A Class of Cayley Digraph Structures Induced by Groups

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Received: January 4, 2012 Accepted: January 19, 2012 Published: April 1, 2012

doi:10.5539/jmr.v4n2p28 URL: http://dx.doi.org/10.5539/jmr.v4n2p28

Abstract

In this paper we introduce the Cayley digraph structure. This can be considered as a generalization of Cayley digraph. We prove that all Cayley digraph structures are vertex transitive. Many graph theoretic properties are studied in terms of algebraic properties.

Keywords: Cayley digraph, Vertex-transitive graph, Hasse-diagram

AMS Subject Classification Number (2000): 05C25

1. Digraph Structure

A *binary relation* on a set V is a subset E of $V \times V$. A *digraph* is a pair (V, E) where V is a nonempty set (called vertex set) and E is a binary relation on V.

We extend the above definition as follows:

Definition 1.1 Let V be a nonempty set and let $E_1, E_2, \ldots, E_n$ be mutually disjoint binary relations on G. Then the $(n+1)$-tuple $G = (V; E_1, E_2, \ldots, E_n)$ is called a digraph structure. The elements of V are called vertices and the elements of $\cup E_i$ are called edges. In case, a digraph structure with only one binary relation is a digraph. So a digraph structure is a generalization of a digraph.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) *trivial* if $E_i = \emptyset$ for all i, (ii) *reflexive* if for all $x \in G$, $(x, x) \in E_i$ for some i, (iii) *symmetric* if $E_i = E_i^{-1}$ for all i, (iv) *transitive* if for every i and j, $E_i \circ E_j \subseteq E_k$ for some k, (v) a *hasse diagram* if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of V, $(v_i, v_{i+1}) \in \cup E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$ for all i, and (vi) *complete* if $\cup E_i = V \times V$. A walk of length k in a digraph structure is an alternating sequence $W = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$, where $e_i = (v_i, v_{i+1}) \in \cup E_i$. A walk W is called a path if all the vertices are distinct. We use notation $(v_0, v_1, v_2, \ldots, v_n)$ for the walk W. A walk is a called a circuit if its first and last vertices are the same, but no other vertex is repeated. A weak path is a sequence $(v_0, v_1, v_2, \ldots, v_n)$ of distinct vertices of G such that $(v_i, v_{i+1}) \in (\cup E_i) \cup (\cup E_i)^{-1}$. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) *connected (strongly connected)* if v is connected to u for all $u, v \in V$, (ii) *locally connected* iff for every pair of vertices $u, v \in V$ there is a path from v to u whenever there is a path from u to v, (iii) *semi connected* for every pair of vertices u, v, there is a path from u to v or a path from v to u, and (iv) weakly connected if any two vertices can be joined by a weak path, that is, the digraph structure $(V; E_1 \cup E_1^{-1}, E_2 \cup E_2^{-1}, \ldots, E_n \cup E_n^{-1})$ is connected. A weakly connected digraph structure $(V; E_1, E_2, \ldots, E_n)$ with out any circuits is called a tree.

The distance between two vertices x and y in a digraph structure G is the length of the shortest path between x and y, denoted $d(x, y)$. Let $G = (V; E_1, E_2, \ldots, E_n)$ be finite connected digraph structure. Then the diameter of G is defined as $d(G) = \max_{u,v \in G} d(x, y)$.

Two digraph structures $(V_1; E_1, E_2, \ldots, E_n)$ and $(V_2; R_1, R_2, \ldots, R_n)$ are said to be isomorphic if there exits a bijective function $f\colon V \to V$ such that $(x, y) \in \cup E_i \Leftrightarrow (f(x), f(y)) \in \cup R_i$. An isomorphism of a graph structure onto itself is called an automorphism. A graph structure $(V; E_1, E_2, \ldots, E_n)$ is said to be vertex-transitive if, given any two vertices a and b of V, there is some graph automorphism $f : V \to V$ such that $f(a) = b$. Let $(V; E_1, E_2, \ldots, E_n)$ be a graph structure and let

$v \in V$. Then the out-degree of u is $|\{v \in V : (u, v) \in \cup E_i\}|$ and in-degree of u is $|\{v \in V : (v, u) \in \cup E_i\}|$.

2. Cayley Digraph Structure

Let G be a group and S be a subset of G. The cayley digraph of G with respect to S is defined as the digraph $X = (G, E)$, where E is a binary relation on G, such that

$$(x, y) \in E \quad \text{if and only if there is some } s \in S\text{, such that } y = xs \text{ (E. Dobson, 2006)}$$

Informally, the vertices of the cayley digraphs are group elements, and two vertices are connected with an edge if and only if the second vertex is the product of an element from S and the first vertex. The cayley digraph of G with respect to S is denoted by Cay(G, S). The set S is called the connection set of Cay(G, S).

We define cayley graph structure as follows:

Definition 2.1 Let G be a group and $S_1, S_2, \ldots, S_n$ be mutually disjoint subsets of G. Then cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $X = (G; E_1, E_2, \ldots, E_n)$, where

$$E_i = \{(x, y) : x^{-1}y \in S_i\}$$

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of X. The cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ is denoted by Cay$(G; S_1, S_2, \ldots, S_n)$. In case, a digraph structure with only one connection set is the usual cayley digraph. So a cayley digraph structure is a generalization of the cayley digraph.

Examples of Cayley Digraph Structures

Example 2.1 Let $G = \mathbb{Z}$, the additive group of integers and let $S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{4\}$. Then the cayley digraph structure Cay$(G; S_1, S_2, S_3, S_4)$ is shown in figure 1.

Example 2.2 Let $G = \mathbb{Z} \oplus \mathbb{Z}$, the direct sum of the group of integers and let $S_1 = \{(1, 0)\}, S_2 = \{(0, 1)\}, S_3 = \{(1, 1)\}$. Then the cayley digraph structure Cay$(G; S_1, S_2, S_3)$ is shown in figure 2.

In this paper we may use the following notations. Let Cay$(G; S_1, S_2, \ldots, S_n)$ be a cayley digraph structure.

(1) Let A_k be the set of all k products of the form $S_i S_j \cdots S_k$. Then $[S]$ is defined as

$$[S] = \bigcup_k A_k.$$

(2) Let $A = \{S_i \cup S_i^{-1} : i = 1, 2, \ldots, n\}$ and let B_k be the set of all finite products of elements from A taken k at a time. Then we define

$$[[S]] = \bigcup_k B_k$$

Theorem 2.1 *If G is a group and let $S_1, S_2, \ldots, S_n$ are mutually disjoint subsets of G, then the cayley digraph structure Cay$(G; S_1, S_2, \ldots, S_n)$ is vertex-transitive.*

Proof: Let a and b be any two arbitrary elements in G. Define a mapping $\varphi : G \to G$ by

$$\varphi(x) = ba^{-1}x \text{ for all } x \in G.$$

This mapping defines a permutation of the vertices of Cay$(G; S_1, S_2, \ldots, S_n)$. It is also a graph automorphism. To see this, note that

$$\begin{aligned}(x, y) \in \cup E_i &\Leftrightarrow (x, y) \in E_i \text{ for some } i\\ &\Leftrightarrow x^{-1}y \in S_i \text{ for some } i\\ &\Leftrightarrow (ba^{-1}x)^{-1}(ba^{-1}y) \in S_i \text{ for some } i\\ &\Leftrightarrow (\varphi(x), \varphi(y)) \in \cup E_i.\end{aligned}$$

Also we note that $\varphi(a) = ba^{-1}a = b$. Hence Cay$(G; S_1, S_2, \ldots, S_n)$ is vertex-transitive.

Corollary 2.1 *Cay$(G; S_1, S_2, \ldots, S_n)$ is a trivial graph if and only if $S_i = \emptyset$ for all i.*

Proof: By definition, Cay$(G; S_1, S_2, \ldots, S_n)$ is trivial $\Leftrightarrow E_i = \emptyset$ for all i. This implies that $S_i = \emptyset$ for all i.

Corollary 2.2 *Cay$(G; S_1, S_2, \ldots, S_n)$ is reflexive (each vertex has a loop) if and only if $1 \in \cup S_i$.*

Proof: Assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is reflexive. Then for every $x \in G$, $(x, x) \in E_i$ for some i. This implies that $x^{-1}x \in S_i$ for some i. That is, $1 \in S_i$ for some i.

Conversely, assume that $1 \in S_i$ for some i. This implies for each $x \in G$, $(x, x) \in E_i$ for some i. That is, $(x, x) \in \cup E_i$ for all $x \in G$.

Corollary 2.3 *Cay$(G; S_1, S_2, \ldots, S_n)$ is symmetric if and only if $S_i = S_i^{-1}$ for all i.*

*Proof:*First, assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is symmetric. Let $a \in S_i$. Then $(1, a) \in E_i$. Since Cay$(G; S_1, S_2, \ldots, S_n)$ is symmetric $(a, 1) \in E_i$. This implies that $a^{-1} \in S_i$. That is $a \in S_i^{-1}$. Hence $S_i \subseteq S_i^{-1}$. Similarly, we can prove that $S_i^{-1} \subseteq S_i$.

Conversely, if $S_i = S_i^{-1}$ for all i, then we can prove that Cay$(G; S_1, S_2, \ldots, S_n)$ is symmetric.

Corollary 2.4 *Cay$(G; S_1, S_2, \ldots, S_n)$ is a transitive if and only if for every i, j, $S_iS_j \subseteq S_k$ for some k.*

Proof: First, assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is transitive. Let $x \in S_iS_j$ for some i and j. Then $x = z_1z_2$ for some $z_1 \in S_i$ and $z_2 \in S_j$. This implies that $(1, z_1) \in E_i$ and $(z_1, z_1z_2) \in E_j$. Since Cay$(G, S_1, S_2, \ldots, S_n)$ is transitive $(1, z_1z_2) \in E_k$ for some k. That is $z_1z_2 \in S_k$. Hence $S_iS_j \subseteq S_k$ for some k.

Conversely assume that for each i, j, $S_iS_j \subseteq S_k$ for some k. Let $(1, x), (x, y) \in \cup E_i$. Then $x \in S_i$ for some i and $x^{-1}y \in S_j$ for some j. This implies that $y \in S_iS_j$. Since $S_iS_j \subseteq S_k$ for some k, $(1, y) \in S_k$.

Corollary 2.5 *Cay$(G; S_1, S_2, \ldots, S_n)$ is complete if and only if $G = \cup S_i$.*

*Proof:*Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is complete. Then for every $x \in G$, we have $(1, x) \in \cup E_i$. This implies that $x \in S_i$ for some i. This implies that $G = \cup S_i$.

Conversely, assume that $G = \cup S_i$. Let x and y be two arbitrary elements in G such that $y = xz$. Then $z \in G$. This implies that $z \in S_i$ for some i. That is, $(1, z) \in \cup E_i$. That is $(x, xz) = (x, y) \in \cup E_i$. This shows that Cay$(G; S_1, S_2, \ldots, S_n)$ is complete.

Corollary 2.6 *Cay$(G; S_1, S_2, \ldots, S_n)$ is a union of complete graphs if and only if each S_i is a sub group of G.*

Corollary 2.7 *Cay$(G; S_1, S_2, \ldots, S_n)$ is connected if and only if $G = [S]$.*

Proof: Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is connected and let $x \in G$. Let

$$(1, y_1, y_2, \ldots, y_n, x)$$

be a path leading from 1 to x. Then we have,

$$y_1 \in S_i, y_1^{-1}y_2 \in S_j, \cdots, y_n^{-1}x \in S_k$$

This implies that $x \in A$ for some $A \in [S]$. Since x is arbitrary, $G = [S]$.

Conversely, assume that $G = [S]$. Let $x \in G$. Then $x \in S_iS_j \cdots S_k$ for some $i, j, \ldots$ and k. This implies that $x = s_is_j \ldots s_k$ for some $i, j \ldots$ and k. Then clearly, $(1, s_i, s_is_j, \ldots, s_is_j \ldots s_k)$ is a path from 1 to x. Hence Cay$(G; S_1, S_2, \ldots, S_n)$ is connected.

Corollary 2.8 *Cay$(G; S_1, S_2, \ldots, S_n)$ is weakly connected if and only if $G = [[S]]$*

Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is weakly connected. Let $x \in G$. Then there exists a weak path say:

$$(1, x_1, x_2, \ldots, x_n, x)$$

from 1 to x. This implies that

$$\begin{aligned} &x_1 \in S_i \cup S_i^{-1} \text{ for some } i \\ &x_1^{-1}x_2 \in S_j \cup S_j^{-1} \text{ for some } j \\ &\vdots \\ &x_n^{-1}x \in S_k \cup S_k^{-1} \text{ for some } k \end{aligned}$$

This implies that $x \in [[S]]$. Since x is arbitrary, $G = [[S]]$.
Conversely, assume that $G = [[S]]$. Let x and y be any two elements in G. Then the equation $y = xz$ has a unique solution $z \in G$. This implies that $z \in (S_i \cup S_i^{-1})(S_j \cup S_j^{-1}) \cdots (S_k \cup S_k^{-1})$ for some $i, j, \ldots, k$. That is

$$z = x_1x_2x_3 \ldots x_k$$

where $x_i \in (S_i \cup S_i^{-1})$. This implies that

$$(1, x_1, x_1x_2, x_1x_2x_3, \ldots, x_1x_2x_3 \ldots x_k)$$

is a weak path from 1 to z. That is

$$(x, xx_1, xx_1x_2, xx_1x_2x_3, \ldots, xx_1x_2x_3 \ldots x_k)$$

is a weak path from x to y. Hence G is weakly connected.

Corollary 2.9 *$Cay(G; S_1, S_2, \ldots, S_n)$ is a quasi ordered set if and only if*

$$(i) 1 \in S_1 \cup S_2 \cdots \cup S_n,$$
$$(ii) \text{for every} (i, j), S_iS_j \subseteq S_k \text{ for some } k.$$

Corollary 2.10 *$Cay(G; S_1, S_2, \ldots, S_n)$ if a partially ordered set if and only if*

$$(i) 1 \in S_1 \cup S_2 \cdots \cup S_n,$$
$$(ii) \text{for every} (i, j), S_iS_j \subseteq S_k \text{ for some } k,$$
$$(iii) \cup (S_i \cap S_i^{-1}) = \{1\}$$

Corollary 2.11 *Let A_n is the set of n products of the form $S_{i_1}S_{i_2} \cdots S_{i_n}$. Then $Cay(G; S_1, S_2, \ldots, S_n)$ is a hasse- diagram if and only if $C \cap S_i = \emptyset$ for all i and for all $C \in A_n$.*

Proof: Suppose the condition holds. Let $x_0, x_1, \ldots, x_n$ be $(n+1)$ elements in G such that $(x_i, x_{i+1}) \in \cup E_i$ for $i = 0, 1, \ldots, n-1$. This implies that

$$\begin{aligned} x_0^{-1}x_1 &\in S_{i_1} \text{ for some } i_1 \\ x_1^{-1}x_2 &\in S_{i_2} \text{ for some } i_2 \\ &\vdots \\ x_{n-1}^{-1}x_n &\in S_{i_n} \text{ for some } i_n \end{aligned}$$

This implies that $x_0x_n^{-1} \in S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_n} \in A_n$. Since $C \cap S_i = \emptyset$ for all i and for all $C \in A_n$, $(x_0, x_n) \notin \cup E_i$.

Conversely assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is a hasse diagram. We will show that $C \cap S_i = \emptyset$ for all i and for all $C \in A_n$. Let $S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_n}$ be any element in A_n. Let $x \in S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_n}$. Then $x = s_{i_1}s_{i_2}s_{i_3} \ldots s_{i_n}$ for some $s_{i_k} \in S_{i_k}$. This implies that

$$(1, s_{i_1}, s_{i_2}s_{i_3}, \ldots, x)$$

is a path from 1 to x. Since Cay$(G; S_1, S_2, \ldots, S_n)$ is a hasse- diagram $x \notin S_i$ for any i. That is, $A_n \cap S_i = \emptyset$ for all i.

Corollary 2.12 *The out-degree of $Cay(G; S_1, S_2, \ldots, S_n)$ is the cardinal number $|S_1 \cup S_2 \cup \cdots \cup S_n|$.*

Proof: Since Cay$(G; S_1, S_2, \ldots, S_n)$ is vertex-transitive it suffices to consider the out degree of the vertex $1 \in G$. Observe that

$$\begin{aligned} \rho(1) &= \{u : (1, u) \in E\} \\ &= \{u : u \in S_i \text{ for some } i\} \\ &= S_1 \cup S_2 \cup \cdots \cup S_n \end{aligned}$$

Hence

$$|\rho(1)| = |S_1 \cup S_2 \cup \cdots \cup S_n|.$$

Corollary 2.13 *The in-degree of $Cay(G; S_1, S_2, \ldots, S_n)$ is the cardinal number $|S_1^{-1} \cup S_2^{-1} \cup \cdots \cup S_n^{-1}|$.*

Proof: Since Cay$(G; S_1, S_2, \ldots, S_n)$ is vertex-transitive it suffices to consider the in degree of the vertex $1 \in G$. Observe that

$$\begin{aligned} \sigma(1) &= \{u : (u, 1) \in E\} \\ &= \{u : u^{-1} \in S_i \text{ for some } i\} \\ &= S_1^{-1} \cup S_2^{-1} \cup \cdots \cup S_n^{-1} \end{aligned}$$

Hence

$$|\sigma(1)| = |S_1^{-1} \cup S_2^{-1} \cup \cdots \cup S_n^{-1}|.$$

Corollary 2.14 *Cay$(G; S_1, S_2, \ldots, S_n)$ is self dual if G is commutative.*

Proof: Note that the mapping $\varphi : G \to G$ defined by $\varphi(x) = x^{-1}$ is a bijective map. Moreover

$$\begin{aligned}(x, y) \in \cup E_i &\Leftrightarrow (x, y) \in E_i \text{ for some } i \\ &\Leftrightarrow x^{-1}y \in E_i \Leftrightarrow yx^{-1} \in E_i \\ &\Leftrightarrow (y^{-1})^{-1}x^{-1} \in E_i \\ &\Leftrightarrow (y^{-1}, x^{-1}) \in E \\ &\Leftrightarrow (\varphi(x), \varphi(y)) \in E_i^{-1} \subseteq E^{-1}.\end{aligned}$$

This implies that Cay$(G; S_1, S_2, \ldots, S_n)$ is isomorphic to its dual.

Corollary 2.15 *If $G \setminus (S_1 \cup S_2 \cup \cdots \cup S_n)$ is a subgroup of G then Cay$(G; S_1, S_2, \ldots, S_n)$ is a bipartite graph structure.*

Corollary 2.16 *For $k = 1, 2, 3, \ldots$ let A_k be the set of all k products of the form $S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_k}$. If Cay$(G; S_1, S_2, \ldots, S_n)$ has finite diameter, then the diameter of Cay$(G; S_1, S_2, \ldots, S_n)$ is the least positive integer n such that*

$$G = \bigcup_{A \in A_n} A.$$

Proof: Let n be the smallest positive integer such that $G = \bigcup_{A \in A_n} A$. We will show that the diameter of Cay$(G; S_1, S_2, \ldots, S_n)$ is n. Let x and y be any two arbitrary elements in G such that $y = xz$. Then $z \in G$. This implies that $z \in A$ for some A in A_n. But then z has a representation of the form $z = s_{i_1}s_{i_2} \cdots s_{i_n}$. This implies that

$$(1, s_{i_1}, s_{i_1}s_{i_2}, \ldots, z)$$

is path of n edges from 1 to z. That is

$$(x, xs_{i_1}, xs_{i_1}s_{i_2}, \ldots, y)$$

is a path of length n from x to y. This shows that $d(x, y) \leq n$. Since x and y are arbitrary,

$$\max_{x,y \in G} d(x, y) \leq n$$

Therefore the diameter of Cay$(G; S_1, S_2, \ldots, S_n)$ is less than or equal to n. On the other hand let the diameter of Cay$(G; S_1, S_2, \ldots, S_n)$ be k. Let $x \in G$ and $d(1, x) = k$. Then we have $x \in B$ for some $B \in A_k$. That is

$$G = \bigcup_{A \in A_k} A.$$

Now by the minimality of k, we have $n \leq k$. Hence $k = n$.

Corollary 2.17 *Cay$(G; S_1, S_2, \ldots, S_n)$ is a tree if and only if*

$$\begin{aligned}&(i)\ \ G = [[S]] \\ &(ii)\ \ 1 \notin A \ \textit{for all}\ A \in A_k,\ \ k = 1, 2, 3, \cdots\end{aligned}$$

Acknowledgment

The authors would like to thank the referee for his valuable suggestions which has helped us to improve this paper.

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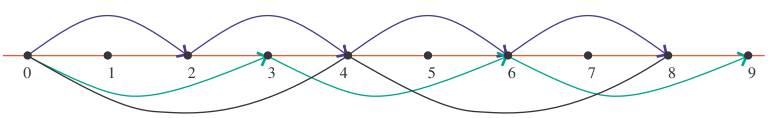

Figure 1. Cay($G = \mathbb{Z}; S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{4\}$)

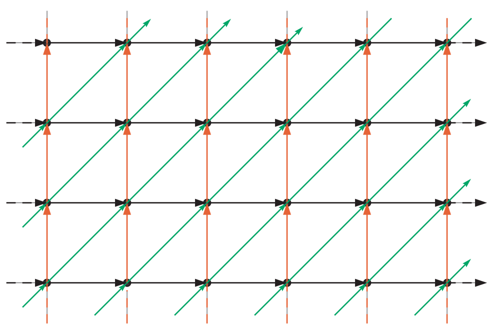

Figure 2. Cay($G = \mathbb{Z} \oplus \mathbb{Z}; S_1 = \{(1, 0)\}, S_2 = \{(0, 1)\}, S_3 = \{(1, 1)\}$)