Oscillation Properties for Second-Order Half-Linear Dynamic Equations on Time Scales

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Received: February 3, 2012 Accepted: February 21, 2012 Published: April 1, 2012

doi:10.5539/jmr.v4n2p90 URL: http://dx.doi.org/10.5539/jmr.v4n2p90

The research is financed by the Natural Science Foundation of Hunan Province of P. R. China (Grant No. 11JJ3010)

Abstract

In this paper, we are concerned with the oscillation of the second-order half-linear dynamic equation

$$\left(a(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} + q(t)|x(t)|^{\gamma-1}x(t) = 0$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma > 0$ is a constant. By using a generalized Riccati substitution, the Pötzsche chain rule and a Hardy-Littlewood-Pólya inequality, we obtain some sufficient conditions for the oscillation of the equation and improve and extend some known results in which $\gamma > 0$ is a quotient of odd positive integers. We also give some examples to illustrate our main results.

Keywords: Oscillation property, Second-order half-linear dynamic equation, Time scale

1. Introduction

Since Hilger (1990) introduced the theory of time scales, many authors have expounded on various aspects of this new theory; see the books (Bohner & Peterson, 2001, 2003) and the papers (Agarwal *et al.*, 2007; Bohner & Saker, 2004; Chen, 2010; Chen & Liu, 2008; Došlý & Hilger 2002; Erbe *et al.*, 2008; Hassan, 2008; Hassan, 2009; Karpuz, 2009; Medico & Kong, 2004; Saker, 2005; Zhang, 2011). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$ (see Hilger, 1990; Bohner & Peterson, 2001, 2003), and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential equations and of difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice–once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale. In this way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. Therefore, not only can the theory of dynamic equations unify the theories of differential equations and difference equations, but it is also able to extend these classical cases to cases "in between," e.g., to the so-called q-difference equations. Dynamic equations on time scales have a lot of applications in population dynamics, quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. Bohner and Peterson (2001) summarizes and organizes much of time scale calculus. For advances of dynamic equations on time scales, we refer the reader to (Bohner & Peterson, 2003).

In recent years, there has been a large number of papers devoted to the oscillation and asymptotic behavior of dynamic equations on time scales, and we refer to (Karpuz, 2009; Hassan, 2009; Chen, 2010; Chen & Liu, 2008; Medico & Kong, 2004; Bohner & Saker, 2004; Došlý & Hilger 2002; Saker, 2005; Agarwal *et al.*, 2007; Hassan, 2008; Erbe *et al.*, 2008) and the references cited therein. For the second-order half-linear dynamic equations

$$\left(a(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + q(t)x^{\gamma}(t) = 0 \tag{1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma > 1$ is an odd positive integer, a and q are positive rd-continuous functions defined on the time scale interval $[t_0, \infty)$, Saker (2005) obtained several oscillation criteria.

Later, Agarwal *et al.* (2007) supposed that $\gamma > 1$ is a quotient of odd positive integers and got several sufficient conditions for the oscillation of all solutions of (1). Agarwal *et al.* (2007) improved and extended the results of Saker (2005).

Very recently, Hassan (2008) supposed that $\gamma > 0$ is a quotient of odd positive integers and established some oscillation criteria of (1). Hassan (2008) improved and extended the results of Saker (2005) and Agarwal *et al.* (2007).

However, the results of Saker (2005), Agarwal *et al.* (2007) and Hassan (2008) cannot be applied to the following second-order half-linear dynamic equations

$$\left(a(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t)\right)^\Delta + q(t)|x(t)|^{\gamma-1}x(t) = 0 \tag{2}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma > 0$ is a constant, a and q are positive rd-continuous functions defined on the time scale interval $[t_0, \infty)$. Therefore, it is of great interest to study the oscillation of (2) when $\gamma > 0$ is a constant.

In this paper, we establish some oscillation criteria for (2) by applying a generalized Riccati substitution, the Pötzsche chain rule and a Hardy-Littlewood-Pólya inequality. Our results improve and extend the results of Saker (2005), Agarwal *et al.* (2007) and Hassan (2008). Some examples are shown to illustrate our main results.

Since we are interested in the oscillation of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$. By a solution of (2) we mean a nontrivial real function $x \in C^1_{rd}[t_x, \infty)$ such that $a|x^\Delta|^{\gamma-1}x^\Delta \in C^1_{rd}[t_x, \infty)$ for a certain $t_x \geq t_0$ and satisfying (2) on $[t_x, \infty)$. Our attention is restricted to those solutions of (2) which exist on the half-line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_*\} > 0$ for any $t_* \geq t_x$. A solution x of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (2) is said to be oscillatory if all its solutions are oscillatory.

We shall need the following lemma to prove our main results.

Lemma 1.1 (Hardy *et al.*, 1988) *If X and Y are nonnegative, then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1)Y^\lambda \quad for \quad \lambda > 1,$$

where the equality holds if and only if X = Y.

2. Main Results

Theorem 2.1 *Suppose that*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{1/\gamma} \Delta t = \infty \tag{3}$$

holds. Furthermore, assume that there exists a positive function $\eta \in C^1_{rd}([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left\{\eta(s)q(s) - \frac{a(s)[(\eta^\Delta(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^\gamma(s)}\right\}\Delta s = \infty, \tag{4}$$

where $(\eta^\Delta(s))_+ := \max\{\eta^\Delta(s), 0\}$. *Then every solution of (2) is oscillatory.*

Proof: Suppose that x is a nonoscillatory solution of (2). Without loss of generality, we may assume that x is an eventually positive solution of (2). Therefore, there exists $t_1 \in [t_0, \infty)$ such that

$$x(t) > 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{5}$$

From (2) and (5) we have

$$\left(a(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t)\right)^\Delta = -q(t)x^\gamma(t) < 0 \quad \text{for} \quad t \in [t_1, \infty), \tag{6}$$

which implies that $a(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t)$ is strictly decreasing on $[t_1, \infty)$ and is eventually of one sign. We claim

$$x^\Delta(t) > 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{7}$$

Assume not, then there exists $t_2 \in [t_1, \infty)$ such that $x^\Delta(t_2) \leq 0$. Hence, we obtain $a(t_2)|x^\Delta(t_2)|^{\gamma-1}x^\Delta(t_2) \leq 0$. Take $t_3 > t_2$. Since $a(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t)$ is strictly decreasing on $[t_1, \infty)$, it is clear that $a(t_3)|x^\Delta(t_3)|^{\gamma-1}x^\Delta(t_3) < a(t_2)|x^\Delta(t_2)|^{\gamma-1}x^\Delta(t_2)$. Therefore, for $t \in [t_3, \infty)$ we get $a(t)|x^\Delta(t)|^{\gamma-1}x^\Delta(t) \leq a(t_3)|x^\Delta(t_3)|^{\gamma-1}x^\Delta(t_3) := c < 0$. Thus, we obtain $x^\Delta(t) \leq -(-c)^{\frac{1}{\gamma}}\left(\frac{1}{a(t)}\right)^{1/\gamma}$ for $t \in [t_3, \infty)$. By integrating both sides of the last inequality from t_3 to t, we get $x(t) - x(t_3) \leq -(-c)^{\frac{1}{\gamma}} \int_{t_3}^{t} \left(\frac{1}{a(s)}\right)^{1/\gamma} \Delta s$ for $t \in [t_3, \infty)$. Noticing (3) and letting $t \to \infty$, we see $\lim_{t\to\infty} x(t) = -\infty$. This contradicts (5). Hence, (7) holds. It follows from (6) and (7) that

$$\left(a(t)(x^\Delta(t))^\gamma\right)^\Delta = -q(t)x^\gamma(t) < 0 \quad \text{for} \quad t \in [t_1, \infty). \tag{8}$$

Define the function w by the generalized Riccati substitution

$$w(t) = \eta(t)\frac{a(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \quad \text{for} \quad t \in [t_1, \infty). \tag{9}$$

It is easy to see that $w(t) > 0$ for $t \in [t_1, \infty)$. By the product rule and then the quotient rule for the delta derivative of the product and the quotient of two delta differentiable functions (see Bohner & Peterson, 2001, p. 7, Theorem 1. 20), from (9) we get

$$w^{\Delta} = [a(x^{\Delta})^{\gamma}]^{\Delta}\frac{\eta}{x^{\gamma}} + (a(x^{\Delta})^{\gamma})^{\sigma}\Big(\frac{\eta}{x^{\gamma}}\Big)^{\Delta} = [a(x^{\Delta})^{\gamma}]^{\Delta}\frac{\eta}{x^{\gamma}} + (a(x^{\Delta})^{\gamma})^{\sigma}\Big[\frac{\eta^{\Delta}}{(x^{\sigma})^{\gamma}} - \frac{\eta(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}\Big] \quad \text{on} \quad [t_1, \infty), \tag{10}$$

where σ is the forward jump operator on $\mathbb{T}$ and $(a(x^{\Delta})^{\gamma})^{\sigma} := (a(x^{\Delta})^{\gamma}) \circ \sigma$. Therefore, from (8)–(10) we have

$$w^{\Delta} = -\eta q + \frac{\eta^{\Delta}}{\eta^{\sigma}}w^{\sigma} - \eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} \quad \text{on} \quad [t_1, \infty). \tag{11}$$

By the Pötzsche chain rule (Bohner & Peterson, 2001, p. 32, Theorem 1. 87) and then using the fact that $x(t)$ is strictly increasing on $[t_1, \infty)$, for $t \in [t_1, \infty)$ we obtain

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\Big\{\int_0^1 [x(t) + h\mu(t)x^{\Delta}(t)]^{\gamma-1}dh\Big\}x^{\Delta}(t) = \gamma\Big\{\int_0^1 [(1-h)x(t) + hx^{\sigma}(t)]^{\gamma-1}dh\Big\}x^{\Delta}(t)\\ &\ge \begin{cases} \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t), & 0 < \gamma \le 1,\\ \gamma(x(t))^{\gamma-1}x^{\Delta}(t), & \gamma > 1,\end{cases}\end{aligned} \tag{12}$$

where $\mu(t) := \sigma(t) - t$ is the graininess function on $\mathbb{T}$. If $0 < \gamma \le 1$, it follows from (11) and (12) that

$$w^{\Delta} \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}\cdot\gamma(x^{\sigma})^{\gamma-1}x^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} = -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}}{(x^{\sigma})^{\gamma+1}}\cdot\frac{(x^{\sigma})^{\gamma}}{x^{\gamma}}x^{\Delta} \quad \text{on} \quad [t_1, \infty). \tag{13}$$

If $\gamma > 1$, it follows from (11) and (12) that

$$w^{\Delta} \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}\cdot\gamma x^{\gamma-1}x^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} = -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}}{(x^{\sigma})^{\gamma+1}}\cdot\frac{x^{\sigma}}{x}x^{\Delta} \quad \text{on} \quad [t_1, \infty). \tag{14}$$

Since $t \le \sigma(t)$ and $x(t)$ is strictly increasing on $[t_1, \infty)$, we have $x(t) \le x^{\sigma}(t)$. Therefore, from (13) and (14) we get

$$w^{\Delta} \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta\frac{(a(x^{\Delta})^{\gamma})^{\sigma}}{(x^{\sigma})^{\gamma+1}}x^{\Delta} \quad \text{on} \quad [t_1, \infty) \quad \text{for} \quad \gamma > 0. \tag{15}$$

Since $a(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_1, \infty)$ and $t \le \sigma(t)$, we have $(a(x^{\Delta})^{\gamma})(t) \ge (a(x^{\Delta})^{\gamma})^{\sigma}(t)$ and $x^{\Delta}(t) \ge \frac{[(a(x^{\Delta})^{\gamma})^{\sigma}(t)]^{1/\gamma}}{a^{1/\gamma}(t)}$. Hence, from (15) we obtain

$$w^{\Delta} \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta a^{-\frac{1}{\gamma}}\frac{[(a(x^{\Delta})^{\gamma})^{\sigma}]^{1+\frac{1}{\gamma}}}{(x^{\sigma})^{\gamma+1}} \quad \text{on} \quad [t_1, \infty). \tag{16}$$

From (9) and (16) we find

$$w^{\Delta}(t) \le -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta a^{-\frac{1}{\gamma}}\Big(\frac{w^{\sigma}}{\eta^{\sigma}}\Big)^{1+\frac{1}{\gamma}} = -\eta q + \frac{(\eta^{\Delta})_+}{\eta^{\sigma}}w^{\sigma} - \gamma\eta a^{-\frac{1}{\gamma}}(\eta^{\sigma})^{-\frac{\gamma+1}{\gamma}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}} \quad \text{on} \quad [t_1, \infty). \tag{17}$$

Take $\lambda = \frac{\gamma+1}{\gamma}$ and define $X \ge 0$ and $Y \ge 0$ by $X^{\lambda} := \gamma\eta a^{-\frac{1}{\gamma}}(\eta^{\sigma})^{-\frac{\gamma+1}{\gamma}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}}$ and $Y^{\lambda-1} := \frac{a^{1/(\gamma+1)}(\eta^{\Delta})_+}{\lambda(\gamma\eta)^{1/\lambda}}$, then by Lemma 1.1 and (17) we conclude

$$w^{\Delta}(t) \le -\eta(t)q(t) + \frac{a(t)[(\eta^{\Delta}(t))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^{\gamma}(t)} \quad \text{for} \quad t \in [t_1, \infty).$$

Integrating both sides of the last inequality from t_1 to t, we obtain

$$w(t) - w(t_1) \le -\int_{t_1}^{t}\Big\{\eta(s)q(s) - \frac{a(s)[(\eta^{\Delta}(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^{\gamma}(s)}\Big\}\Delta s \quad \text{for} \quad t \in [t_1, \infty).$$

Since $w(t) > 0$ for $t \in [t_1, \infty)$, we have

$$\int_{t_1}^{t} \left\{ \eta(s)q(s) - \frac{a(s)[(\eta^{\Delta}(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^{\gamma}(s)} \right\} \Delta s \leq w(t_1) - w(t) < w(t_1) \quad \text{for} \quad t \in [t_1, \infty).$$

Thus, we get

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left\{ \eta(s)q(s) - \frac{a(s)[(\eta^{\Delta}(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^{\gamma}(s)} \right\} \Delta s \leq w(t_1) < \infty,$$

which contradicts (4). Hence, the proof is complete.

The following theorem gives a Philos-type oscillation criterion for (2).

Theorem 2.2 *Assume that (3) holds. Furthermore, suppose that there exist a positive function $\eta \in C^1_{rd}([t_0, \infty), \mathbb{R})$ and a function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$, such that*

$$H(t, t) = 0 \quad \text{for} \quad t \geq t_0, \quad H(t, s) > 0 \quad \text{for} \quad (t, s) \in \mathbb{D}_0,$$

where $\mathbb{D}_0 := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$, and H has a nonpositive rd-continuous delta partial derivative $H^{\Delta_s}(t, s)$ on $\mathbb{D}_0$ with respect to the second variable and satisfies

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left\{ H(t, s)\eta(s)q(s) - \frac{a(s)[\eta(\sigma(s))h_+(t, s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[H(t, s)\eta(s)]^{\gamma}} \right\} \Delta s = \infty, \tag{18}$$

where σ is the forward jump operator on $\mathbb{T}$ and $h_+(t, s) := \max\{H^{\Delta_s}(t, s) + H(t, s)\frac{(\eta^{\Delta}(s))_+}{\eta(\sigma(s))}, 0\}$, here $(\eta^{\Delta}(s))_+ := \max\{\eta^{\Delta}(s), 0\}$. Then all solutions of (2) are oscillatory.

Proof: Assume that x is a nonoscillatory solution of (2). Without loss of generality, assume that x is an eventually positive solution of (2). Define again the function w by (9). Proceeding as in the proof of Theorem 2.1, we see that (17) holds. Multiplying (17) by $H(t, s)$ and integrating from t_1 to t, we find

$$\begin{aligned}\int_{t_1}^{t} H(t, s)\eta(s)q(s)\Delta s \leq &-\int_{t_1}^{t} H(t, s)w^{\Delta}(s)\Delta s + \int_{t_1}^{t} H(t, s)\frac{(\eta^{\Delta}(s))_+}{\eta^{\sigma}(s)} w^{\sigma}(s)\Delta s \\ &- \int_{t_1}^{t} H(t, s)V(s)(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} \Delta s \quad \text{for} \quad t \in [t_1, \infty),\end{aligned} \tag{19}$$

where $V(s) := \gamma\eta(s)a^{-\frac{1}{\gamma}}(s)(\eta^{\sigma}(s))^{-\frac{\gamma+1}{\gamma}}$. Applying the integration by parts formula

$$\int_{c}^{d} F(s)G^{\Delta}(s)\Delta s = \Big[F(s)G(s)\Big]_c^d - \int_{c}^{d} F^{\Delta}(s)G(\sigma(s))\Delta s,$$

for $t \in [t_1, \infty)$ we get

$$\begin{aligned}-\int_{t_1}^{t} H(t, s)w^{\Delta}(s)\Delta s &= \Big[-H(t, s)w(s)\Big]_{s=t_1}^{s=t} + \int_{t_1}^{t} H^{\Delta_s}(t, s)w^{\sigma}(s)\Delta s \\ &= H(t, t_1)w(t_1) + \int_{t_1}^{t} H^{\Delta_s}(t, s)w^{\sigma}(s)\Delta s.\end{aligned} \tag{20}$$

Substituting (20) in (19), for $t \in [t_1, \infty)$ we obtain

$$\begin{aligned}&\int_{t_1}^{t} H(t, s)\eta(s)q(s)\Delta s \\ &\leq H(t, t_1)w(t_1) + \int_{t_1}^{t} \left\{ \Big[H^{\Delta_s}(t, s) + H(t, s)\frac{(\eta^{\Delta}(s))_+}{\eta^{\sigma}(s)}\Big] w^{\sigma}(s) - H(t, s)V(s)(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} \right\} \Delta s \\ &\leq H(t, t_1)w(t_1) + \int_{t_1}^{t} \Big[h_+(t, s)w^{\sigma}(s) - H(t, s)V(s)(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}}\Big] \Delta s,\end{aligned} \tag{21}$$

where $h_+(t, s)$ is defined as in Theorem 2.2. Take $\lambda = \frac{\gamma+1}{\gamma}$ and define $X \geq 0$ and $Y \geq 0$ by

$$X^{\lambda} := H(t, s)V(s)(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} \quad \text{and} \quad Y^{\lambda-1} := \frac{h_+(t, s)}{\lambda[H(t, s)V(s)]^{1/\lambda}},$$

then by Lemma 1.1 and (21) we find

$$\int_{t_1}^{t} H(t,s)\eta(s)q(s)\Delta s \leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \Theta(t,s)\Delta s \quad \text{for} \quad t \in [t_1,\infty), \tag{22}$$

where $\Theta(t,s) := \frac{a(s)[\eta(\sigma(s))h_+(t,s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1}[H(t,s)\eta(s)]^{\gamma}}$. Since $H^{\Delta_s}(t,s) \leq 0$ on $\mathbb{D}_0$, we obtain $H(t,t_1) \leq H(t,t_0)$ for $t > t_1 \geq t_0$. Hence, for $t > t_1 \geq t_0$, it follows from (22) that

$$\int_{t_1}^{t} \Big[H(t,s)\eta(s)q(s) - \Theta(t,s)\Big]\Delta s \leq H(t,t_1)w(t_1) \leq H(t,t_0)w(t_1). \tag{23}$$

For $t > s \geq t_0$, we have $0 < H(t,s) \leq H(t,t_0)$ and $0 < \frac{H(t,s)}{H(t,t_0)} \leq 1$. Thus, from (23) we get

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t} \Big[H(t,s)\eta(s)q(s) - \Theta(t,s)\Big]\Delta s &= \frac{1}{H(t,t_0)}\left(\int_{t_0}^{t_1} + \int_{t_1}^{t}\right)\Big[H(t,s)\eta(s)q(s) - \Theta(t,s)\Big]\Delta s \\ &\leq \int_{t_0}^{t_1} \frac{H(t,s)}{H(t,t_0)}\eta(s)q(s)\Delta s + w(t_1) \\ &\leq \int_{t_0}^{t_1} \eta(s)q(s)\Delta s + w(t_1) \quad \text{for} \quad t > t_1 \geq t_0.\end{aligned}$$

Therefore, we find

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \Big[H(t,s)\eta(s)q(s) - \Theta(t,s)\Big]\Delta s \leq \int_{t_0}^{t_1} \eta(s)q(s)\Delta s + w(t_1) < \infty,$$

which implies a contradiction to (18). Thus, this completes the proof.

Next, we consider the case when

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{1/\gamma}\Delta t < \infty \tag{24}$$

holds. It is clear that (24) implies that (3) does not hold.

Theorem 2.3 *Suppose that (24) holds. Let η be defined as in Theorem 2.1 such that (4) holds. Furthermore, assume that for every constant $C \geq t_0$,*

$$\int_{C}^{\infty} \left[\frac{1}{a(t)}\int_{C}^{t} q(s)\Delta s\right]^{1/\gamma}\Delta t = \infty. \tag{25}$$

Then every solution of (2) is oscillatory or converges to zero as $t \to \infty$.

Proof: Assume that x is a nonoscillatory solution of (2). Without loss of generality, assume that x is an eventually positive solution of (2). Define again the function w by (9). There are two cases for the sign of $x^{\Delta}(t)$. The proof when $x^{\Delta}(t)$ is eventually positive is similar to that of Theorem 2.1 and hence is omitted.

Next, assume that $x^{\Delta}(t) \leq 0$ holds eventually. Proceeding as in the proof of Theorem 2.1, we obtain that (5) and (6) hold. Thus we get $\lim_{t\to\infty} x(t) := L \geq 0$ and $x(t) \geq L$. Furthermore, there exists $t_2 \in [t_1,\infty)$ such that

$$x^{\Delta}(t) \leq 0 \quad \text{for} \quad t \in [t_2,\infty). \tag{26}$$

We now claim $L = 0$. Assume not, i.e., $L > 0$, then from (6) and (26) we get

$$-\Big(a(t)(-x^{\Delta}(t))^{\gamma}\Big)^{\Delta} \leq -L^{\gamma}q(t) \quad \text{for} \quad t \in [t_2,\infty).$$

Integrating both sides of the last inequality from t_2 to t, we have

$$x^{\Delta}(t) \leq -L\Big[\frac{1}{a(t)}\int_{t_2}^{t} q(s)\Delta s\Big]^{1/\gamma} \quad \text{for} \quad t \in [t_2,\infty).$$

Integrating both sides of the last inequality from t_2 to t, we obtain

$$x(t) \leq x(t_2) - L\int_{t_2}^{t}\left[\frac{1}{a(u)}\int_{t_2}^{u} q(s)\Delta s\right]^{1/\gamma}\Delta u \quad \text{for} \quad t \in [t_2,\infty).$$

Letting $t \to \infty$, from (25) we get $\lim_{t\to\infty} x(t) = -\infty$. This contradicts (5). Therefore, we have $L = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

Theorem 2.4 *Suppose that (24) holds. Let η and H be defined as in Theorem 2.2 such that (18) holds. Furthermore, assume that for every constant $C \geq t_0$, (25) holds. Then every solution of (2) is oscillatory or tends to zero as $t \to \infty$.*

Proof: Assume that x is a nonoscillatory solution of (2). Without loss of generality, assume that x is an eventually positive solution of (2). Define again the function w by (9). There are two cases for the sign of $x^\Delta(t)$. The proof when $x^\Delta(t)$ is eventually positive is similar to that of Theorem 2.2 and hence is omitted.

Next, assume that $x^\Delta(t) \leq 0$ holds eventually. In this case, the proof is similar to that of the proof of Theorem 2.3 and therefore is omitted. The proof is complete.

Remark 2.1 From Theorems 2.1–2.4, we can obtain many different sufficient conditions for the oscillation of (2) with different choices of the functions η and H.

3. Some Examples

Example 3.1 Consider the second-order half-linear dynamic equation

$$\left(t^{\gamma-1}|x^\Delta(t)|^{\gamma-1}x^\Delta(t)\right)^\Delta + t^\beta|x(t)|^{\gamma-1}x(t) = 0 \quad \text{for} \quad t \in \mathbb{T}, \tag{27}$$

where $\gamma > 0$ and $\beta \geq -2$ are constants. In (27), $a(t) = t^{\gamma-1}$ and $q(t) = t^\beta$. Take $t_0 > 0$, then we have

$$\int_{t_0}^\infty \left(\frac{1}{a(t)}\right)^{1/\gamma} \Delta t = \int_{t_0}^\infty \frac{1}{t^{1-\frac{1}{\gamma}}} \Delta t = \infty,$$

which implies that (3) holds. We will apply Theorem 2.1, and it remains to satisfy the condition (4). If $\beta = -2$, then we get $\lim_{s\to\infty}[s^{\beta+2} - \frac{1}{(\gamma+1)^{\gamma+1}}] = 1 - \frac{1}{(\gamma+1)^{\gamma+1}} > 0$. If $\beta > -2$, then we find $\lim_{s\to\infty}[s^{\beta+2} - \frac{1}{(\gamma+1)^{\gamma+1}}] = \infty$. Thus, by taking $\eta(s) = s$ we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \int_{t_0}^t \left\{\eta(s)q(s) - \frac{a(s)[(\eta^\Delta(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^\gamma(s)}\right\}\Delta s &= \limsup_{t\to\infty} \int_{t_0}^t \left[s^{\beta+1} - \frac{s^{\gamma-1}}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^t \frac{1}{s}\left[s^{\beta+2} - \frac{1}{(\gamma+1)^{\gamma+1}}\right]\Delta s \\
&= \infty,
\end{aligned}$$

which implies that (4) holds. Therefore, by Theorem 2.1 every solution of (27) is oscillatory.

Example 3.2 Consider the second-order half-linear dynamic equation

$$\left(t^{\gamma+1}|x^\Delta(t)|^{\gamma-1}x^\Delta(t)\right)^\Delta + (t+\sigma(t))|x(t)|^{\gamma-1}x(t) = 0 \quad \text{for} \quad t \in \mathbb{T}, \tag{28}$$

where $\gamma > 0$ is a constant. In (28), $a(t) = t^{\gamma+1}$ and $q(t) = t + \sigma(t)$. Take $t_0 > 0$, then we obtain

$$\int_{t_0}^\infty \left(\frac{1}{a(t)}\right)^{1/\gamma} \Delta t = \int_{t_0}^\infty \frac{1}{t^{1+\frac{1}{\gamma}}} \Delta t < \infty,$$

which implies that (24) holds. To apply Theorem 2.3, it remains to satisfy the conditions (4) and (25). Take $\eta(s) = 1$, then we get

$$\begin{aligned}
\limsup_{t\to\infty} \int_{t_0}^t \left\{\eta(s)q(s) - \frac{a(s)[(\eta^\Delta(s))_+]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\eta^\gamma(s)}\right\}\Delta s &= \limsup_{t\to\infty} \int_{t_0}^t (s+\sigma(s))\Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^t (s^2)^\Delta \Delta s \\
&= \limsup_{t\to\infty} (t^2 - t_0^2)\Delta s = \infty,
\end{aligned}$$

which implies that (4) holds. For every constant $C \geq t_0$, we can find $0 < M < 1$ and $t_M \geq C$ such that $t - C \geq Mt$ for $t \in [t_M, \infty)$. Thus, we conclude

$$\begin{aligned}
\int_C^\infty \left[\frac{1}{a(t)}\int_C^t q(s)\Delta s\right]^{1/\gamma} \Delta t &= \int_C^\infty \left[\frac{1}{t^{\gamma+1}}\int_C^t (s+\sigma(s))\Delta s\right]^{1/\gamma} \Delta t = \int_C^\infty \left(\frac{t^2 - C^2}{t^{\gamma+1}}\right)^{1/\gamma} \Delta t \\
&= \int_C^\infty \left[\frac{(t+C)(t-C)}{t^{\gamma+1}}\right]^{1/\gamma} \Delta t \geq (2CM)^{\frac{1}{\gamma}} \int_{t_M}^\infty \frac{1}{t}\Delta t = \infty,
\end{aligned}$$

which yields that (25) holds. Hence, by Theorem 2.3 every solution of (28) is oscillatory or converges to zero as $t \to \infty$.

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