = 0.4$) and 0.16 (for $\alpha = 0.6$) and the effort for producing the said amounts are 95.1 units and 129.6 units respectively. Also for the optimal harvesting effort E^* , the optimal steady state is globally asymptotically stable. Optimal catch are $(0.2136 + 0.0722) = 0.2858$ (for $\alpha = 0.4$) and $(0.2236 + 0.0649) = 0.2885$ (for $\alpha = 0.6$). Optimal efforts for producing the said optimal catch are 48.06 and 74.511 respectively. It clearly implies that the higher level of effort causes overfishing which, in turn, causes lower stock. As a consequence, even higher level of effort in later years does not get adequate quantity of catch.

3. Results and discussion of the model using catch functions

$h_i = q_i E_i \alpha_i x_i$, $i = 1, 2$ given in (6)

In recent years, serious stock decline or collapse is a major risk for most of the world fisheries. Marine reserve may be a possible way to reduce this risk by increasing the equilibrium stock size and other effects such as preserving spawning and nursery growths. It may influence biomass growth in various ways. It is for instance possible that marine reserves, by providing some of the fish with a sanctuary from distributing fishing activity may actually enhance biomass growth. The same effect may occur if marine reserves manage to protect fast growing immature fish or conserve habitat variables important for fish survival and growth.

The management goal is overall profit maximization with the profit function

$$\pi = p_1 q_1 E_1 \alpha_1 x_1 + p_2 q_2 E_2 \alpha_2 x_2 - c_1 E_1 - c_2 E_2. \quad (32)$$

The mathematical formulation is given by

$$\text{maximize } \int_0^\infty \pi e^{-\delta t} dt$$

subject to

$$\frac{dx_1}{dt} = r_1 x_1 \left(1 - \frac{x_1}{a_1 x_2}\right) - b_1 x_1 - q_1 E_1 \alpha_1 x_1 \quad (33)$$

$$\frac{dx_2}{dt} = r_2 x_2 \left(1 - \frac{x_2}{a_2 x_1}\right) - q_2 E_2 \alpha_2 x_2,$$

$$0 \leq E_i \leq E_{imax}.$$

We have applied the maximum principle to the optimal control problem (as in section 2) and the parameter values of Table 1.

Equilibrium profits for different closure alternatives are given below.

<Table 4-8>

It is already established that marine reserves are a practical means of managing resource populations and are therefore, beneficial for conserving the ecological environment and resource populations (Kar & Matsuda, 2008). Using the data

from North-East Atlantic cod fishery, our analysis shows that, if it is insisted that a closure is to take place for one of the two sub-stocks, it will be optimal to reduce fishing on the immature sub-stock rather than the mature sub-stock. This supports the case for greater focus on regulation of harvest upon juvenile fish, as may be claimed to be the case in most fisheries in recent times. For the cod fishery, there is very little difference in resulting profits from a fishery with no reserves and a fishery where half of the immature fish stock is projected.

Introduction of marine reserves uniformly increases the optimal equilibrium and reduce the profitability of harvesting for a given biomass. Both effects serve to increase the equilibrium biomass level. Due to these opposing forces, the impact of marine reserves on optimal fishing effort level is indeterminate. Thus for the cod fishery, introduction of marine reserves is more likely to reduce net benefits from the fishery. Thus, inspite of a certain increase in stock biomass, introducing marine reserves in the cod fishery seems rather not attractive in general.

4. Conclusions

Past experience has shown that over fishing may cause extinction or near extinction of different species of fishes, like Antarctic blue whales, Peruvian anchoveta and Canadian cod fishery etc. North-East Atlantic cod stocks have suffered from both over fishing and food shortages, as stocks of their prey species herring and capelin have fluctuated. The whole stock is so unstable that the threat of a collapse is imminent unless fisheries scientists recommendations are followed. According to ICES, the current fishing levels are unstable. ICES has consequently called for a recovery plan to protect these stocks. In 2003, ICES stated that there is a high risk of stock collapse if current exploitation levels continue and recommended a zero catch of Atlantic cod in the year 2004. In the year 2004, estimated cod stock size was 1.6 million tones. Experts from the International Council for the Exploitation of Seas (ICES) have proposed that cod catches should be limited to 50% of catch levels in 2006 to enable young fish to mature and build up the population.

We assume that one of the objectives of the fishery is to find an optimal harvesting policy which ensures a lasting exploitation of the resource and other is to maximize the benefit resulting from the harvesting.

If we analyze our results and compare these with the real situation that exists in the last year of the period, it is clear that the North-East Atlantic cod fishery is being exploited in an inefficient way, from an economic view point as well as with the conservation of the resource. By comparing optimal stock and harvest with its actual stock and harvest, this study indicates the fact that both stocks and harvest at the current level is much lower than its optimal level. Over fishing during the earlier period may have had some consequences on population dynamics of the species. The biomass level in 2004 was significantly lower than the optimal equilibrium level. Therefore in order to achieve optimal steady solution, it is necessary to proper regulation of the fishery and reduction of fishing pressure on it.

The results of the study presented in this paper conclude that (i) North-East Atlantic cod fishery is not managed and utilized optimally, (ii) present condition of less biomass stock indicates that the danger of depletion of the resource can not be ruled out, (iii) if corrective measures are taken, NEAC stock would take less time to reach its optimal steady state, (iv) The quickest possible means of reaching an efficient solution is by means of so-called bang-bang controls, i.e., not exercising fishing effort in the fishery until the resource recovers its optimum levels, (v) Herring and capelin are the most important prey for Atlantic cod, so the survival of their stocks also vital for cod. Thus the interaction between cod and its preys like capelin and herring are of fundamental importance to the dynamic of the processes which govern the fish production in the region.

In the present model many important variables such as the environmental effects and interaction with other species are disregarded. Hence, the results obtained in this study should be taken with care.

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Table 1. Estimated values of the parameters

Parameters	r_1	r_2	a_1	a_2	b_1	q_1	q_2	p_1	p_2
Values	0.5003	0.6728	8.7608	1.1880	0.2023	0.006650	0.001175	7579	8655
Parameters	c_1	c_2	δ						
Values	18.602103	1.452341	0.01						

Table 2.

α	x_1^*	x_2^*	E^*	E_{max}	E_{min}
0.4	1.113845	1.279392	48.06224	112.3152	0
0.6	1.128144	1.235587	74.51134	166.2006	0

Table 3.

Figure no.	Co-ordinates of		Path type	Time taken to reach the end point
	Initial point	End point		
$(\alpha = 0.4)$ Figure 12	(0.2, 0.3)	(1.114, 1.278)	Optimal	11.67
		(1.1150, 1.2791)	Suboptimal	20
$(\alpha = 0.6)$ Figure 13	(0.2, 0.3)	(1.129, 1.255)	Optimal	10.83
		(1.1346, 1.236)	Suboptimal	20.8

Table 4.

	$\alpha_1 = 1, \alpha_2 = 1$	$\alpha_1 = 0.5, \alpha_2 = 0.5$
Profit	6660.24	5938.74
Optimal equilibrium	(3.71, 1.38)	(3.85, 1.47)

Table 5. $\alpha_1 = 1$

α_2	0.3	0.7	0.8	0.9
Profit	5343.18	6415.10	6517.08	6596.56
Optimal equilibrium	(3.58, 1.43)	(3.69, 1.39)	(3.7, 1.38)	(3.73, 1.38)

Table 6. $\alpha_2 = 1$

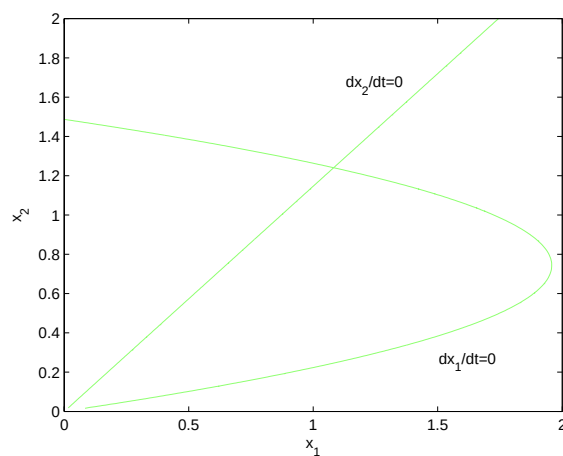
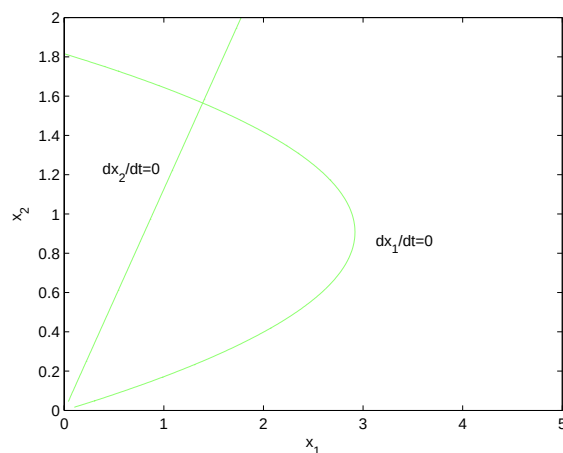
α_1	0.3	0.4	0.5	0.6	0.9
Profit	6451.15	6447.46	6509.1	6555.04	6641.52
Optimal equilibrium	(4.15, 1.54)	(4.00, 1.48)	(3.91, 1.45)	(3.84, 1.42)	(3.73, 1.38)

Table 7. $\alpha_1 = 0.5$

α_2	0.1	0.3	0.4	0.5	0.6	0.9	1
Profit	1790.16	5191.5	5656.73	5938.59	6127.92	6445.09	6508.81
Optimal equilibrium	(3.08, 1.74)	(3.77, 1.51)	(3.82, 1.48)	(3.85, 1.47)	(3.87, 1.46)	(3.90, 1.45)	(3.91, 1.45)

Table 8. $\alpha_2 = 0.5$

α_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Profit	5798.16	4901.05	5938.77	5984.93	6020.59	6048.8	6071.6
Optimal equilibrium	(4.1, 1.57)	(3.95, 1.51)	(3.85, 1.47)	(3.79, 1.45)	(3.74, 1.45)	(3.71, 1.42)	(3.68, 1.41)

Figure 1. Isoclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ for $\alpha = 0.4$ Figure 2. Isoclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ for $\alpha = 0.6$

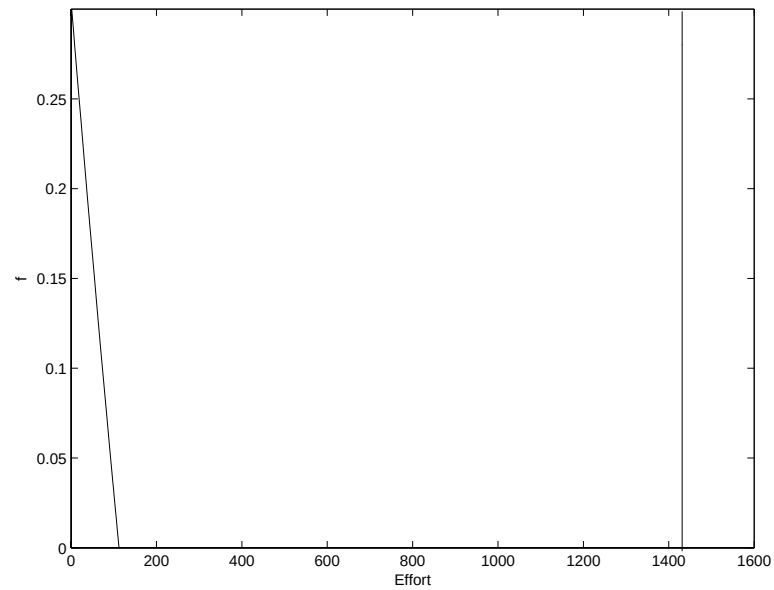


Figure 3. This figure shows that $f(E) > 0$ for $E < 112.3$ and so it make some sense (here $\alpha = 0.4$.)

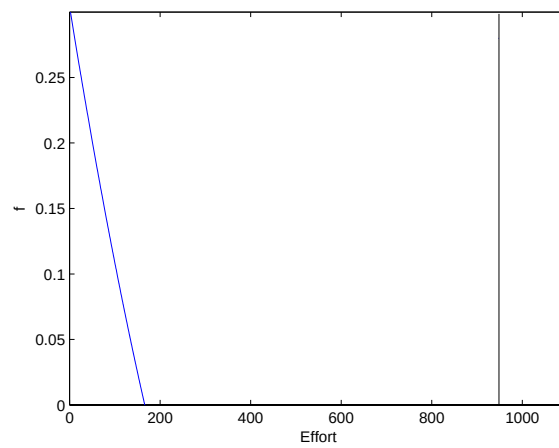


Figure 4. This figure shows that $f(E) > 0$ for $E < 166.2$ and so it make some sense (here $\alpha = 0.6$)

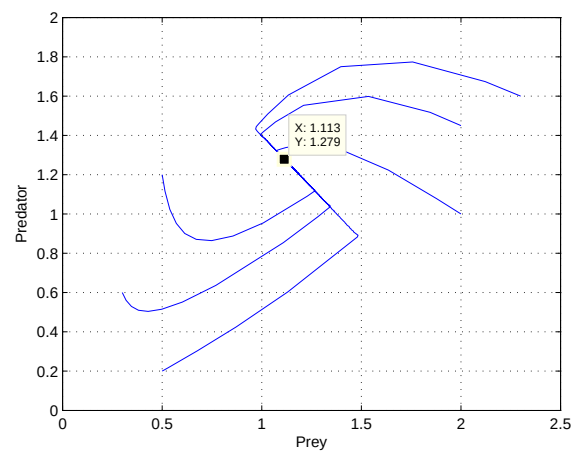


Figure 5. Phase plane trajectories with $\alpha = 0.4$.

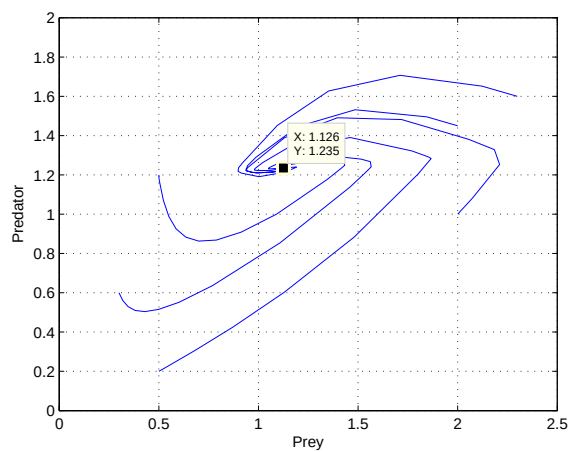


Figure 6. Phase plane trajectories with $\alpha = 0.6$.

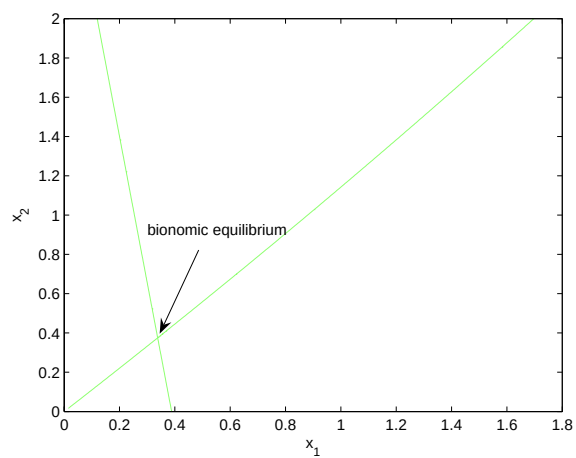


Figure 7. Phase diagram showing the unique bionomic equilibrium exists for $\alpha = 0.4$.

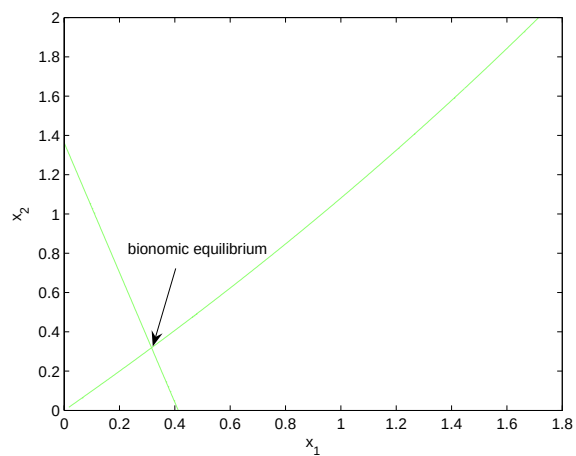


Figure 8. Phase diagram showing the unique bionomic equilibrium exists for $\alpha = 0.6$.

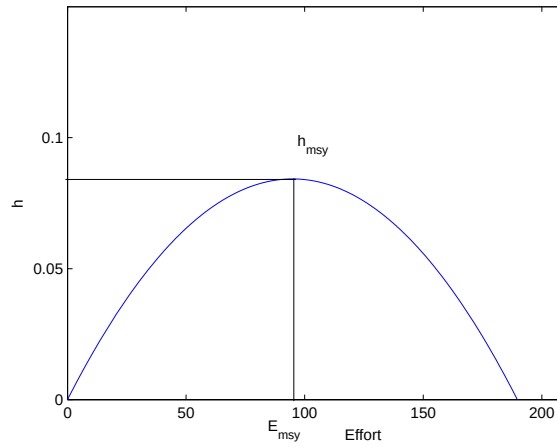


Figure 9. The Maximum Sustainable Yield (MSY) for $\alpha = 0.4$. Here $E_{MSY} = 95.10$ units and $h_{MSY} = 0.084$.

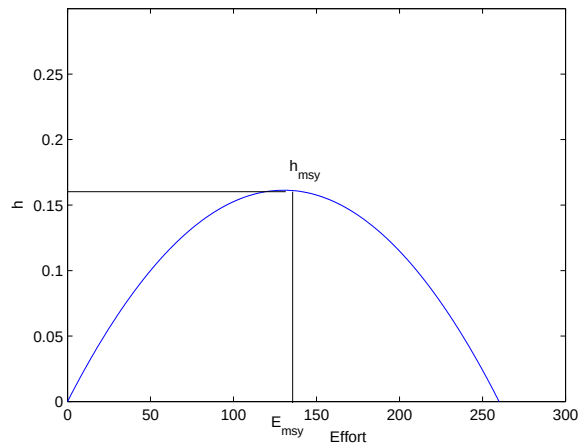


Figure 10. The Maximum Sustainable Yield (MSY) for $\alpha = 0.6$. Here $E_{MSY} = 129.6$ units and $h_{MSY} = 0.16$.

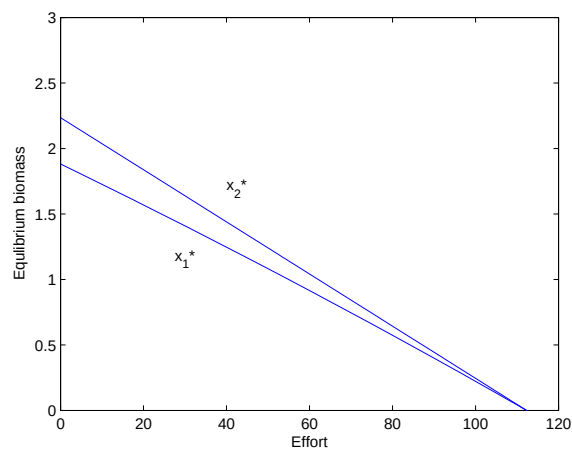


Figure 11a.

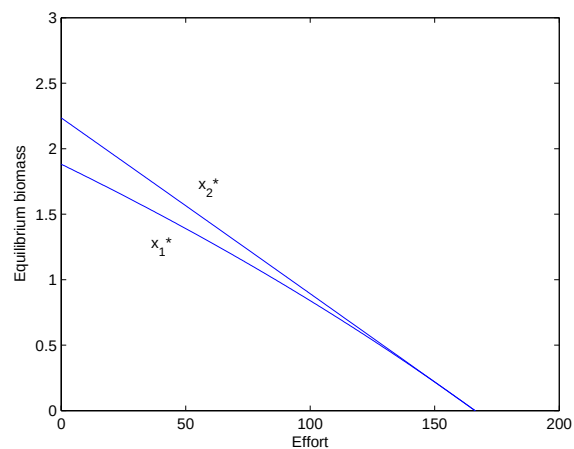
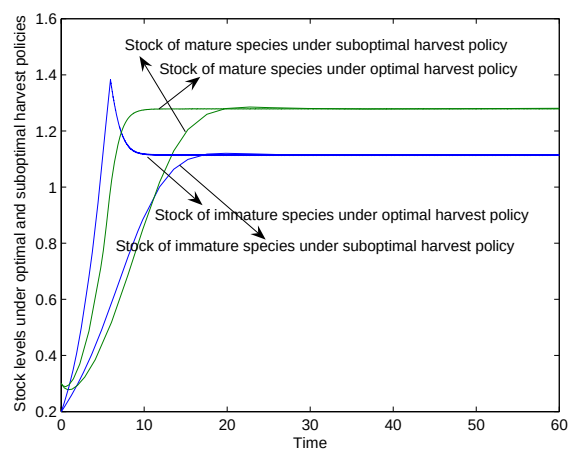
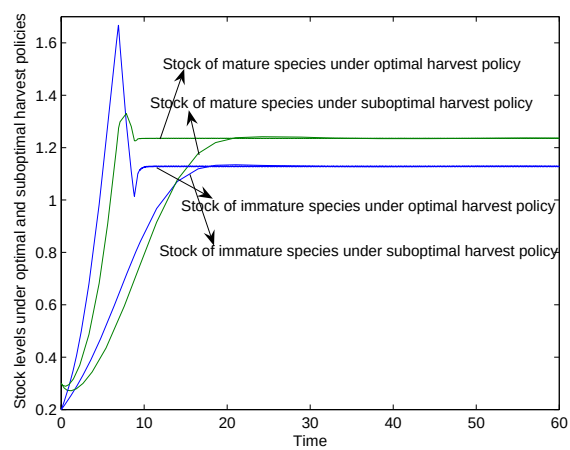


Figure 11b.

Figure 12. Optimal and suboptimal approach paths for $\alpha = 0.4$ Figure 13. Optimal and suboptimal approach paths for $\alpha = 0.6$