Supersymmetric Hypermatrix Lie Algebra and Hypermatrix Groups Generated by the Dihedral Set D_3

Jacob M. Schreiber

5 Sharet street, Ramat-Gan, Israel

Tel: 972-3-673-4060 E-mail: ora1963@gmail.com

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Abstract

This work is an investigation into the structure and properties of supersymmetric hypermatrix Lie algebra generated by elements of the dihedral group D_3. It is based on previous work on the subject of supersymmetric Lie algebra (Schreiber, 2012).

In preview work I used several new algebraic tools; namely cubic hypermatrices (including special arrangements of such hypermatrices) and I obtained an algebraic structure associated with the basis of the Lie algebra sl_2, and I showed that the basis elements sl_2 are generators of infinite periodic hypermatrix Lie algebraic structures with semisimple sub-algebras. The generated algebra has been shown to be an extended Lie hypermatrix algebra that has a classical Lie algebra decomposition composed of hypermatrices with periodic properties. The generators of higher dimensional Lie algebra were shown to be special supersymmetric, anti-symmetric and certain skew-symmetric hypermatrices. The present work takes a different look at the structure of periodic hypermatrix Lie algebra by using elements generating the classical dihedral group D_3. Using cubic dihedral symmetric hypermatrices (type: even-even, odd-odd, even-odd odd-even permutation) to generate Lie hypermatrix algebra I show that the extended dihedral algebra is a Lie hypermatrix algebras with special hypermatrix group properties, semisimple, symmetric, skew-symmetric, anti-symmetric, and anti-clockwise symmetric properties.

Keywords: Anti-symmetric, Anti-clockwise-symmetry, Basis, Dihedral, Generator, Hypermatrices, Ideal, Lie algebra, Semisimplicity, Skew-symmetry, Supersymmetry

1. Introduction

In preview work a hypermatrix Lie algebra generated by the basis elements of sl_2 has been shown to be an extended Lie hypermatrix algebra that has a classical Lie algebra decomposition (Bourbaki, 1980; Humphreys, 1972; Jacobson, 1962; Serre, 1987); specifically a periodic set of Lie algebras composed of hypermatrices has been generated.

In the present work I study the structure of certain periodic hypermatrix Lie algebras by using generators of the classical dihedral group D_3 structured from the simultaneously turning and reflecting pairs of triangles and relations among hypermatrices. Using cubic dihedral symmetric hypermatrices I show that a dihedral D_3 hypermatrices Lie algebra can be extended to higher dimension Lie hypermatrix algebras with special semisimple, symmetric, skew-symmetric, antisymmetric, and anti-clockwise symmetric properties.

2. The HD$_3$ Lie algebra of Two Triangles

The group elements associated with the reflection, and turning of a triangle or its permutation are classically represented by using the following set of matrices:

$$t_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, t_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, t_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, t_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, t_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Under matrix multiplication it has the non-commutative classical group structure shown in <Table 1>.

2.1 The first extension of D_3

We consider the first extension HD_3 Lie hypermatrix algebra (defined below and see also; Schreiber, 2012) of simultaneously turning and reflecting 36 pairs of triangles

The triangles are represented by pairs of matrices situated on the back and front of hypermatrix $W_{i,j}$, such that:

$$W_{1;1} = \begin{pmatrix} t_1 \\ t_1 \end{pmatrix},\ W_{1;2} = \begin{pmatrix} t_2 \\ t_1 \end{pmatrix},\ W_{1;3} = \begin{pmatrix} t_3 \\ t_1 \end{pmatrix},\ W_{1;4} = \begin{pmatrix} t_4 \\ t_1 \end{pmatrix},\ W_{1;5} = \begin{pmatrix} t_5 \\ t_1 \end{pmatrix},\ W_{1;6} = \begin{pmatrix} t_6 \\ t_1 \end{pmatrix},$$

and in general all 36 hypermatrices are represented by the relations

$$W_{i;j} = \begin{pmatrix} t_j \\ t_i \end{pmatrix},\ \ i = 1, ..., 6; j = 1, ..., 6.$$

We label each hypermatrix even-even, even-odd, odd-even, odd-odd according to the number of permutations in each W_i-hypermatrix components. The matrix t_1 is even, t_2 is even, t_3 is odd, t_4 is even, t_5 is odd, and t_6 is odd. For example t_6 has the permutation $(2, 1, 3)$ with one entry smaller than the first and the sub permutation $(1, 3)$ with no smaller entry than the first element (2), hence t_6 has an odd permutation. Accordingly, $W_{1,1}$ is an even-even permutation structured from two identical even permutation matrices. We sum up the 36 elements in four sets, $W_{even-even}$, $W_{odd-odd}$, $W_{even-odd}$, $W_{odd-even}$ as follows:

$$\begin{aligned} W_{even-even} &= \{W_{1;1}, W_{2;2}, W_{4;4}, W_{1;4}, W_{4;1}, W_{2;4}, W_{4;2}, W_{1;2}, W_{2;1}\} \\ W_{odd-odd} &= \{W_{3;3}, W_{5;5}, W_{6;6}, W_{3;5}, W_{5;3}, W_{3;6}, W_{6;3}, W_{5;6}, W_{6;5}\} \\ W_{even-odd} &= \{W_{1;3}, W_{1;5}, W_{1;6}, W_{2;3}, W_{2;5}, W_{2;6}, W_{4;3}, W_{4;5}, W_{4;6}\} \\ W_{odd-even} &= \{W_{3;1}, W_{5;1}, W_{6;1}, W_{2;3}, W_{5;2}, W_{6;2}, W_{3;4}, W_{5;4}, W_{6;4}\} \end{aligned} \tag{1}$$

The even elements in D_3 are represented by t_1, t_2, t_4 and the odd elements by t_3, t_5, t_6. In general for even elements we have a commutation relation $e_i e_j = e_j e_i$ characteristics of the alternating even elements in the symmetric subgroup of S_3, see also <Table 1>.

Definition 1 Lie Algebra of Hypermetrices (Schreiber, 2012). Consider the space {W} over a field F, with an operation $WW \in W^*$. Note that W^* is the first extension, e.g., if $W_{i,j}$ is a two sheet hypermatrix W^* is a 4-sheet hypermatrix.

Denote by (W_i, W_j) the hypermatrix Lie bracket over F; the set $\{W^*\}$ constitutes a Lie hypermatrix algebra if the following conditions are satisfied:

A) $WW \in W^*$ where $(W_{si}, W_{sj}) \in W^*$,W_{si} a component sheet of W^{*k}, i.e., $(W, W) \in \{\times, +, -, W^*\} \in L_{inear}\{W^*\}$ - a linear combination in W_i^* sheets.

B) 1) the bracket operation is bilinear.

2) $(W, W) = 0^*$ for all $W \in \{W\}$.

C) $(W_i, (W_j, W_k)) + (W_j, (W_k, W_i)) + (W_k, (W_i, W_j)) = 0^{**}, \forall W_i, W_j, W_k \in \{W\}$. $**$ - is the second extension under hypermatrix multiplication.

The hypermatrix algebra has to be closed in terms of its components, and with respect to the field operations, in the sense that the component sheets $\{W_{si} \in W\}$ are well defined in the extended space, under the bracket operation $\{WW \in W^*\}$.

Next I describe some of the bracket products in the hypermatrix dihedral set $(W_{i,j}, W_{k,l})$ in terms of triangle permutation properties (even or odd permutation) and symmetric characteristics of the hypermatrix products.

2.2 Odd permutation

For the Lie bracket operation of the odd-odd components e.g., hypermatrices $W_{3,3}, W_{3,5}$ we have

$$(W_{3;3}, W_{3;5}) = W_{3;3}W_{3;5} - W_{3;5}W_{3;3} = \begin{pmatrix} & t_3t_3 & \\ t_3t_5 & & t_3t_5 \\ & t_3t_3 & \end{pmatrix} - \begin{pmatrix} & t_5t_3 & \\ t_3t_3 & & t_5t_5 \\ & t_3t_3 & \end{pmatrix} = \begin{pmatrix} & (t_3t_3 - t_5t_3) & \\ (t_3t_5 - t_3t_3) & & (t_3t_5 - t_5t_5) \\ & (0) & \end{pmatrix}$$

$$= \begin{pmatrix} & \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} & \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} & & \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \\ & (0) & \end{pmatrix}$$

In short

$$(W_{3;3}, W_{3;5}) = W_{0,t_4-t_1,t_1-t_2,t_4-t_2} \tag{2}$$

Sub indices such as W_{t4-t1} indicate the difference of $t4 - t1$ matrices of D_3. We see that $(W_{3;3}, W_{3;5})$ has partial sheet symmetry in one central plane direction such that $(W_{3;3}, W_{3;5})^t = (W_{3;3}, W_{3;5})$ for components W_2 and W_4 in $(W_{3;3}, W_{3;5})$ there is symmetry about the hypermatrix horizontal line.

2.3 Even-even permutation and anti-symmetric sheet arrangement

$$(W_{4;2}, W_{2;4}) = W_{4;2} \times W_{2;4} - W_{2;4} \times W_{4;2} = \left(\begin{array}{ccc} & \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} & \\ \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{pmatrix} & & (0) \\ & (0) & \end{array}\right)$$

$$(W_{4;2}, W_{2;4}) = W_{0,t_2-t_4,t_4-t_2,0} \tag{3}$$

We see that $(W_{4;2}, W_{2;4})$ has trace zero and it has partial sheet skew symmetry in one central plane direction so that $(W_{4;2}, W_{2;4})^t = -(W_{4;2}, W_{2;4})$ for inherent sheet components. Sheet elements W_{s2} and W_{s4} are skew-symmetric and have trace zero.

2.4 Odd permutation and anti-symmetric sheet arrangement

$$(W_{3;5}, W_{5;3}) = W_{3;5} \times W_{5;3} - W_{5;3} \times W_{3;5} = \left(\begin{array}{ccc} & (0) & \\ (0) & & \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} \\ & \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix} & \end{array}\right)$$

$$(W_{3;5}, W_{5;3}) = W_{t_2-t_4,0,0,t_4-t_2} \tag{4}$$

We see that $(W_{3;5}, W_{5;3})$ has trace zero and it has partial sheet skew symmetry in one central plane direction so that $(W_{3;5}, W_{5;3})^t = -(W_{3;5}, W_{5;3})$ for the sheet components. W_1 and W_4 have trace zero and are skew-symmetric.

2.5 Even-odd permutation and anti-symmetric sheet arrangement

$$(W_{2;5}, W_{5;2}) = W_{2;5} \times W_{5;2} - W_{5;2} \times W_{2;5} = \left(\begin{array}{ccc} & \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{pmatrix} & \\ \begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & -1 & 1\\ 1 & 0 & -1\end{pmatrix} & & \begin{pmatrix}0 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & -1\end{pmatrix} \\ & \begin{pmatrix}0 & 1 & -1\\ 1 & -1 & 0\\ -1 & 0 & 1\end{pmatrix} & \end{array}\right) \text{skew} - \text{symmetric}$$

$$(W_{2;5}, W_{5;2}) = W_{t_6-t_3,t_4-t_1,t_1-t_4,t_3-t_6} \tag{5}$$

We see that $(W_{2;5}, W_{5;2})$ has trace zero components and it has partial sheet skew symmetry in one plane direction so that $(W_{2,5}, W_{5,2})^t = -(W_{2,5}, W_{5,2})$ for inherent sheet components.

2.6 Even-odd sheet arrangement

For the Lie bracket operation of the even-odd components hypermatrices $W_{1;3}$, and $W_{1;5}$ we have

$$(W_{1;3}, W_{1;5}) = W_{1;3} \times W_{1;5} - W_{1;5} \times W_{1;3} = \left(\begin{array}{ccc} & \begin{pmatrix}-1 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & 0\end{pmatrix} & \\ \begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 1\\ -1 & 1 & 0\end{pmatrix} & & \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} \\ & (0) & \end{array}\right)$$

$$(W_{1;3}, W_{1;5}) = W_{0,t_5-t_3,t_3-t_5,t_4-t_2} \tag{6}$$

We see that $(W_{1;3}, W_{1;5})$ has trace zero. Sheets W_{s2} and W_{s3} are symmetric, and Ws4 is skew-symmetric $W_{s1} = 0$.

The multiplication table for the 1296 hypermatrices of D_3^* are represented here according to the evenness and oddness of the permutation of the triangles and the Lie bracket characteristics; it has certain algebraic symmetric characteristics summarized in table 2 for the multiplication of the two triangle hypermatrices and in table 3 for the two triangle anti-symmetric hypermatrix product, <Table 2>, <Table 3>.

Theorem 1 *A1) Lie bracket products of dihedral antisymmetric D_3 hypermatrices of type (even, even) in the two triangle configuration generate trace-zero hypermatrices with skewsymmetric sheet arrangements of the hypermatrices.*

A2) The Lie bracket products of dihedral antisymmetric D_3 hypermatrices of type (even, even) as in the two triangle configuration, generate a semisimple hypermatrix Lie algebra.

B1) The Lie Bracket products of dihedral antisymmetric D_3 hypermatrices of type (odd, odd) result in trace-zero Lie hypermatrices with skewsymmetric sheet arrangements of the hypermatrices.

B2) The Lie bracket products of dihedral antisymmetric D_3 hypermatrices of type (odd, odd) generate a semisimple hypermatrix Lie algebra.

Proof: A1) Let the hypermatrices be composed of even antisymmetric sheets such that, written here horizontally, $(W(e_i, e_j),\ W(e_j, e_i)) = W(e_ie_j - e_je_i, e_ie_i - e_je_j, e_je_j - e_ie_i, e_je_i - e_ie_j)$ if I-identity element is not among the $e_i - s'$ the differences, by table 1, in W are either $t_4 - t_2$ or $t_2 - t_4$ therefore the result is $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ and since the even sheet elements commute two of the above sheet W_{S_e} entries commute and will vanish while the other two are related by $W_{si} = -W_{sj}$ and are skewsymmetric as hypermatrix sheets. If the identity element is among the $e_i - s'$ only the sign of the sheet will change and, therefore, the result follows.

A2) By Schreiber (2012) for antisymmetric matrices we had rules related to the classical semisimpe Lie algebra

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the antisymmetric hypermatrices with nilpotent components

$$(t_4, t_2) = \left[\begin{pmatrix} -y \\ X \end{pmatrix}, \begin{pmatrix} -x \\ Y \end{pmatrix} \right]$$

and the antisymmetric hypermatrices with nilpotent and symmetric components

$$(t_8, t_6) = \left[\begin{pmatrix} h \\ Y \end{pmatrix}, \begin{pmatrix} y \\ h \end{pmatrix} \right]$$

Generally we have: $(W_{f(h,y,x)}, W_{\otimes f(h,y,x)}) = W^*_{trace\ zero\ hyper-matrix}$ where W^* is a trace zero hypermatrix resulting from the product of nilpotent and trace zero elements in antisymmetric hypermetrics arrangement.

Here $\otimes$ stands for antisymmetric arrangement of any cubic sheet arrangement of the basis elements $f(h, y, x)$. With a mapping, for example $(h, y) \to (y, h)$, $(h, y, x) \to (y, h, x)$, $(h, y, x, 5h - x) \to (y, h, 5h - x, x)$ etc'.

In conclusion we may sum up with rule 2 stating that:

$$\{(WH_{f(h,y,x)}, WH_{\otimes f(h,y,x)})\} = \{WH_{semi\ skew-symmetric\ hypermatrixalgebra}\} \tag{7}$$

The Lie product of hypermetrics (composed of h, y, x elements) set in antisymmetric hypermatrices, result in skew-symmetric trace zero hypermatrices, or semi skew-symmetric hypermatrices. The generalization to any size hypermatrices follows by induction on dimension.

In order to show that the resulting Lie algebra is semisimple it is required to show that the hypermatrix Lie algebra $\{W\}$ has no proper abelian ideals other then $\{0\}$.

Since the resulting products of antisymmetric even-even hypermatrices have skew-symmetric sheet arrangement the hypermatrices do not commute and the only ideals are the $\{0\}$ ideal, therefore, the resulting Lie algebra is a semisimple Lie algebra.

B1) Although the odd elements do not commute their product results in an even element (see table 1), therefore, the argument for part B1 is similar to the argument in part A1 and the proof follows in similar lines to those in part A1.

B2) Although the odd elements do not commute their product results in an even element (see table 1), therefore, the argument for part B2 is similar to the argument in part A2 and the proof follows in similar lines to those of part A2. The resulting products constitute semisimple hypermatrix Lie algebra (schriber, 2012).

3. Symmetric Hypermatrices Dihedral Type

The symmetric set of hypermatrices $W_{symmetric} = \{W_{1,1}, W_{2,2}, W_{3,3}, W_{4,4}, W_{5,5}, W_{6,6}\}$ composed from the dihedral group elements $t_i,\ i = 1, ..., 6$ has special unique properties.

Under bracket multiplication it has the non-commutative classical group structure as shown in <Table 4>.

The matrices with even permutation elements (t_1, t_4, t_2) constitute a normal alternating subgroup of the dihedral group D_3. Therefore, the hypermatrices with even-even permutation elements $\{W_{1,1}, W_{2,2}, W_{4,4}\}$ constitute a normal alternating hypermatrix subgroup (defined below) of the dihedral Lie hypermatrix algebra $D_3^*\{W_{1,1}, W_{2,2}, W_{3,3}, W_{4,4}, W_{5,5}, W_{6,6}\}_{symmetric}$. The Lie algebra D_3^* has trace zero, and it is skew symmetric. By definition D_3^* is the extended Lie hypermatrix algebra of D_3, while $D_n^{*\times n}$ is the nth-extended Lie hypermatrix algebra of D_n.

3.1 Lie supersymmetric hypermatrix group

Definition 2 A multiplicative cubic-hypermatrix group is a set $\{W\}$ of hypermatrices with a multiplicative operation and the following properties:

a) Closure in the extended hypermatrix set. $W_i \times W_j \in \{W^*\},\ \forall W_i, W_j \in \{W\}$, W^*-the higher dimensional extended cubic-hypermatrix. Closure is maintained with respect to the field and components matrix sheets. The hypermatrix size and the number of sheets do not remain constant.

b) Identity element. $\exists$ an element $I \ni W_i \times I \in W_i^*, \forall W_i, I \in \{W\}$.

c) Invertability. $\forall W_i, W \in \{W\}$, $\exists$ an element $W \ni W_i \times W = I^*, I^* \in \{W^*\}$.

d) Associativity. $(W_i \times W_j) \times W_k = W_i \times (W_j \times W_k), W_i, W_j, W_k \in \{W\}$.

Associativity of hypermatrices is maintained when the pair order multiplication of hypermatrices is maintained, and when pair order is maintained the associativity property of hypermatrices is derived from matrix associativity.

Definition 3 A subset of hypermatrix Lie algebra is a hypermatrix subgroup if it is a hypermatrix group as a set, i.e., it satisfies all the conditions of a hypermatrix group.

4. The Determinant of Hypermatrices and Conditions for Invertability

The necessary and sufficient conditions for the existence of an invertible hypermatrix W^{-1} to an hypermatrix $W \in \{W\}$, are given by the following theorem in which DW is the determinant of W.

Theorem 2 $\forall W_i \in \{W\}$ *if* $DW = 0$ *and for all sheets* S_i *of* W $DW_{si} \neq 0$, $\forall i$ *then* W^{-1} *the inverse element of* W *exist and*

$$WW^{-1} = W^{-1}W = I^* \tag{8}$$

4.1 The determinant of the hypermatrix $W_{2\times2\times2}$

Let W be defined by $\alpha = \begin{pmatrix} \alpha_{111} & \alpha_{121} \\ \alpha_{211} & \alpha_{221} \end{pmatrix}, \beta = \begin{pmatrix} \alpha_{112} & \alpha_{122} \\ \alpha_{212} & \alpha_{222} \end{pmatrix}$ where $W = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$. The determinant D of the hypermatrix $W_{2\times2\times2}$ is defined as follows:

$DW = sgn(1,1,1;2,2,2)\alpha_{111}\alpha_{222} + sgn(1,1,2;2,2,1)\alpha_{112}\alpha_{221} + sgn(1,2,2;2,1,1)\alpha_{122}\alpha_{211} + sgn(1,2,1;2,1,2)\alpha_{121}\alpha_{212} = det(W_{major-d}) - det(W_{minor-d})$ = Signed sum of transversals. See (Cayley, 1843).

The general definition of hypermatrix determinant is given by

Definition 2 The determinant of an n-dimensional cubic hypermatrix with $k-$sheets $W_{s1,s2,,sk} \in \{W^{*\times n}, *\times n$ *is the nth extension*$\}$ is defined by the signed sum of transversing products

$$\sum_{(\sigma_1,...,\sigma_n)\in S_n,(S_{\sigma1},...,S_{\sigma n})\in S_n} sgn \prod \alpha_{1\sigma S_{\sigma1}} ... \alpha_{n\sigma n S_{\sigma n}} \tag{10}$$

Summation is over all arrangements of $(S_1, ..., S_{\sigma n})$ sub-sheets in the cubic hypermatrices and sheets.

Proof of theorem 2 The idea of the proof is as follows: since the general case of a hypermatrix determinant has to reduce to the case of a single matrix in terms of conditions for invertability. The individual sheet sub-matrices must be non-trivial and invertible because their product with the inverse hypermatrix results with identity sheet sub-matrices. If the hypermatrix is a symmetric structure by the definition of the determinant of a hypermatrix the determinant of an (even-even) dihedral hypermatrix has value zero and an inverse. When the hypermatrix W with the above characteristics exists it must have an inverse since its individual components has inverses i.e., $WW^{-1} = (I, I, I, I, I, ..., I) = I^{**}$ for all W, the elements product in the brackets are arranged right to left.

Lemma 1 *Assume $DW \neq 0$ and the sheets of W are identical then W is not necessarily invertible.*

Proof: In the proof I consider one of the matrix sheets in W and without loss of generality if $DW_{s2} \neq 0$, $W_{s1} \neq W_{s2}$ and if W_{s2} is invertible W is not invertible because I^* cannot exist, this follows because $WW^{-1} = I^*$ if and only if $W_{s1} = W_{s2}$ and both sheets are not trivial or nilpotent. This could be checked directly and the lemma holds by induction for any finite dimension of W.

Therefore, $DW = 0$ is a necessary condition for the invertability of W but not a sufficient condition. Generalization for higher dimension of W is by induction and induction on dimension of hypermatrices.

Lemma 2 *If $DW = 0$ and $W_{si} \neq W_{sj}$, $\forall i, j$ W is not necessarily invertible.*

Proof: To get $DW = 0$ we might choose one invertible sheet e.g., W_{si} and one trivial sheet $W_{sj} = 0$ or two nilpotent non-identical sheets then possibly $D(W_{major}) \neq D(W_{minor})$ and I^* cannot exist because $WW^{-1} = I^*$ if and only if $W_{s1} = W_{s2}$ and they must be invertible as matrices.

In conclusion we have shown that in order for the hypermatrix W to have an inverse element W^{-1} all the components sheets of the hypermatrix need to be identical non-trivial and invertible sub-matrices of the hypermatrix W. The determinant of the hypermatrix W has to vanish, i.e., $DW = 0$. We sum up the bracket multiplication of the (even, even), (odd, odd) elements and their properties in <Table 5>.

Theorem 3 *The invertible (even, even) dihedral set elements of any order constitute a commutative multiplicative hypermatrix group.*

Proof: By table 1 the multiplication is closed in the matrix set, each element has an inverse in an hypermatrix set, and multiplication is commutative, here associativity follows from the commutativity of the hypermatrix (..., even, even...even) set.

Theorem 4 *The invertible (odd, odd) dihedral set elements of any order constitute a skew-symmetric Lie hypermatrix algebra.*

Proof: By table 3 the invertible (odd, odd) dihedral set is Lie hypermatrix algebra and by induction on dimension it is a skew-symmetric Lie hypermatrix algebra of the invertible (odd, odd) dihedral set.

5. The Extended Hypermatrix Lie Algebra Associated with Antisymmetric and Symmetric Hypermatrices

The (even, even), (even,even) hypermatrix products of the dihedral algebra in table 4 with identical symmetric sheet elements constitute a hypermatrix group and a commutative Lie hypermatrix algebra. Each extension of the set of group elements constitutes a group structure in a higher cubic hypermatrix set. The set of elements with trace zero is a sub-Lie hypermatrix algebra which generates a higher cubic dimension semisimple Lie hypermatrix algebra, which has global skew-symmetric properties (Schreiber, 2012). Next we consider the third extension of the symmetric elements.

5.1 The third extension

Next, consider the extension of the elements in table 4 under the Lie bracket operation:

Skew-symmetric elements type (odd, odd, odd, odd)

Consider the products of the (odd, odd, odd, odd) elements $W_{3,3;5,5}$ and $W_{3,3;6,6}$. These elements are symmetric and their product is skew-symmetric. If we define the elements $W_{3,3;5,5} = (t_3^*, t_5^*)$, then the 4-dimensional hypermatrix product $W_{\alpha,\alpha,\alpha,\alpha}$, is represented by hypermatrix with four sheets $\alpha = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and similarly for $W_{3,3;6,6} = (t_3^*, t_6^*)$, we have the hypermatrix $W_{\beta,\beta,\beta,\beta}$, with four identical sheets $\beta = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$, therefore, we find that the bracket product $(W_{\alpha,\alpha,\alpha,\alpha}, W_{\beta,\beta,\beta,\beta}) =$ $\{(\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha,\beta), (\alpha, \beta)\} = W(0)^{**}$ that follows from $\alpha^t = -\beta$ and $\alpha^t = -\alpha$, and therefore, $(\alpha, -\alpha^t) = 0$, which holds for all skew-symmetric elements. Hence this Lie product is represented by a trivial 16- sheet symmetric hypermatrix set in a four dimensional cube, the bracket entries are arranged right to left.

Similarly, $(W_{6,6;5,5}, W_{6,6;3,3}) = W(0)^{**}$.

Symmetric and skew-symmetric elements

If we consider the products of the symmetric element $W_{3,3;6,6}$ and skew-symmetric element $W_{2,2;6,6}$ we find that their Lie product result in a hypermatrix with skew-symmetric sheets. The element $W_{3,3;6,6} = (t_3^*, t_6^*)$ is represented by a 4-dimensional hypermatrix $W_{\gamma,\gamma,\gamma,\gamma}$ set in cubic hypermatrix, with $\gamma = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ and the element $W_{2,2;6,6} = (t_2^*, t_6^*)$ is represented by the hypermatrix $W_{\omega,\omega,\omega,\omega}$ with $\omega = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and we find that the Lie bracket product is $(W_{\gamma,\gamma,\gamma,\gamma}, W_{\omega,\omega,\omega,\omega}) = \{(\gamma,\omega), (\gamma,\omega), (\gamma,\omega), (\gamma,\beta), (\gamma,\omega), (\gamma,\omega), (\gamma,\beta), (\gamma,\omega), (\gamma,\omega), (\gamma,\beta\omega), (\gamma,\beta), (\gamma,\omega), (\gamma,\omega), (\gamma,\omega), (\gamma,\omega), (\gamma,\omega)\} = W(\begin{pmatrix} 2 & -4 & 2 \\ -4 & 2 & 2 \\ 2 & 2 & -4 \end{pmatrix})^{**}$. We obtain a16- sheet symmetric hypermatrix set in a four dimensional cube with symmetric matrices and hypermatrices that have trace zero.

Symmetric elements type (odd, odd, even, even) and (even, even, odd, odd)

Consider $W_{3,3;2,2}$ and $W_{5,5;4,4}$ arranged anti-symmetrically as (odd,odd, even,even) and (odd,odd, even,even) products defined by $(W_{\tau,\tau,\tau,\tau}, W_{\sigma,\sigma,\sigma,\sigma}) = \{(\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma), (\tau,\sigma)\} = W(\begin{pmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{pmatrix})^{**}$. We obtain a16- sheet symmetric hypermatrix set in a four dimensional hypermatrix with skew-symmetric hypermatrix sheets that have trace zero.

While for the (odd, odd), (even, even) product $W_{3,3;2,2}$ and the (even, even), (odd, odd), product $W_{4,4;5,5}$ we have $(W_{3,3;2,2}, W_{4,4;5,5}) = W(\begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix})^{**}$. We obtain a16- sheet symmetric hypermatrix set in a four dimensional cube with anti-clockwise-symmetric (acs) hypermatrix sheets i.e., $(W_{3,3;2,2}, W_{4,4;5,5})^{acs} = (W_{3,3;2,2}, W_{4,4;5,5})$. For $W^*_{symmetric}$ type odd, even and even, odd we have $(W^*_{symm}, W^*_{symm}) = W^{**acs}(symm) = W^{**}$.

Symmetric elements type (odd, odd, even, even) and (even, even, odd, odd)

$(W_{3,3;4,4}, W_{2,2;6,6}) = W(\begin{pmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -3 & 3 & 0 \end{pmatrix})^{**}$.

Anti-symmetric elements type (odd, odd, even, even)

For the anti-symmetric hypermatrices of type (odd, odd, even, even) $W_{3,3;2,2}$, under Lie product with the (even, even, odd, odd) $W_{2,2;3,3}$ we have $(W_{3,3;2,2}, W_{2,2;3,3}) = W(\begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix})^{**}$ set in a16- sheet symmetric hypermatrix with symmetric sheets.

Anti-symmetric elements type (odd, odd, even, even)

$(W_{5,5;2,2}, W_{2,2;5,5}) = W(0)^{**}$

Anti-symmetric elements type (even, even, odd, odd)

For trivial center elements $W_{6,6;5,5}$ and $W_{5,5;6,6}$ we obtain $(W_{6,6;5,5}, W_{5,5;6,6}) = W(0)^{**}$.

Hence the higher extensions set of the symmetric hypermatrices associated with the dihedral D_3 group arrangement of triangles have a zero trace if the hypermatrics have all even or all odd elements and if they are arranged anti-symmetrically their product results in symmetric sheets and symmetric Lie hypermatrix algebra.

Theorem 5 *a) The extended dihedral D_3^* Lie algebra associated with symmetric and (odd, odd, odd, odd) anti-symmetric, trace zero type, four dimensional W^* hypermatrices generates a semisimple Lie hypermatrix algebra W^{**} with symmetric sheets.*

*b) The extended dihedral D_3^* Lie algebra associated with skew-symmetric trace zero four dimensional W*hypermatrices type W^*(odd, odd, odd, odd) generates a semisimple Lie hypermatrix algebra W^{**} with trivial sheets.*

c) The extended dihedral D_3^* Lie algebra associated with symmetric trace zero four dimensional W^* hypermatrices type W^*(odd, odd, even, even) or W(even, even, odd, odd) generates a trace-zero Lie hypermatrix algebra if the sheets are symmetric and the hypermatrices are anti-symmetric and generates an acs Lie hypermatrix algebra if the sheets are symmetric and the hypermatrices are of general type.*

Proof: a) I have shown above and it can be seen from the resulting products in table 4 that the (even, even) symmetric element and the (odd, odd) anti-symmetric elements of the second extension W^* have trace zero as hypermatrices so when we apply the Lie bracket operation to symmetric hypermetrics we obtain the following W^{**} extension:

$$(W_{i,i;j,j}(t), W_{k,k;l,l}(t)) = (W_\alpha^*, W_\beta^*) = W^{**}((\alpha,\beta), ..., (\alpha,\beta)) \tag{11}$$

a 16 sheet hypermatrix set in a 4-dimensional hypermatrix with symmetric sheets.

In table 4 if the generators elements $\{t_i\}$ have trace zero, there are two possibilities: a) either W^* has symmetric sheets, or b) (odd, odd, odd, odd) type anti-symmetric sheets, in any case the hypermatrices W^{**} are globally symmetric because they are generated by symmetric elements type $W_{i,i;j,j}$ which result in symmetric sheets if generated by an (even, even) = even$^{\times 2}$ hypermatrices $W(t)$ and skewsymmetric sheets if generated by odd$^{\times 2}$ hypermatrices $W(t)$.

b) If we apply the Lie bracket operation to skew-symmetric hypermatrics W^* in table 4 we obtain the following W^{**} extension

$$(W_{i,i;j,j}(t), W_{k,k;l,l}(t)) = (W_\alpha^*, W^*\beta) = W^{**}((\alpha,\beta), ..., (\alpha,\beta)) \tag{12}$$

a 16 sheet hypermatrix set in a cubic 4-dimensional hypermatrix with trivial sheets, see rule 2 (Schreiber, 2012).

$\{(WH_{f(h,y,x)}, WH_{\otimes f(h,y,x)})\} = \{WH_{semi\ skew-symmetric\ hypermatrix\ algebra}\}$ or trace zero algebra $(W_{f(h,y,x)}, W_{\otimes f(h,y,x)}) = W^*_{trace\ zero\ hypermatrix\ algebra}$.

In table 4 if the generators elements $\{t_i\}$ have trace zero, there are two possibilities: either W^* has symmetric sheets, or (odd, odd ,odd, odd) type antisymmetric sheets, as noted above the hypermatrices W^{**} are globally symmetric because they are generated by symmetric elements type $W_{i,i;j,j}$ and have symmetric trivial sheets if generated by even hypermatrices $W(t)$ and symmetric trivial sheets $W^{**}((\alpha,\beta), ..., (\alpha,\beta)) = W(0)^{**}$ if generated by odd hypermatrices $W(t)$.

So the extended dihedral Lie algebra associated with W^* skew-symmetric four dimensional hypermatrices generates the semisimple Lie hypermatrix algebra W^{**} with trivial sheets.

c) The extended dihedral D_3^* Lie algebra associated with skew-symmetric trace zero four dimensional W^* hypermatrices type W^* (odd, odd, even, even) or W^* (even, even, odd, odd) generates a Lie hypermatrix acs-algebra W^{**}. For the hypermatrices $W_{i,i;j,j}$, $W_{k,k;l,l}$ of general type W^* (odd, odd, even, even) or W^* (even, even, odd, odd) we have $(W_{i,i;j,j}, W_{k,k;l,l})^{acs} = (W_{i,i;j,j}, W_{k,k;l,l})$, acs-anti-clockwise-symmetric. For W^* symmetric type (odd, even) and (even, odd) we have $(W^*_{symm}, W^*_{symm}) = W^{**acs}(m) = W^{**}$. In this case we obtain products of Lie hypermatrices with anti-clockwise-symmetric sheets.

The product of the elements in $W_{i,i;j,j}$ and $W_{k,k;l,l}$ are symmetric trace-zero sheets, their Lie product is either trivial or has an anti-clockwise-symmetric (acs) sheet structure because the constituent sub-matrices are pair-wise-symmetric and acs. We may note that the odd elements W_5, W_6 are symmetric and we also have the relations $t_5^{acs} = t_6$, $t_4^{acs} = t_2$. But, their product with acs elements results in a acs type sheet and, therefore, all resulting products must be acs type sheets, and so are the resulting hypermatrices. That follows because the product of acs type sheet with another acs type sheet is an acs type sheet. We note that a linear combination of acs matrices is also an acs matrix, and generally for α, β type acs we have $\alpha^{acs} = \alpha, \beta^{acs} = \beta$, therefore, $(\beta\alpha) = (\alpha^{acs}\beta^{acs})^{acs}$. Also every symmetric matrix is an acs matrix but not every acs matrix is symmetric. Hence the Lie product of acs hypermatrices produces an acs Lie hypermatrix algebra.

If the elements are symmetric and the hypermatrices are anti-symmetric the resulting product is trivial and the resulting Lie hypermatrix algebra has trace zero (Schreiber, 2012); it is not commutative hence it is semisimple.

Open Questions

Describe the extended dihedral $D_n^{*\times n}$ Lie hypermatrix algebra and all iterations associated with symmetric, anti-symmetric, and skew-symmetric sheets composing the Lie hypermatrices algebra of $D_n^{*\times(n-1)}$. $D_n^{*\times n}$ is the nth-extended Lie hypermatrix algebra of the elements of the classical group D_n.

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Table 1. Multiplication table for the dihedral group D_3

X		even	even	even	odd	odd	odd
		t_1	t_4	t_2	t_5	t_3	t_6
even	t_1	t_1	t_4	t_2	t_5	t_3	t_6
even	t_4	t_4	t_2	t_1	t_3	t_6	t_5
even	t_2	t_2	t_1	t_4	t_6	t_5	t_3
odd	t_5	t_5	t_6	t_3	t_1	t_2	t_4
odd	t_3	t_3	t_5	t_6	t_4	t_1	t_2
odd	t_6	t_6	t_3	t_5	t_2	t_4	t_1

The even permutation elements (t_1, t_4, t_2) constitute a normal subgroup of the dihedral group D_3.

Table 2. Multiplication table for two triangle hypermatrices

$W(t_i, t_i), W(t_k, t_l)$	even, even	odd, odd	even, odd	odd, even
even, even Symm (even, even)	Semisimplicity and sheet skew-symmetry	Trace-zero		
odd, odd Symm(odd, odd)	Trace-zero	Semisimplicity symmetry about certain hypermatrix plane lines		
even, odd			Semisimple, sheet skew-symmetry	Semisimple, sheet skew-symmetry
odd, even			Semisimple, sheet skew-symmetry	Semisimple, sheet skew-symmetry

Table 3. Two triangle antisymmetric hypermatrices multiplication table

$(W(t_i; t_i), W(t_j; t_i))$	even, even	odd, odd	even, odd	odd, even
even, even	Semisimplicity and sheet skewsymmetry	X		
odd, odd	X	Semisimplicity, symmetry about certain hypermatrix plane lines		
even, odd			X	Semisimple, Skew-symmetry
odd, even			Semisimple, Skew-symmetry	X

Table 4. Second extension of symmetric elements of the dihedral group D_3^*

		even, even	even, even	even, even	odd, odd	odd, odd	odd, odd
$W(t_i;t_i)$	$W(t_j;t_j)$	$W_{1,1}$	$W_{4,4}$	$W_{2,2}$	$W_{5,5}$	$W_{3,3}$	$W_{6,6}$
even, even	$W_{1,1}$	0^{**}	0^{**}	0^{**}	0^{**}	0^{**}	0^{**}
even, even	$W_{4,4}$	0^{**}	0^{**}	0^{**}	$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$
even, even	$W_{2,2}$	0^{**}	0^{**}	0^{**}	$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$
odd,odd	$W_{5,5}$	0^{**}	$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	0^{**}	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$
odd,odd	$W_{3,3}$	0^{**}	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	0^{**}	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$
odd,odd	$W_{6,6}$	0^{**}	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	0^{**}

* Each entry in the table is a 4-sheet 4 dimensional cubic hypermatrix.

The matrices with even permutation elements (t_1, t_4, t_2) constitute a normal subgroup of the dihedral group D_3. The hypermatrices with even-even permutation elements $\{W_{1,1}, W_{2,2}, W_{4,4}\}$ constitute a normal hypermatrix subgroup of the dihedral Lie hypermatrix algebra $D_3^*\{W_{1,1}, W_{2,2}, W_{3,3}, W_{4,4}, W_{5,5}, W_{6,6}\}$ symmetric. The Lie algebra D_3^* has trace zero, and it is skew symmetric.

Table 5. Two triangle symmetric-invertible hypermatrices multiplication

$(W(t_i;t_i), W(t_j;t_j))$	even, even	odd, odd
even, even	0^*	Trace zero
odd, odd	Trace zero	Semisimple, symmetry about certain hypermatrix lines, Trace-zero