Journal of Mathematics Research; Vol. 4, No. 3; 2012
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

The Upper Bound of Transitive Index of Reducible Tournaments

Xiaogen Chen
School of Information Science and Technology, Zhanjiang Normal University
Zhanjiang 524048, Guangdong, China
Tel: 86-759-318-3739 E-mail: oamc@163.com

Kejian Wu
School of Mathematics and Computational Science, Zhanjiang Normal University
Zhanjiang 524048, Guangdong, China
Tel: 86-759-317-4793 E-mail: KejianWu@163.com

Received: April 5, 2012 Accepted: April 19, 2012 Online Published: May 29, 2012
doi:10.5539/jmr.v4n3p116 URL: http://dx.doi.org/10.5539/jmr.v4n3p116

The research is financed by the National Natural Science Foundation of Guangdong (No. 9151027501000039)

Abstract

In this paper, we show the upper bound of transitive index of reducible tournaments and prove that this upper bound is sharp.

Keywords: Reducible tournaments,Transitive index, Boolean matrix, Primitive exponent

1. Introduction

Let $V = \{v_1, v_2, \cdots, v_n\}$ be a finite set with $n\ (> 1)$ distinct elements. A binary relation on V is defined by a subset R of $V \times V$. The set of all binary relations on V (including the empty relation) is denoted by $\mathfrak{R}_n(V)$. A Boolean matrix is a matrix over the binary Boolean algebra $\{0, 1\}$, where the (Boolean) addition and (Boolean) multiplication in $\{0, 1\}$ are defined as $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$, respectively (we assume $0 < 1$). Let $\mathfrak{B}_n$ denote the set of all $n \times n$ matrices over the Boolean algebra $\{0, 1\}$. The map

$$R \longrightarrow B(R) = (a_{ij}),$$

where $a_{ij} = 1$ if $(v_i, v_j) \in R$ and $a_{ij} = 0$ otherwise, is an isomorphism from $\mathfrak{R}_n(V)$ to $\mathfrak{B}_n$.

Let $D = (V, E)$ be a digraph. Elements of V are referred as vertices and those of E as arcs. In this paper, digraphs are all finite, and loops are permitted but no multiple arcs. Let $\mathfrak{D}_n(V)$ be the set of all such digraphs. Then each matrix in $\mathfrak{B}_n$ can be regarded as the adjacency matrix of $D \in \mathfrak{D}_n(V)$, and each digraph in $\mathfrak{D}_n(V)$ can be regarded as the associated digraph of $A \in \mathfrak{B}_n(V)$. It is well known that there is a bijection between $\mathfrak{R}_n(V)$, $\mathfrak{B}_n$ and $\mathfrak{D}_n(V)$:

$$R \longleftrightarrow B(R) \longleftrightarrow D(R),$$

where $D(R)$ is the graph mapping to the matrix $B(R)$.

In $\mathfrak{R}_n(V)$ a multiplication can be introduced. Let $R_1, R_2 \in \mathfrak{R}_n(V)$. Then $(x, y) \in R_1R_2$ if there is a $z \in V$ such that $(x, z) \in R_1$ and $(z, y) \in R_2$.

A binary R is called transitive if $R^2 \subseteqq R$. $t(R)$ denote the least integer $s \geqq 1$ such that R^s is transitive, i.e. $R^{2s} \subseteqq R^s$. Such a number exists (Schwarz, S., 1970). Let $R \in \mathfrak{R}_n(V)$, $B(R)$ is the matrix corresponding to R. $B(R)$ is called transitive if R is transitive. $t(R)$ is transitive index of $B(R)$ and denoted by $t(B(R))$. It is easy to show that $B \in \mathfrak{B}_n$ is transitive if and only if $B^2 \leq B$. Let $D \in \mathfrak{D}_n(V)$ be the associated digraph of $B \in \mathfrak{B}_n(V)$. D is called transitive if B is transitive, and $t(B)$ is transitive index of D and denote by $t(D)$. Using matrix theoretic techniques the study $t(D)$ can now be turned into the study $t(B)$.

In 1970, Schwarz introduced a concept of the transitive index and gave some results.

For $B \in \mathfrak{B}_n$, if there is a permutation matrix P such that $PBP^T = A$, then we say that B is permutation similar to a matrix A (written $B \sim A$). It is well-known that $B \sim A$ if and only if $D(B)$ is isomorphic to $D(A)$.

A matrix $B \in \mathfrak{B}_n$ is reducible if $B \sim \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$, where B_1 and B_2 are square (non-vacuous). B is irreducible if it is not reducible. A matrix of order 1 is always irreducible. A digraph $D = (V, E)$ is said to be strongly connected (or strong) if there exists a path from u to v for all $u, v \in V(D)$. It is well know that B is irreducible if and only if its associated digraph $D(B)$ is strongly connected.

A Boolean matrix $B \in \mathfrak{B}_n$ is primitive if $B^k = J$ for some positive integer k, where J is the matrix of all $1'$s and the least integer k is called the primitive exponent of B and denoted by $\gamma(B)$. Let $D = (V, E) \in \mathfrak{D}_n(V)$, $u, v \in D = (V, E)$. A walk from u to v is a sequence of not necessarily distinct vertices $u, u_1, \cdots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \cdots, (u_p, v)$. A path is a walk with distinct vertices. A digraph $D = (V, E) \in \mathfrak{D}_n(V)$ is primitive if there exists a positive integer k such that there is a walk of length k from u to v for all $u, v \in V(D)$. The least integer k is called the exponent of D and denoted by $\gamma(D)$.

A tournament is an orientation of a complete graph. The adjacency matrix of tournament is called tournament matrix. Let $\mathfrak{T}_n$ be the set of all tournaments. $T_n \in \mathfrak{T}_n$ is reducible (or irreducible) if the adjacency matrix of T_n is reducible (or irreducible). Notice that a tournament matrix A_n satisfies the equation

$$A_n + A_n^T = J_n - I_n$$

where J_n is the matrix of all $1'$s and I_n is the identity matrix.

If a tournament matrix has a certain property (e.g. reducible), then we shall say that the tournament defined by the matrix also has the property. Tournament properties have been investigated in Ryser, H. J. (1964), Richard A. Brualdi (2006), Bondy, J. A. and Murty, U. S. R. (1976), Zhou Bo and Shen Jian (2002) and Xuemei Ye (2007).

2. Preliminaries

The notation and terminology used in this paper will basically follow those in Liu Bolian (2006). For convenience of the reader, we will include here the necessary definitions and basic results in Moon, J. W. and Pullman, N. J. (1970) and Xuemei Ye (2007). In this paper, digraphs are all finite, and loops are permitted but no multiple arcs.

Let $\mathbb{T} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathbf{T}_l = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l \times l}$, $T^\star_{3m} = \begin{pmatrix} \mathbb{T} & 0 & \cdots & 0 \\ J & \mathbb{T} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \mathbb{T} \end{pmatrix}$, $I^\star_{3m} = \begin{pmatrix} I_3 & 0 & \cdots & 0 \\ J & I_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & I_3 \end{pmatrix}$,
where I_3 is the identity matrix of order 3.

Lemma 2.1 (Richard A. Brualdi, 2006) *Let $A \in \mathfrak{B}_n$.Then*

$$A \sim \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ J & A_2 & 0 & \cdots & 0 \\ J & J & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & A_k \end{pmatrix},$$

where the diagonal blocks $A_1, \cdots, A_k$ are irreducible components of A. Let A_i be $n_i \times n_i$ matrix, $1 \le i \le k$ and $1 \le n_i \le n$. Then k and n_i are uniquely determined by A.

Lemma 2.2 (Moon, J. W. & Pullman, N. J., 1970) *Let A be $n \times n$ tournament matrix with $n \ge 4$. Then A is primitive if and only if A is irreducible.*

It is obvious that 3×3 tournament matrix is not primitive, the primitive exponent of 4×4 irreducible tournament matrix is 9. For $n > 4$, we have

Lemma 2.3 (Moon, J. W., & Pullman, N. J., 1970) *If $n \ge 5$, A_n is $n \times n$ irreducible tournament matrix, then $\gamma(A_n) \le n + 2$.*

Lemma 2.4 (Xuemei Ye, 2007) *Let $\bar{T}_n = (V, E)$ be a digraph and $V = \{1, 2, 3, \cdots, n\}$ with $n \ge 4$.$E = \{(i, i+1) \mid 1 \le i \le n-1\} \cup \{(i, j) \mid 3 \le j + 2 \le i \le n\}$, $\bar{T}_n$ is irreducible tournament. If $n \ge 5$, then $\gamma(\bar{T}_n) = n + 2$.*

Lemma 2.5 (Xuemei Ye, 2007) *Let $n \ge 5$, T_n be an irreducible tournament of order n. Then $\gamma(T_n) = n + 2$ if and only if T_n is isomorphic to $\bar{T}_n$ as Lemma 2.4.*

3. The Main Results

It is evident that if D is primitive digraph then $t(D) = \gamma(D)$. For primitive tournament T_n, its primitive exponent are determined by Moon and Pullman in Moon, J. W. and Pullman, N. J. (1970). In this paper we obtain some results on transitive index of reducible tournaments.

Theorem 3.1 *If A_n is Boolean matrix of reducible tournament with order $n(\geq 8)$, then there exists a positive integer $s \leq n+1$ such that*

$$A_n^s \sim A^\star = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ J & B_2 & 0 & \cdots & 0 \\ J & J & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & B_g \end{pmatrix}$$

where the diagonal blocks B_i is zero matrix of order l_i, $I^\star_{3q_i}$ or matrices of 1's of order $m_i(4 \leq m_i < n)$, $1 \leq i \leq g$. $0 \leq 3q_i, l_i \leq n$, and integer q_i, l_i, m_i, g are uniquely determined by A_n.

Proof. It is obvious that the irreducible tournament matrix of order 1 is zero matrix of order 1, such matrix of order 2 is not exists, and the matrix of order 3 is isomorphic to $\mathbb{T}$. Hence, the diagonal blocks A_i is zero matrix of order 1, $\mathbb{T}$ or irreducible tournament matrix of order m_i with $4 \leq m_i < n$ in Lemma 2.1.

Let $A_i \neq (0)_{1\times 1}$,$A_{i+1} = A_{i+2} = \ldots = A_{i+l_i} = (0)_{1\times 1}$ and $A_{i+l_i+1} \neq (0)_{1\times 1}$(if exists). Then

$$\begin{pmatrix} A_{i+1} & 0 & \cdots & 0 \\ J & A_{i+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{i+l_i} \end{pmatrix} = \mathbf{T}_{l_i} = \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l_i\times l_i}.$$

Let $A_j \neq \mathbb{T}$, $A_{j+1} = A_{j+2} = \ldots = A_{j+q_i} = \mathbb{T}$ and $A_{j+q_i+1} \neq \mathbb{T}$(if exists). Then

$$\begin{pmatrix} A_{j+1} & 0 & \cdots & 0 \\ J & A_{j+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{j+q_i} \end{pmatrix} = T^\star_{3q_i} = \begin{pmatrix} \mathbb{T} & 0 & \cdots & 0 \\ J & \mathbb{T} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \mathbb{T} \end{pmatrix}_{3q_i\times 3q_i}.$$

Let s be a multiple of 3 in $\{n-1, n, n+1\}$. Since A_n is a Boolean matrix of reducible tournament of order n with $n \geq 8$, ${T^\star_{3q_i}}^s = I^\star_{3q_i}$ and ${\mathbf{T}_{l_i}}^s = (0)_{l_i\times l_i}$. If A_i is irreducible tournament matrix of order m_i with $4 \leq m_i < n$ in Lemma 2.1. Then ${A_i}^s = J$. By Lemma 2.3, the conclusion established and the proof is done.

Note that $t(\mathbf{T}_n) = 1$, $t(T^\star_{3n}) = 3$, $n > 1$.

Let T_n be reducible tournament of order n, and let A_n be the adjacency matrix of T_n. Thus $A_2 \sim \mathbf{T}_2$ and $A_3 \sim \mathbf{T}_3$. And we have $t(T_2) = t(T_3) = 1$.

For T_4, it is obtained from Lemma 2.1 that $A_4 \sim \mathbf{T}_4$, $A_4 \sim \bar{A}_4 = \begin{pmatrix} 0 & 0 \\ J & \mathbb{T} \end{pmatrix}$ or $A_4 \sim \tilde{A}_4 = \begin{pmatrix} \mathbb{T} & 0 \\ J & 0 \end{pmatrix}$. Since $t(\mathbf{T}_4) = 1$ and $t(\bar{A}_4) = t(\tilde{A}_4) = 3$, we have $t(T_4) \leq 3$.

For T_5, it follow from Lemma 2.2 that $A_5 \sim \mathbf{T}_5$, $A_5 \sim \tilde{A}_5 = \begin{pmatrix} \mathbf{T}_2 & 0 \\ J & \mathbb{T} \end{pmatrix}$, $A_5 \sim \hat{A}_5 = \begin{pmatrix} \mathbb{T} & 0 \\ J & \mathbf{T}_2 \end{pmatrix}$, $A_5 \sim \bar{\bar{A}}_5 = \begin{pmatrix} \mathbf{T}_1 & 0 & 0 \\ J & \mathbb{T} & 0 \\ J & J & \mathbf{T}_1 \end{pmatrix}$, $A_5 \sim \bar{A}_5 = \begin{pmatrix} 0 & 0 \\ J & B_4 \end{pmatrix}$ or $A_5 \sim \check{A}_5 = \begin{pmatrix} B_4 & 0 \\ J & 0 \end{pmatrix}$, where B_4 is primitive tournament matrix of order 4. It is clear that $t(\mathbf{T}_5) = 1$, $t(\tilde{A}_5) = t(\hat{A}_5) = t(\bar{\bar{A}}_5) = 3$ and $t(\bar{A}_5) = t(\check{A}_5) = 9$. Thus we have $t(T_5) \leq 9$.

Similarly, $t(T_i) \leq 9$ for $i = 6, 7$. Let $\bar{A}_6 = \begin{pmatrix} \mathbf{T}_2 & 0 \\ J & B_4 \end{pmatrix}$ and $\bar{A}_7 = \begin{pmatrix} \mathbf{T}_3 & 0 \\ J & B_4 \end{pmatrix}$, where B_4 is primitive tournament matrix of order 4, and let $\bar{T}_i$ be associated digraph of $\bar{A}_i$ for $i = 6, 7$. It is easy to see that $t(\bar{T}_6) = t(\bar{T}_7) = 9$.

For $n \geq 8$, we have the follow result.

Theorem 3.2 *If $T_n (n \geq 8)$ is reducible tournament then $t(T_n) \leq n + 1$.*

Proof. Let A_n be the adjacency matrix of T_n of order n. By Theorem 3.1, there exists a positive integer $s \leq n + 1$ such that

$$A_n^s \sim A^\star = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ J & B_2 & 0 & \cdots & 0 \\ J & J & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & B_g \end{pmatrix}$$

where the diagonal blocks B_i is zero matrix of orderl_i, $I^\star_{3q_i}$ or matrices of 1's of order m_i with $m_i \geq 4$, $1 \leq i \leq g$, $0 \leq 3q_i, l_i, m_i \leq n$ and the integers q_i, l_i, m_i, g are uniquely determined by A_n. Obviously, $(A^\star)^2 \leq A^\star$, where $A^\star$ is transitive matrix. Hence $t(T_n) = t(A_n) \leq s \leq n + 1$. This completes the proof.

Theorem 3.3 *If $n \geq 8$, then there exists a reducible matrix $T_n^{(1)}$ of order n such that $t(T_n^{(1)}) = n + 1$.*

Proof. Let $T_n^{(1)} = (V, E)$ be a digraph , where $V = \{1, 2, 3, \cdots, n\}, and let E = \{(i, i+1) \mid 2 \leq i \leq n-1\} \cup \{(i, j) \mid 3 \leq j+2 \leq i \leq n\} \cup \{(2, 1)\}$, where (i, j) denote an arc from vertex i to vertex j. It is easy to check that $T_n^{(1)}$ is a reducible tournament. Using Lemma 2.5, we have that the subgraph $\tilde{T}_{n-1} = T_n^{(1)} \backslash \{1\}$ of $T_n^{(1)}$ is a primitive tournament of order $n - 1$ and $\gamma(\tilde{T}_{n-1}) = n - 1 + 2 = n + 1$.

Hence $t(T_n^{(1)}) = n + 1$. we are done.

In Theorem 3.3, the adjacency matrix of $T_n^{(1)}$ is $A_n^{(1)} = \begin{pmatrix} 0 & 0 \\ J & \tilde{A}_{n-1} \end{pmatrix}$, where

$$\tilde{A}_{n-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 0 & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

Let $\tilde{A}_n^{(1)} = \begin{pmatrix} \tilde{A}_{n-1} & 0 \\ J & 0 \end{pmatrix}$, $A_n^{(2)} = \begin{pmatrix} \mathbb{T} & 0 \\ J & \tilde{A}_{n-3} \end{pmatrix}$, and let $\tilde{A}_n^{(2)} = \begin{pmatrix} \tilde{A}_{n-3} & 0 \\ J & \mathbb{T} \end{pmatrix}$. The associated digraph of the matrices $\tilde{A}_n^{(1)}$, $A_n^{(2)}$ and $\tilde{A}_n^{(2)}$ are $\tilde{T}_n^{(1)}$, $T_n^{(2)}$ and $\tilde{T}_n^{(2)}$, respectively.

In fact, we obtain $T_n^{(2)} = (V, E)$, where $V = \{1, 2, 3, \cdots, n\}$ and $E = \{(i, i+1) \mid 4 \leq i \leq n-1\} \cup \{(i, j) \mid 3 \leq j+2 \leq i \leq n\} \cup \{(1, 2), (2, 3), (4, 3)\}$.

Theorem 3.4 *Let T_n be reducible tournament of order n with $n \geq 8$. Then we have the following results.*

(1) Let $n \equiv 0, 1 (mod\ 3)$ then $t(T_n) = n + 1$ if and only if T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

(2) Let $n \equiv 2 (mod\ 3)$ then $t(T_n) = n + 1$ if and only if T_n is isomorphic to $T_n^{(1)}$, $\tilde{T}_n^{(1)}$, $T_n^{(2)}$ or $\tilde{T}_n^{(2)}$.

Proof. Let A_n be the adjacency matrix of the graph T_n.

(1) Suppose $n \equiv 0, 1 (\text{mod } 3)$. If T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$, it follow $t(T_n) = t(T_n^{(1)}) = t(\tilde{T}_n^{(1)}) = n + 1$ from Theorem 3.3.

Conversely, suppose $t(T_n) = n + 1$. If there exists B_i that it is $I^\star_{3q_i}$ for $1 \leq i \leq g$ and $1 \leq 3q_i$, then $s = n$ if $n \equiv 0($ mod 3) and $s = n - 1$ if $n \equiv 1 (\text{mod } 3)$ in Theorem 3.1. Hence s is multiple of 3. By Theorem 3.2, $s < n + 1$ and $t(T_n) = t(A_n) \leq s < n + 1$ which is impossible. By Lemma 2.5 and Theorem 3.1, it follow $A_n \sim \begin{pmatrix} 0 & 0 \\ J & A_0 \end{pmatrix}$ or $A_n \sim \begin{pmatrix} A_0 & 0 \\ J & 0 \end{pmatrix}$, where A_0 is irreducible tournament matrix of order $n - 1$. By Lemma 2.5, we have that T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

(2) Suppose $n \equiv 2$ (mod 3). If T_n is isomorphic to $T_n^{(1)}$, $\tilde{T}_n^{(1)}$, $T_n^{(2)}$ or $\tilde{T}_n^{(2)}$, then $t(T_n) = t(T_n^{(1)}) = t(\tilde{T}_n^{(1)}) = n + 1$ by Theorem 3.3. And it is easy to verify that $t(T_n) = t(T_n^{(2)}) = t(\tilde{T}_n^{(2)}) = t(A_n) = n + 1$.

Conversely, suppose $t(T_n) = n + 1$. If there does not exist B_i that it is $I^{\star}_{3q_i}$ for $1 \le i \le g$ and $1 \le 3q_i$ in Theorem 3.1. Lemma 2.5 and Theorem 3.1 give $A_n \sim \begin{pmatrix} 0 & 0 \\ J & A_0 \end{pmatrix}$, or $A_n \sim \begin{pmatrix} A_0 & 0 \\ J & 0 \end{pmatrix}$, where A_0 is irreducible tournament matrix of order $n - 1$. Using Lemma 2.5, we have that T_n is isomorphic to $T_n^{(1)}$ or $\tilde{T}_n^{(1)}$.

If there exists B_i that it is $I^{\star}_{3q_i}$ for $1 \le i \le g$ and $1 \le 3q_i$ in Theorem 3.1. By Lemma 2.5 and Theorem 3.1, we get $A_n \sim \begin{pmatrix} \mathbb{T} & 0 \\ J & A_0 \end{pmatrix}$, or $A_n \sim \begin{pmatrix} A_0 & 0 \\ J & \mathbb{T} \end{pmatrix}$, where A_0 is irreducible tournament matrix of order $n - 3$. Lemma 2.5 give that T_n is isomorphic to $T_n^{(2)}$ or $\tilde{T}_n^{(2)}$. We are done.

References

Bondy, J. A., & Murty, U. S. R. (1976). *Graph theory with applications*. London: Macmillan.

Liu Bolian. (2006). *Combinatorial matrix theory*, 2nd ed. Bejing: Science Press.

Moon, J. W., & Pullman, N. J. (1970). On the power of tournament matrices. *Comb. theory, 3,* 1-9.

Richard A. Brualdi. (2006). *Combinatorial matrix classes*, 1st ed. New York: Cambridge University Press. http://dx.doi.org/10.1017/CBO9780511721182

Ryser, H. J. (1964). Matrices of zeros and ones in combinatorial mathematics. *Recent advances in matrix theory*, pp. 103-124. Madison: University of Wisconsin Press.

Schwarz, S. (1970). On the semigroup of binary relations on a finite set. *Czechoslovak mathematical journal, 95,* 632-670.

Xuemei Ye. (2007). Characterization of the tournament with primitive exponent reaching its secondary value. *Journal of mathematical research and exposition, 4,* 715-718.

Zhou Bo, & Shen Jian. (2002). On generalized exponents of tournaments. *Taiwanese J. Math., 6,* 565-572.