On Boundness Conditions for the Set of Feasible Points of Systems of Linear Inequalities

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Abstract

In a linear programming problem involving maximization (or minimization) of the objective function the set of feasible points is often required to be bounded above (or below). A criterion based on the simplex method which requires the constraints coefficients of the entering variable to be zero or negative for the set of feasible points to be unbounded is often used. In this paper, the necessary and sufficient conditions for the set of feasible points of the system of linear inequalities to be bounded are stated and proved. These conditions which do not require the knowledge of the entering variable are illustrated with examples.

Keywords: Bounded set, Constraint matrix, Diverging hyperplane, Feasible set, Simplex method

1. Introduction

The Boundedness of a set of feasible points of a system of linear inequalities is paramount in the solution of linear programming problems. The operations researchers always aim at developing models that would have finite optimal solutions. Unfortunately, some linear programming problems often results in an unbounded optimal feasible solution which occurs due to two major reasons, namely, when the set of feasible points is unbounded, and when the gradient of the ray directing into the cone giving the rays is orthogonal to the diverging hyperplane Eiselt, *et al.* (1987).

The existence of the unbounded optimal solutions in practical problems always indicates that a mistake has been made in modeling. It would therefore be unwise to formulate problems that has unbounded optimal solutions. For a finite optimal solution to exists, the set of feasible points is often required to be bounded above in the case of a maximization problem or bounded below in the case of a minimization problem.

The bounded set of feasible points is not always apparent, especially, when it is not possible to display the set graphically. Thus it is important for the Operations Researchers to know beforehand the conditions for the problems to be bounded. A criterion based on the simplex method which requires the constraints coefficients of the entering variable to be zero or negative for the set of feasible points to be unbounded is often used. Our interest in this paper is to derive the necessary and sufficient conditions for the system of linear inequalities to have a bounded set of feasible points.

2. Preliminaries

Consider the system of m linear inequalities in n variables $x_1,\ x_2,\cdots,x_n$. That is,

$$\left.\begin{aligned}\sum_{j=1}^{n} a_{ij}x_j \leq b_i,\ i = 1,2,\cdots,m\\ x_j \geq 0,\ j = 1,2,\cdots,n\end{aligned}\right\} \tag{1}$$

Let

$$H_i = \left\{(x_1,\ x_2,\cdots,x_n) : \sum_{j=1}^{n} a_{ij}x_j \leq b_i,\ x_j \geq 0,\ j = 1,2,\cdots,n\right\},\quad i = 1,2,\cdots,m. \tag{2}$$

be the halfspace for each constraint i. Then the set,

$$F_r = \bigcap_{i=1}^{m} H_i \tag{3}$$

may be empty or nonempty. If F_r is nonempty it defines the set of feasible points.

The set F_r is said to be bounded if there exists a finite number $c \in R$ so that

$$\|(x_1,\ x_2,\cdots,x_n)\| < c \tag{4}$$

for every $(x_1,\ x_2,\cdots,x_n) \in F_r$ (Fikhtengol's, 1979).

3. Conditions for the Set of Feasible Points to be Bounded

Consider the following linear programming problem

$$\left.\begin{aligned}minimize\ Z = \sum_{j=1}^{n} c_jx_j\\ \sum_{j=1}^{n} a_{ij}x_j \leq b_i,\ i = 1,2,\cdots,m\\ x_j \geq 0,\ j = 1,2,\cdots,n\end{aligned}\right\} \tag{5}$$

The following lemma provides the simplex criterion for the objective function of linear programming problem (5) to be unbounded.

Lemma 1 (Shapiro, 1979) *If, for a basic feasible system, there is a nonbasic variable x_s with the properties $\bar{c}_s < 0$ and $\bar{a}_{is} \leq 0$, for $i = 1,2,\cdots,m$, then the objective function of the linear programming problem can be driven to $-\infty$.*

Theorem 3.1 (Effanga, 2009) *Consider the constraint matrix*

$$A = (a_{ij}),\ i = 1,2,\cdots,m;\ j = 1,2,\cdots,n$$

If there exists a column $j*$, $1 \leq j* \leq n$, such that $a_{ij*} \leq 0$, for all $1 \leq i \leq m$, then the set of feasible points F_r is unbounded.

Proof: If $(x_1,\ x_2,\cdots,x_n) \in F_r$, then

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \forall i,\ 1 \leq i \leq m$$

Writing the above constraints as

$$\sum_{\substack{j=1\\ j\neq j*}}^{n} a_{ij}x_j + a_{ij*}x_{j*} \leq b_i, \forall i,\ 1 \leq i \leq m$$

and adding a positive number λ to x_{j*}, we obtain

$$\sum_{j\neq j*}^{n} a_{ij}x_j + a_{ij*}(x_{j*} + \lambda) \leq b_i,\ 1 \leq i \leq m$$

since $a_{ij*} \leq 0, \forall i, 1 \leq i \leq m$. Hence $(x_1, x_2, \cdots, x_{j*} + \lambda, \cdots, x_n) \in F_r \ \forall \lambda > 0$. For if F_r is bounded, there exists a number $c \in R^+$ so that

$$\|(x_1, x_2, \cdots, x_n)\| \leq c, \ \forall (x_1, x_2, \cdots, x_n) \in F_r$$

Using a P-norm defined as follows

$$\|(x_1, x_2, \cdots, x_n)\|_P = \left(\sum_{j=1}^{n} |x_j|^P \right)^{\frac{1}{P}},$$

for $P = 1$,

$$\|(x_1, x_2, \cdots, x_n)\| = \left(\sum_{j=1}^{n} |x_j| \right).$$

If $x_j \geq 0, \forall j, 1 \leq j \leq n$, then

$$\|(x_1, x_2, \cdots, x_n)\| = \sum_{j=1}^{n} x_j,$$

$$\|(x_1, x_2, \cdots, x_{j*} + \lambda, \cdots, x_n)\| = \|(x_1, x_2, \cdots, x_n)\| + \lambda \leq c + \lambda, \ \forall \lambda > 0.$$

This implies that

$$\|(x_1, x_2, \cdots, x_{j*} + \lambda, \cdots, x_n)\| \leq c,$$

or

$$\|(x_1, x_2, \cdots, x_{j*} + \lambda, \cdots, x_n)\| > c, \ \forall \lambda > 0.$$

So F_r is unbounded.

Theorem 1 is equivalent to the unboundedness criterion for the simplex algorithm in Shapiro (1979), Hillier and Lieberman (2005), Eieselt *et al.* (1987) or Taha (2005). This condition is only necessary but not sufficient for the detection of the unboundedness of the set of feasible points.

Corollary 1 *If F_r is bounded, then*

$$\max_{1 \leq i \leq m} \{a_{ij}\} > 0, \ 1 \leq j \leq n.$$

Proof: If $\max_{1 \leq i \leq m} \{a_{ij}\} \leq 0$, for any $j*$, then $a_{ij*} \leq 0$, for all $1 \leq i \leq m$. This implies that F_r is unbounded by Theorem 1. Hence, if F_r is bounded, then, $\max_{1 \leq i \leq m} \{a_{ij}\} > 0$, for all $1 \leq j \leq n$.

Example 1 Consider the set of feasibility points defined by the simultaneous linear inequalities

$$-2x_1 + x_2 \leq 2 \qquad (I)$$

$$-x_1 - 3x_2 \leq 3 \qquad (II)$$

$$x_1 \geq 0, \ x_2 \geq 0$$

The above set of feasibility points is unbounded since $a_{11} < 0$ and $a_{21} < 0$ (See figure 1)

<Figure 1>

The following example illustrates the fact that theorem 1 is not a sufficient condition for the set of feasible points to be unbounded.

Example 2 Consider the set of feasibility points defined by the simultaneous linear inequalities

$$-2x_1 + x_2 \leq 2 \qquad (I)$$

$$x_1 - 3x_2 \leq 3 \qquad (II)$$

$$x_1 \geq 0, \ x_2 \geq 0$$

The set of feasible points represented by the above system of linear inequalities is unbounded, but there is no column of the constraints matrix with negative entries throughout (see figure 2).

<Figure 2>

Example 3 Consider the set of feasible points represented by the system of linear inequalities below,

$$-2x_1 + x_2 \leq 2 \qquad (I)$$

$$x_1 - 3x_2 \leq 3 \qquad (II)$$

$$2x_1 + x_2 \leq 6 \qquad (III)$$

$$x_1 \geq 0,\ x_2 \geq 0$$

The above set of feasible points is bounded (see figure3), hence,

$$\max_{1\leq i\leq 3}\{a_{i1}\} = 2 > 0 \textit{ and } \max_{1\leq i\leq 3}\{a_{i2}\} = 1 > 0.$$

<Figure 3>

Examples 2 and 3 illustrate the fact that corollary 1 is not a sufficient condition for the set of feasible points to be bounded. We now present a necessary and sufficient condition for the set of feasible points to be bounded.

Theorem 3.2 (Effanga, 2009) The constraints, $Ax \leq b$, determines a bounded set of feasible points, if and only if there exists a $c \in R^n$, $c > 0$ and $T \in R^+$ such that $Ax \leq b \Rightarrow cx \leq T$, $\forall x \in F_r = \{x : Ax \leq b\}$.

*Proof:*Let $Ax \leq b$ be bounded, then there exists $T \in R^+$ so that

$$||x|| = \sum_{j=1}^{n} x_j \leq T,\ \forall x \in F_r$$

Thus, for $0 < c_j \leq 1$, $\forall j$, $1 \leq j \leq n$,

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} x_j \leq T.$$

For $c_j > 1$, $\forall j$, $1 \leq j \leq n$, $\exists T^* \in R^n \ni ||x|| \leq T^*$ with $T^* = cT$ and $c = \max_{1\leq j\leq n}\{c_j\}$. Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} c x_j \leq T^*.$$

Conversely, we now show that if $Ax \leq b \Rightarrow cx \leq T$, $\forall x \in F_r$, then F_r is bounded.

Without loss of generality, we assume that there exists a row A_{i*} of the constraints matrix A such that $c_j = a_{i*j} > 0$, $1 \leq j \leq n$ and $1 \leq i* \leq m$ and $b_{i*} = T$.

From the constraints $Ax \leq b$, we get

$$\left(\min_{1\leq j\leq n} a_{i*j}\right) ||x|| \leq \sum_{j=1}^{n} a_{i*j} x_j = A_{i*}x \leq b_{i*} \leq \max_{1\leq i\leq m} |b|_i$$

$$\min_{1\leq j\leq n} a_{i*j} \geq \min_{1\leq j\leq n} \min\{1, a_{i*j}\} = M$$

Hence

$$||x|| \leq \frac{\max_{1\leq j\leq m} |b_j|}{M}$$

This implies that F_r is bounded.

Theorem 2 provides a necessary and sufficient condition for the set of feasible points of linear system of inequalities to be bounded.

Example 4 Consider the set of feasibility points represented by the simultaneous linear inequalities below,

$$x_1 - x_2 \leq 2 \qquad (I)$$

$$-x_1 + 24x_2 \leq 6 \qquad (II)$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Multiplying (I) by 1.5 and adding to (II) yields,

$$0.5x_1 + 0.5x_2 \leq 9.$$

Thus,

$$Ax \leq b \Rightarrow cx \leq T,$$

where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 6 \end{pmatrix},\ c = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \text{ and T = 9.}$$

Hence, the set of feasible points is bounded (see figure 4).

<Figure 4>

Example 5 Consider the set of feasibility points represented by the simultaneous linear inequalities below,

$$x_1 - x_2 \leq 2 \quad (I)$$

$$-x_1 + 2x_2 \leq 6 \quad (II)$$

$$x_1 + x_2 \geq 3 \quad (III)$$

$$x_1 \geq 0,\ x_2 \geq 0$$

On multiplying (III) by -1 yields

$$-x_1 - x_2 \leq -3 \quad (IV)$$

Multiplying (I) by 1.5 and adding to (II) yields

$$0.5x_1 + 0.5x_2 \leq 9 \quad (V)$$

Adding (II) and (IV) we have

$$-2x_1 + x_2 \leq 3 \quad (VI)$$

Multiplying (V) by 5 and adding to (VI) we have

$$0.5x_1 + 3.5x_2 \leq 48$$

Thus,

$$Ax \leq b \Rightarrow cx \leq T,$$

where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ -1 & -1 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix},\ c = \begin{pmatrix} 0.5 & 3.5 \end{pmatrix} \text{ and T = 48.}$$

Hence, the set of feasible points is bounded (see shaded region in figure 5).

<Figure 5>

4. Conclusion

The unboundedness condition given in Theorem 1 is equivalent to the simplex criterion for the unboundedness of a set of feasible points of the system of linear inequalities. In simplex algorithm the entering variable need to be known before the unboundedness can be detected, and in most cases this will happen after a number of iterations have been performed. In our case no knowledge of the entering variable is required to determine whether the set of feasible points is unbounded or not. A necessary and sufficient condition for the system of linear inequalities to be bounded given in Theorem 2 can only be applied when system of linear inequalities are stated with a less than or equal to sign. Fortunately, any system of linear inequalities can be put in this form, thus making the condition valid for any system of linear inequalities.

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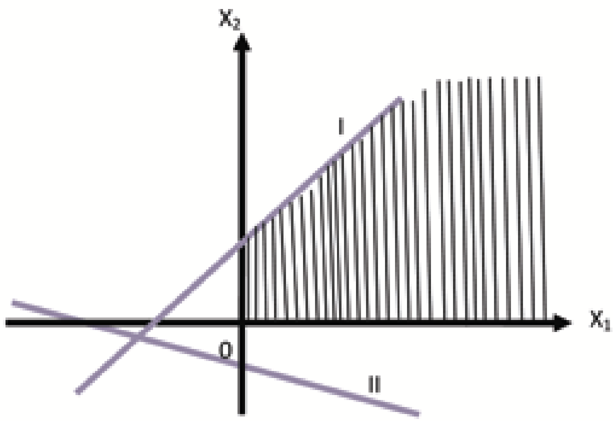

Figure 1. Showing the unbounded set of feasible points (shaded)

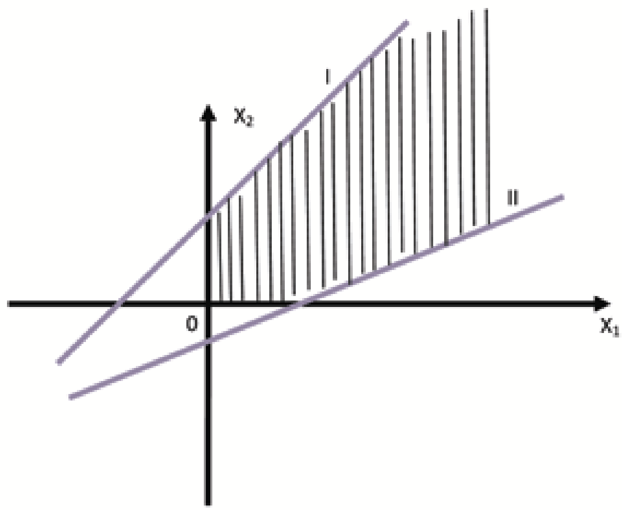

Figure 2. Showing the unbounded set of feasible points (shaded)

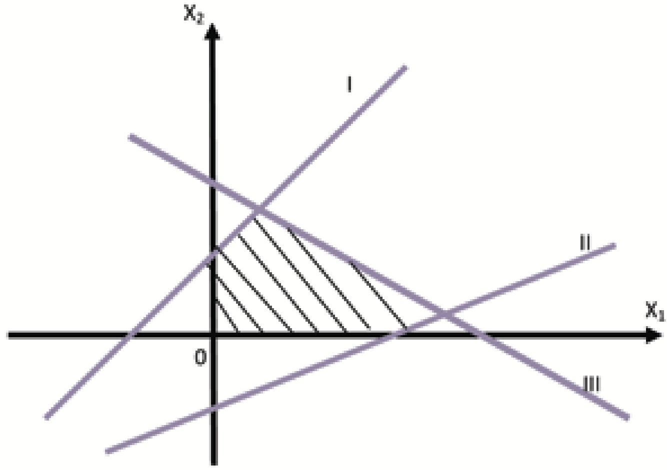

Figure 3. Showing the bounded set of feasible points (shaded)

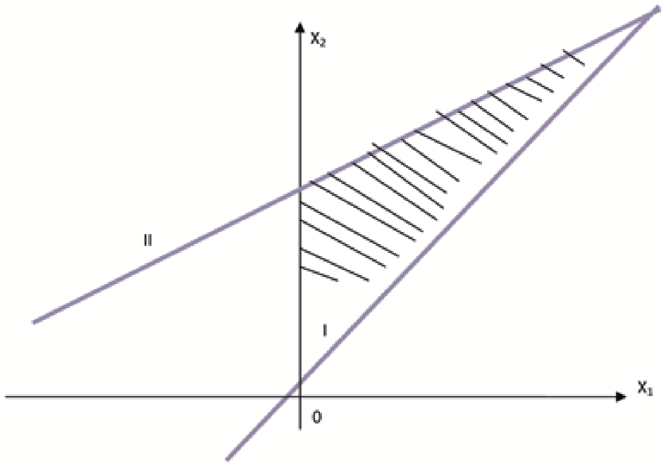

Figure 4. Showing the bounded set of feasible points (shaded)

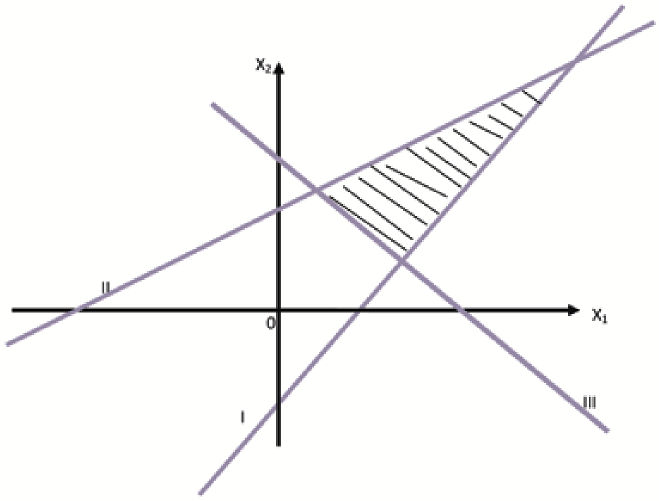

Figure 5. Showing the bounded set of feasible points (shaded)