On Dislocated Fuzzy Topologies and Fixed Points

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Abstract

The dislocated topology associated with a Dislocated Fuzzy Quasi Metric Space ($DFqM - Space$) is discussed and a generalised common fixed point theorem for two mappings $f : X \to X$ and $T : X^k \to X$ in a $DFqM - Space$ is proved. Our result extends and generalises many well known results.

Keywords: Dislocated fuzzy topology, Dislocated fuzzy quasi metric space, Coincidence and common fixed points, Occasionally weakly compatible

1. Introduction

Zadehs introduction (Zadeh, 1965) of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In the last two decades there were a tremendous growth in fuzzy mathematics. Many fixed point theorems for contractions in fuzzy metric spaces and quasi fuzzy metric spaces appeared (see Cho, Sedghi, & Shobe, 2009; Fang, 1992; George & Veeramani, 1997; Gregory & Romuguera, 2002; Gregory & Sapena, 2002; Mihet, 2004; Mishra, Sharma, & Singh, 1994; Radu, 2002; Romaguera, Sapena, & Tirado, 2007; Sedghi, Turkoghlu, & Shobe, 2007; Sharma, 2002; Vasuki, 1998; Vasuki & Veeramani, 2003). The role of topology in logic programming has come to be recognized in recent years. In particular topological methods are employed in order to obtain fixed point semantics for logic programs. In classical approach to logic programming semantics in which positive or definite positive programs are considered (those in which negation does not occur) Knaster - Tarski fixed point theorem can be applied to obtain a least fixed point of an operator called the single step or immediate consequence operator. However when the syntax is enhanced in the sense that negation is allowed, the approach using Knaster - Tarski theorem does not work. In such cases the Banach contraction mapping theorem for complete metric spaces is an alternative to Knaster - Tarski fixed point theorem. However topological spaces which arise in the area of denotational semantics are often not Hausdorff and so spaces which are weaker than metric spaces in a topological sense had to be cosidered. Motivated by this fact Hitzler and Seda (Hitzler & Seda, 2000) introduced the concept of dislocated metric space and studied the dislocated topologies which is a generalisation of the conventinal topologies and can be thought of as underlying the notion of dislocated metrics. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. Later George and Khan (Reny George & Khan, 2005) introduced the concept of dislocated fuzzy metric spaces and studied the associated topologies. Alaca (Alaca, 2010) introduced the concept of Dislocated Fuzzy Quasi Metric Space ($DFqM - Space$)in the sense of Kramosil and Michalek as well as George and Veeramani and discussed the topologies associated with it which is conventional in nature. In this paper we have discussed the dislocated fuzzy topologies associated with a $DFqM - Space$ and also proved a common fixed point theorem of Presic type which extends and generalises the well known Banach contraction principle in a Fuzzy metric space and also fuzzyfies other known results.

2. Preliminaries

Let (X, d) be a metric space and $T : X \to X$ and $f : X \to X$ be mappings. Let $C(T, f)$ denote the set of all coincidence points of the mappings f and T, that is $C(T, f) = \{u : fu = Tu\}$.

Definition 2.1 The mappings f and T are said to be weakly compatible if and only if they commute at their coincidence points.

Remark 2.2 Clearly mappings f and T are not weakly compatible if and only if there exist some $u \in C(T, f)$, such that $fTu \neq Tfu$. Obviously, if $C(T, f) = \phi$ then f and T are weakly compatible.

Definition 2.3 The mappings f and T are said to be occasinally weakly compatible (*owc*) if and only if they commute at some coincidence point of f and T, i.e. $fTu = Tfu$ for some $u \in C(T, f)$.

Remark 2.4 Occasionally weakly compatible mappings requires the set of coincidence points to be non empty. In other words if $C(T, f) = \phi$ then mappings f and T cease to be *owc*. Hence *owc* mappings cannot be considered as a generalisation of weakly compatible mappings. Hence we put it in the following way.

Definition 2.5 The mappings f and T are said to be occasionally weakly compatible (*owc*) if and only if $fTu = Tfu$ for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

Definition 2.6 f is said to be coincidentally idempotent with respect to T if and only if f is idempotent at the coincidence points of f and T.

Definition 2.7 The mapping f is said to be occasionally coincidentally idempotent (*oci*) if and only if $ffu = fu$ for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

Definition 2.8 (Hitzler & Seda 2000) Let X be a set. A relation $\succ \subseteq X \times P(X)$ is called a *d-membership relation* on X if it satisfies the following property :
$x \succ A$ and $A \subseteq B$ implies $x \succ B$ $\forall x \in X$, and $A, B \in P(X)$, where P(X) is the power set of X.
If $x \succ A$ we read it as x is below A.

Definition 2.9 (Hitzler & Seda 2000) Let X be a set, $\succ$ be a d-membership relation on X. For each $x \in X$, let Ux be the collection of all subsets of X satisfying the following conditions:

(N1) if $U \in Ux$ then $x \succ U$.

(N2) if $U, V \in Ux$ then $U \cap V \in Ux$.

(N3) if $U \in Ux$ then there is a $V \subset U$ with $V \in Ux$, such that for all $y \succ V$ we have $U \in Uy$.

(N4) if $U \in Ux$ and $U \subset V$ then $V \in Ux$.

Then (Ux, $\succ$) is called a d-neighborhood system for x and each $U \in Ux$ is called a d-neighborhood of x.

Remark 2.10 If $U = \{Ux : x \in X\}$ then $(X, U, \succ)$ is called a dislocated topological space or a d-topological space.

Remark 2.11 Note that points can have empty d-neighborhoods and the above definition is exactly the definition of a topological neighbourhood system if the relation $\succ$ is replaced by the membership relation $\in$.

3. Dislocated Fuzzy Quasi Metric Space and Topologies

In this section we will define Dislocated Fuzzy Quasi Metric Space and discuss the topologies associated with it.

Definition 3.1 (Scweizer & Sklar 1960) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $([0, 1], *)$ is an abelian monoid with unit one such that, for all a, b, c, d in [0, 1], $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$.

Definition 3.2 Let X be any non empty set, * be a continuous t-norm and $M : X^2 \times [0, \infty) \to [0, 1]$ be a fuzzy set. Consider the following conditions:

For all $x, y, z \in X$ and $t, s \in [0, \infty)$

$FM1$ $M(x, y, 0) = 0$

$FM2$ $M(x, x, t) = 1$

$FM3$ $M(x, y, t) = 1$ and $M(y, x, t) = 1 \Rightarrow x = y$

$FM4$ $M(x, y, t) = M(y, x, t)$

$FM5$ $M(x, y, t + s) \geq M(x, z, t) + M(z, y, s)$

$FM6$ $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous

$FM7$ $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous

If M satisfies conditions $FM1$ to $FM6$ then $(X, M, *)$ is called a *Fuzzy Metric Space* (Kramosil & Michalec 1975). If M satisfies conditions $FM1$ and $FM3$ to $FM6$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Metric Space in the sense of Kramosil and Michalek (in short $D_{KM}FM - Space$)* (Reny George & Khan 2005). If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions $FM1$ and $FM3$ to $FM5$ and $FM7$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Metric Space in the sense of George and Veeramani (in short $D_{GV}FM - Space$)*(Reny George & Khan 2005). If M satisfies conditions $FM1, FM3, FM5$ and $FM6$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Quasi Metric Space in the sense of Kramosil and Michalek (in short $D_{KM}FqM - Space$)*(Alaca 2010). If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions $FM3, FM5$ and $FM7$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Quasi Metric Space in the sense of George and Veeramani (in short $D_{GV}FqM - Space$)* (Alaca 2010).

Example 3.3 Let $X = R$; Define $a * b = ab$, M(x, y, t) = $[exp^{\frac{|x-y|+2|x|+|y|}{t}}]^{-1}$ for all $(x, y) \in X \times X$, $t \in (0, \infty)$. Then $(X, M, *)$ is a $D_{GV}FqM - Space$).

For all $(x, y) \in X \times X$, $t \in (0, \infty)$ let $M^{\ddagger}(x, y, t) = min\{M(x, y, t), M(y, x, t)\}$. Clearly if $(X, M, *)$ is a $D_{GV}FqM - Space$ ($D_{KM}FqM - Space$) then $(X, M^{\ddagger}, *)$ is a $D_{GV}FM - Space$ ($D_{KM}FM - Space$). Obviously each $D_{GV}FqM - Space$ can be cosidered as a $D_{KM}FqM - Space$ by defining $M(x, y, 0) = 0$ for all $x, y \in X$ (see Gregory & Romuguera 2004). Hereafter by a Dislocated Fuzzy Quasi Metric Space ($DFqM - Space$) we mean a $D_{GV}FqM - Space$ or a $D_{KM}FqM - Space$.

Definition 3.4 Let $(X, M, *)$ be a $DFqM - Space$. We define a left open ball (L-open ball) with centre x and radius $r(0 < r < 1)$ in X as $B_L(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$, for all $t \in (0, \infty)$. We define a right open ball(R-open ball) with centre x and radius r $(0 < r < 1)$ in X as $B_R(x, r, t) = \{y \in X : M(y, x, t) > 1 - r\}$, for all $t \in (0, \infty)$. We define an open ball with centre x and radius r $(0 < r < 1)$ in X as $B(x, r, t) = \{y \in X : M^{\ddagger}(x, y, t) > 1 - r\}$, for all $t \in (0, \infty)$.

Obviously $B(x, r, t) = B_L(x, r, t) \bigcap B_R(x, r, t)$ and its not necessary that $x \in B(x, r, t)$ for all $x \in X$.

Proposition 3.5 *Let $(X, M, *)$ be a $DFqM - Space$. Define a d-membership relation $\succ$ by $x \succ A$ if and only if there exists $r(0 < r < 1)$ such that $B(x, r, t)) \subseteq A$. For each $x \in X$, let Ux be the collection of subsets A of X such that $x \succ A$. Then $(Ux, \succ)$ is a d-neighborhood system for x.*

Proof: $(N1)$ Clearly if $U \in Ux$ then $x \succ U$ and vice versa.

$(N2)$ Let $U, V \in Ux$, i.e. $x \succ U$ and $x \succ V$. Then there exists $0 < r_1 < 1$ and $0 < r_2 < 1$ such that $B(x, r_1, t) \subseteq U$ and $B(x, r_2, t) \subseteq V$. Suppose $r_1 < r_2$. Then we have $1 - r_1 > 1 - r_2$. Let $y \in B(x, r_1, t)$. Then $M^{\ddagger}(x, y, t) > 1 - r_1 \Rightarrow M^{\ddagger}(x, y, t) > 1 - r_2 \Rightarrow y \in B(x, r_2, t)$. Thus $B(x, r_1) \subseteq B(x, r_2, t)$. Hence $B(x, r_1, t) = B(x, r_1, t) \cap B(x, r_2, t) \subseteq U \cap V \Rightarrow x \succ U \cap V$.

$(N3)$ Let $U \in Ux$ i.e. $x \succ U$. Then, there exists $0 < r < 1$ such that $B(x, r, t) \subseteq U$. Also $B(x, r, t) \in Ux$. Let $y \in B(x, r, t)$ be arbitrary. Then there exist $0 < r_1 < 1$ such that $B(y, r_1, t) \subseteq B(x, r, t)$. But $y \succ B(y, r_1, t)$. Thus we have $y \succ B(y, r_1, t) \subseteq B(y, r, t) \subseteq U \Rightarrow y \succ U$. Thus $U \in Uy$.

$(N4)$ This is obvious.

Remark 3.6 The above construction yields the usual dislocated fuzzy topology associated with a $DFqM - Space$ and consequently we may define a dislocated topology T_D on X as $T_D = \{A \subset X : x \succ A$ for some $x \in X$ and $\succ$ is as defined in Proposition 3.5. The topology T_D is called the Dislocated Fuzzy topology associated with the $DFqM - Space$ $(X, M, *)$. Clearly open balls in a $DFqM - Space$ is open in the dislocated topology T_D. Collection of open balls in a $DFqM - Space$ does not in general yield a conventional topology.

Proceeding on the same lines as in the proof of above proposition we have the following.

Proposition 3.7 *Let $(X, M, *)$ be a $DFqM - Space$. Define a d-membership relation $\succ_L$ by $x \succ_L A$ if and only if there exists r $(0 < r < 1)$ such that $B_L(x, r, t)) \subseteq A$. For each $x \in X$, let U_Lx be the collection of subsets A of X such that $x \succ_L A$. Then $(U_Lx, \succ_L)$ is a d-neighborhood system for x.*

Proposition 3.8 *Let $(X, M, *)$ be a $DFqM - Space$. Define a d-membership relation $\succ_R$ by $x \succ_R A$ if and only if there exists $r(0 < r < 1)$ such that $B_R(x, r, t)) \subseteq A$. For each $x \in X$, let U_Rx be the collection of subsets A of X such that $x \succ_R A$. Then $(U_Rx, \succ_R)$ is a d-neighborhood system for x.*

Definition 3.9 A sequence x_n in a $DFqM - Space$ $(X, M, *)$ is said to be bi-convergent to a point $x \in X$ if and only if $Lim_{n\to\infty} M^{\ddagger}(x_n, x, t) = 1$ for all $t > 0$. In this case we say that limit of the sequence x_n is x.

Definition 3.10 A sequence x_n in a $DFqM - Space$ $(X, M, *)$ is said to be Left (Right) Cauchy sequence if and only if $Lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ $(Lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1)$ for all $t > 0$, $p > 0$.

Definition 3.11 A sequence x_n in a $DFqM-Space$ $(X, M, *)$ is said to be bi-Cauchy if and only if $Lim_{n\to\infty}M^{\ddagger}(x_n, x_{n+p}, t) = 1$ for all $t > 0$, $p > 0$.

Definition 3.12 A $DFqM-Space$ is said to be Left (Right) complete if and only if every Left (Right) Cauchy sequence in it is bi-convergent.

Definition 3.13 A $DFqM-Space$ is said to be bi-complete if and only if every bi-Cauchy sequence in it is bi-convergent.

Remark 3.14 Clearly a sequence x_n in a $DFqM-Space$ $(X, M, *)$ is bi- Cauchy sequence if and only if sequence x_n is a Cauchy sequence in the $DFM-Space$ $(X, M^{\ddagger}, *)$. A $DFqM-Space$ $(X, M, *)$ is bi-Complete if and only if the $DFM-Space$ $(X, M^{\ddagger}, *)$ is complete.

Proposition 3.15 *Limit of a sequence in a DFqM − S pace* $(X, M, *)$ *is unique.*

Proof: Let x_n be a sequence in X and suppose u and v are two limits of x_n. Then we have $M^{\ddagger}(u, v, t) \geq M^{\ddagger}(u, x_n, t/2) * M^{\ddagger}(x_n, v, t/2)$. Taking the limit as $n \to \infty$ we have $M^{\ddagger}(u, v, t) \geq 1 * 1 = 1$. Hence $u = v$.

Proposition 3.16 *Let* $(X, M, *)$ *be a DFqM−S pace(DFM−S pace) and* x_n *be a sequence in X. If sequence* x_n *bi-converges (converges) to* $x \in X$ *then* $M(x, x, t) = 1$ *for all* $t > 0$.

Proof: We have $M(x, x, t) \geq M(x, x_n, t/2)*M(x_n, x, t/2)$ for all n. Taking the limit as $n \to \infty$ we have $M(x, x, t) \geq 1*1 = 1$.

Proposition 3.17 *Let* $(X, M, *)$ *be a DFqM − S pace (DFM − S pace),* $f, g : X \to X$ *be mappings. If* $fz = gz$ *and* $M^{\ddagger}(fgz, gfz, t) = 1$ $(M(fgz, gfz, t) = 1)$ *for some* $z \in X$ *and* $t \in [0, \infty)$*, then* $M(ffz, ffz, t) = 1$ *for all* $t \in [0, \infty)$. *Proof:* Since $M^{\ddagger}(fgz, gfz, t) = 1$ we have $fgz = gfz$. Therefore $M(ffz, ffz, t) = M(fgz, fgz, t) = M(fgz, gfz, t) = 1$.

4. Common Fixed Point Theorems

Let $(X, M, *)$ be a $DFqM-Space$, $T : X^k \to X$ and $f : X \to X$ be mappings.A point $z \in X$ is said to be a *coincidencepoint* of f and T if $T(z, z, ..., z) = fz$. z is said to be a *coomonfixedpoint* of f and T if $T(z, z, ..., z) = fz = z$. Let $C(T, f)$ denote the set of all coincidence points of the mappings f and T, that is $C(T, f) = \{z : fz = T(z, z, ..., z)\}$.

Definition 4.1 The mappings f and T in a $DFqM-Space$ $(DFM-Space)$ are said to be weakly compatible if and only if $M^{\ddagger}(T(fz, fz, ..., fz), f(T(z, z, ..., z), t) = 1$ $(M(T(fz, fz, ..., fz), f(T(z, z, ..., z), t) = 1)$ for all $z \in C(T, f)$ and $t \in [0, \infty)$.

Definition 4.2 The mappings f and T in a $DFqM-Space$ $(DFM-Space)$ are said to be occasinally weakly compatible (owc) if and only if $M^{\ddagger}(T(fz, fz, ..., fz), f(T(z, z, ..., z), t) = 1$ $(M(T(fz, fz, ..., fz), f(T(z, z, ..., z), t) = 1)$for some $z \in C(T, f)$ and $t \in [0, \infty)$, whenever $C(T, f) \neq \phi$.

Definition 4.3 f is said to be coincidentally idempotent with respect to T if and only if f is idempotent at the coincidence points of f and T.

Definition 4.4 The mapping f is said to be occasionally coincidentally idempotent (oci)with respect to T if and only if $ffu = fu$ for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

Now we present our main results as follows:

Theorem 4.5 *Let* $(X, M, *)$ *be a DFM − S pace, k a positive integer,* $f : X \longrightarrow X$ *and* $T : X^k \longrightarrow X$ *be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{1}$$

$$M(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}), qt) \geq Min\{M(fx_1, fx_2, t), M(fx_2, fx_3, t), ...$$

$$M(fx_k, fx_{k+1}, t)\}, \tag{2}$$

where $x_1, x_2, ...x_{k+1}$ *are arbitrary elemants in X,* $0 < q < \frac{1}{2}$ *and* $t \in [0, \infty)$

$$f(X) \text{ is complete} \tag{3}$$

$$lim_{t\to\infty}M(x, y, t) = 1. \tag{4}$$

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$*. Further f and T has a common fixed point provided the pair* (f, T) *are weakly compatible.*

Proof: Let $x_1, x_2, ..., x_k$ be arbitrary elemants in X. By (1) we define sequence $< y_n >$ in $f(X)$ as follows :

$$y_{n+k} = f(x_{n+k}) = T(x_n, x_{n+1}, ..., x_{n+k-1}), \text{for } n = 1, 2, ...$$

Let $\alpha_n = M(y_n, y_{n+1}, qt)$. By the method of mathematical induction we will prove that

$$\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2 \tag{5}$$

where $\theta = \frac{1}{q}$, $K = Min\{\frac{\theta(1+\sqrt{\alpha_1})}{(1-\sqrt{\alpha_1})}, \frac{\theta^2(1+\sqrt{\alpha_2})}{(1-\sqrt{\alpha_2})}, ..., \frac{\theta^k(1+\sqrt{\alpha_k})}{(1-\sqrt{\alpha_k})}\}$.

Clearly from the definition of K, we see that (5) is true for $n = 1, 2, ..., k$. Let the k inequalities $\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \alpha_{n+1} \geq \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, ..., \alpha_{n+k-1} \geq \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2$, be the induction hypothesis. Then we have

$$\begin{aligned}
\alpha_{n+k} &= M(y_{n+k}, y_{n+k+1}, qt)\\
&\geq M(T(x_n, x_{n+1}, ...x_{n+k-1}), T(x_{n+1}, x_{n+2}, ...x_{n+k}), qt)\\
&\geq Min\{M(fx_n, fx_{n+1}, t), M(fx_{n+1}, fx_{n+2}, t), ...M(fx_{n+k-1}, fx_{n+k}, t)\}\\
&\geq Min\{\alpha_n, \alpha_{n+1}, ..., \alpha_{n+k-1}\}\\
&\geq Min\{\left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, ...\left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\}\\
&= \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\\
&\geq \left(\frac{K-\theta^{n+k}}{K+\theta^{n+k}}\right)^2.
\end{aligned}$$

Thus inductive proof of (5) is complete.

Now for $p \in N$ and $t \in [0, \infty)$, we have

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, \frac{t}{2}) \star M(y_n, y_{n+1}, \frac{t}{2^2}) \star ... \star M(y_{n+p-1}, y_{n+p}, \frac{t}{2^p}) \geq \left(\frac{K-2^n}{K+2^n}\right)^2 \star \left(\frac{K-2^{2n}}{K+2^{2n}}\right)^2 \star ... \star \left(\frac{K-2^{np}}{K+2^{np}}\right)^2$
$\to 1 \star 1 \star ... \star 1 = 1$, as $n \to \infty$. Hence $< y_n >$ is a Cauchy sequence in $f(X)$ and since $f(X)$ is complete, there will exist z in $f(X)$ such that $lim_{n\to\infty}y_n = z$. Let $z = f(u)$ for some $u \in X$. Then we have
$M(T(u, u, ..., u), fu, t) = lim_{n\to\infty}M(T(u, u, ..., u), y_{n+k}, t)$
$= lim_{n\to\infty}M(T(u, u, ..., u), T(x_n, x_{n+1}, ..., x_{n+k-1}), t)$
$\geq lim_{n\to\infty}M(T(u, u, ..., u), T(u, u, ..., x_n), \frac{t}{2}) \star M(T(u, u, ..., x_n), T(u, u, ..., x_n, x_{n+1}), \frac{t}{2^2}) \star M(T(u, u, ..., x_n, x_{n+1}),$
$T(u, u, ..., x_n, x_{n+1}, x_{n+2}), \frac{t}{2^3}) \star ... \star M(T(u, x_n, ..., x_{n+k-2}), T(x_n, x_{n+1}, ..., x_{n+k-1}), \frac{t}{2^{k-1}})$
$\geq lim_{n\to\infty}M(fu, fx_n, t) \star Min(M(fu, fx_n, t), M(fx_n, fx_{n+1}, t)) \star ... \star Min(M(fu, fx_n, t), M(fx_n, fx_{n+1}, t)),$
$...M(fx_{n+k-2}, fx_{n+k-1}, t)) \to 1$ i.e. $fu = T(u, u, ..., u)$, and hence $C(T, f) \neq \phi$.

Now suppose the pair (f, T) are weakly compatible. Then we have
$M(ffu, fu, qt) = M(fT(u, u, ..., u), fu, qt) = M(T(fu, fu, ..., fu), fu, qt)$. Also we have
$M(T(fu, fu, ..., fu), fu, qt) = M(T(fu, fu, ..., fu), T(u, u, ..., u), qt)$
$\geq M(T(fu, fu, ..., fu), T(fu, fu, ..., fu, u), \frac{qt}{2}) \star M(T(fu, fu, ..., fu, u), T(fu, fu, ..., u, u), \frac{qt}{2^2}) \star M(T(fu, fu, ..., u, u),$
$T(fu, fu, ..., u, u, u), \frac{qt}{2^3}) \star ... \star M(T(fu, u, ..., u), T(u, u, ..., u), \frac{qt}{2^{k-1}})$

By (2), Propositions 3.14 and 3.15 we get
$M(T(fu, fu, ..., fu), fu, t) \geq M(ffu, fu, \frac{t}{2}) \star M(ffu, fu, \frac{t}{2^2}) \star ... \star M(ffu, fu, \frac{t}{2^{k-1}})$
$= M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2}) \star M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2^2}) \star ... \star M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2^{k-1}})$
$\geq M(ffu, fu, \frac{t}{2q}) \star M(ffu, fu, \frac{t}{2^2q}) \star ... \star M(ffu, fu, \frac{t}{2^{k-1}q})$

Repeating the above process n times we get
$M(T(fu, fu, ..., fu), fu, t) \geq M(ffu, fu, \frac{t}{2^nq^n}) \star M(ffu, fu, \frac{t}{2^{n+1}q^n}) \star ... \star M(ffu, fu, \frac{t}{2^{n+k-2}q^n})$

Taking the limit as $n \to \infty$ we get
$M(T(fu, fu, ..., fu), fu, t) \geq 1 \star 1 \star ....1 = 1$. Hence $fu = ffu = T(fu, fu, ..., fu)$.

In the next result we will remove the condition $lim_{t\to\infty}M(x, y, t) = 1$ and also increase the range of q.

Theorem 4.6 *Let $(X, M, *)$ be a $DFM-Space$, k a positive integer, $f : X \longrightarrow X$ and $T : X^k \longrightarrow X$ be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{6}$$

$M(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}), qt) \geq Min\{M(fx_1, fx_2, t), M(fx_2, fx_3, t), ...$

$$M(fx_k, fx_{k+1}, t)\}, \tag{7}$$

where $x_1, x_2, ..., x_{k+1}$ *are arbitrary elemants in X,* $0 < q < 1$ *and* $t \in [0, \infty)$

$$f(X) \text{ is complete} \tag{8}$$

Then f *and* T *has a coincidence point, i.e.* $C(f, T) \neq \phi$*. Further* f *and* T *has a common fixed point provided either* f *is oci with respect to* T *and the pair* (f, T) *are weakly compatible or* f *is coincidentally idempotent with respect to* T *and the pair* (f, T) *are occasionally weakly compatible.*

Proof: Proceeding as in the proof of previous theorem, we can show that $C(T, f) \neq \phi$. Now suppose f is *oci* with respect to T and the pair (f, T) are weakly compatible. Then there will exist $z \in C(f, T)$ such that $ffz = fz$ and also $f(T(z, z, ..., z)) = T(fz, fz, ..., fz)$.

Thus we have $fz = ffz = f(T(z, z, ..., z)) = T(fz, fz, ..., fz)$, i.e. fz is a common fixed point of f and T. The proof follows on the same lines in the other case also.

Theorem 4.7 *Let* $(X, M, *)$ *be a* $DFqM - Space$*, k a positive integer,* $f : X \longrightarrow X$ *and* $T : X^k \longrightarrow X$ *be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{9}$$

$$M(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}), qt) \geq Min\{M(fx_1, fx_2, t), M(fx_2, fx_3, t), ...$$
$$M(fx_k, fx_{k+1}, t)\} \tag{10}$$

where $x_1, x_2, ...x_{k+1}$ *are arbitrary elemants in X,* $0 < q < \frac{1}{2}$ *and* $t \in [0, \infty)$

$$f(X) \text{ is L-complete} \tag{11}$$

$$lim_{t\to\infty} M(x, y, t) = 1 for all x, y \in X. \tag{12}$$

Then f *and* T *has a coincidence point, i.e.* $C(f, T) \neq \phi$*. Further* f *and* T *has a common fixed point provided the pair* (f, T) *are weakly compatible.*

Proof: Proceeding as in Theorem 4.5 we see that $M(y_n, y_{n+p}, t) \to 1$, as $n \to \infty$ and so $< y_n >$ is a L-Cauchy sequence in $f(X)$ and since $f(X)$ is L-complete, there will exist z in $f(X)$ such that $lim_{n\to\infty} y_n = z$. Rest of the proof goes on the same lines as that of Theorem 4.5.

Theorem 4.8 *Let* $(X, M, *)$ *be a* $DFqM - Space$*, k a positive integer,* $f : X \longrightarrow X$ *and* $T : X^k \longrightarrow X$ *be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{13}$$

$$M(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}), qt) \geq Min\{M(fx_1, fx_2, t), M(fx_2, fx_3, t), ...$$
$$M(fx_k, fx_{k+1}, t)\} \tag{14}$$

where $x_1, x_2, ..., x_{k+1}$ *are arbitrary elemants in X,* $0 < q < 1$ *and* $t \in [0, \infty)$

$$f(X) \text{ is L-complete} \tag{15}$$

Then f *and* T *has a coincidence point, i.e.* $C(f, T) \neq \phi$*. Further* f *and* T *has a common fixed point provided either* f *is oci with respect to* T *and the pair* (f, T) *are weakly compatible or* f *is coincidentally idempotent with respect to* T *and the pair* (f, T) *are occasionally weakly compatible.*

Proof: The proof follows on the same line as in the previous theorem and Theorem 4.6.

Theorem 4.9 *Let* $(X, M, *)$ *be a* $DFqM - Space$*, k a positive integer,* $f : X \longrightarrow X$ *and* $T : X^k \longrightarrow X$ *be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{16}$$

$$M(T(x_1, x_2, ..., x_k), T(x_{k+1}, x_1, x_2, ..., x_{k-1}), qt) \geq Min\{M(fx_1, fx_{k+1}, t), M(fx_2, fx_1, t), ...$$
$$M(fx_k, fx_{k-1}, t)\} \tag{17}$$

where $x_1, x_2, ...x_{k+1}$ *are arbitrary elemants in X,* $0 < q < \frac{1}{2}$ *and* $t \in [0, \infty)$

$$f(X) \text{ is R-complete} \tag{18}$$

$$lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X. \tag{19}$$

Then f and T has a coincidence point, i.e. $C(f,T) \neq \phi$. Further f and T has a common fixed point provided the pair (f,T) are weakly compatible.

Proof: Let $x_1, x_2, ..., x_k$ be arbitrary elemants in X. By (16) we define sequence $< y_n >$ in $f(X)$ as follows:

$$y_{n+k} = f(x_{n+k}) = T(x_n, x_{n+1}, ..., x_{n+k-1}), \text{ for } n = 1, 2, ...$$

Let $\alpha_n = M(y_{n+1}, y_n, qt)$. By the method of mathematical induction we will prove that

$$\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2 \tag{20}$$

where $\theta = \frac{1}{q}$, $K = Min\{\frac{\theta(1+\sqrt{\alpha_1})}{(1-\sqrt{\alpha_1})}, \frac{\theta^2(1+\sqrt{\alpha_2})}{(1-\sqrt{\alpha_2})}, ..., \frac{\theta^k(1+\sqrt{\alpha_k})}{(1-\sqrt{\alpha_k})}\}$.

Clearly from the definition of K, we see that (4.4) is true for $n = 1, 2, ..., k$. Let the k inequalities $\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \alpha_{n+1} \geq \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, ...\alpha_{n+k-1} \geq \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2$, be the induction hypothesis. Then we have

$$\begin{aligned}
\alpha_{n+k} &= M(y_{n+k+1}, y_{n+k}, qt)\\
&\geq M(T(x_{n+1}, x_{n+2}, ..., x_{n+k}), T(x_n, x_{n+1}, x_{n+2}, ..., x_{n+k-1}), qt)\\
&\geq Min\{M(fx_{n+1}, fx_n, t), M(fx_{n+2}, fx_{n+1}, t), ...M(fx_{n+k}, fx_{n+k-1}, t)\}\\
&\geq Min\{\alpha_n, \alpha_{n+1}, ...\alpha_{n+k-1}\}\\
&\geq Min\{\left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, ..., \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\\
&= \left(\frac{K-\theta^{n+k-1}}{K+\theta^{n+k-1}}\right)^2\\
&\geq \left(\frac{K-\theta^{n+k}}{K+\theta^{n+k}}\right)^2
\end{aligned}$$

Thus inductive proof of (20) is complete.

Now for $p \in N$ and $t \in [0, \infty)$, we have
$M(y_{n+p}, y_n, t) \geq M(y_{n+p}, y_{n+p-1}, \frac{t}{2}) \star M(y_{n+p-1}, y_{n+p-2}, \frac{t}{2}) \star ... \star M(y_{n+1}, y_n, \frac{t}{2^2})$
$\geq \left(\frac{K-2^n}{K+2^n}\right)^2 \star \left(\frac{K-2^{2n}}{K+2^{2n}}\right)^2 \star ... \star \left(\frac{K-2^{np}}{K+2^{np}}\right)^2$
$\to 1 \star 1 \star ... \star 1 = 1$, as $n \to \infty$. Hence $< y_n >$ is a R-Cauchy sequence in $f(X)$ and since $f(X)$ is R-complete, there will exist z in $f(X)$ such that $lim_{n\to\infty} y_n = z$. Let $z = f(u)$ for some $u \in X$. Then we have
$M(T(u, u, ..., u), fu, t) = lim_{n\to\infty} M(T(u, u, ..., u), y_{n+k}, t)$
$= lim_{n\to\infty} M(T(u, u, ...u), T(x_n, x_{n+1}, ...x_{n+k-1}), t)$
$\geq lim_{n\to\infty} M(T(u, u, ..., u), T(x_{n+k-1}, u, ..., u), \frac{t}{2}) \star M(T(x_{n+k-1}, u, ..., u), T(x_{n+k-2}, x_{n+k-1}, u, ..., u), \frac{t}{2^2}) \star$
$M(T(x_{n+k-2}, x_{n+k-1}, u, ..., u), T(x_{n+k-3}, x_{n+k-2}, x_{n+k-1}, ..., u), \frac{t}{2^3}) \star ... \star M(T(x_{n+1}, ..., x_{n+k-1}, u), T(x_n, x_{n+1}, ..., x_{n+k-1}), \frac{t}{2^{k-1}})$
$\geq lim_{n\to\infty} M(fu, fx_{n+k-1}, t) \star Min(M(fu, fx_{n+k-1}, t), M(fx_{n+k-1}, fx_{n+k-2}, t)) \star$
$... \star Min(M(fu, fx_{n+k-1}, t), M(fx_{n+k-1}, fx_{n+k-2}, t)), ..., M(fx_{n+1}, fx_n, t)) \to 1$.

Similarly it can be shown that $M(fu, T(u, u, ..., u), t) = 1$. Hence $fu = T(u, u, ..., u)$, and so $C(T, f) \neq \phi$.

Now suppose the pair (f, T) are weakly compatible. Then we have
$M(ffu, fu, qt) = M(fT(u, u, ..., u), fu, qt) = M(T(fu, fu, ..., fu), fu, qt)$. Also we have
$M(T(fu, fu, ..., fu), fu, qt) = M(T(fu, fu, ..., fu), T(u, u, ..., u), qt)$
$\geq M(T(fu, fu, ..., fu), T(u, fu, ..., fu), \frac{qt}{2}) \star M(T(u, fu, ..., fu), T(u, u, ..., fu, fu), \frac{qt}{2^2}) \star M(T(u, u, ..., fu, fu),$
$T(u, u, u, ..., fu, fu), \frac{qt}{2^3}) \star ... \star M(T(u, u, ..., fu), T(u, u, ..., u), \frac{qt}{2^{k-1}})$

By (4.17), Proposition 3.14 and 3.15 we get
$M(T(fu, fu, ..., fu), fu, t) \geq M(ffu, fu, \frac{t}{2}) \star M(ffu, fu, \frac{t}{2^2}) \star ... \star M(ffu, fu, \frac{t}{2^{k-1}})$
$= M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2}) \star M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2^2}) \star ... \star M(T(fu, fu, ..., fu), T(u, u, ..., u), \frac{t}{2^{k-1}})$
$\geq M(ffu, fu, \frac{t}{2q}) \star M(ffu, fu, \frac{t}{2^2 q}) \star ... \star M(ffu, fu, \frac{t}{2^{k-1} q})$.

Repeating the above process n times we get
$M(T(fu, fu, ..., fu), fu, t) \geq M(ffu, fu, \frac{t}{2^n q^n}) \star M(ffu, fu, \frac{t}{2^{n+1} q^n}) \star ... \star M(ffu, fu, \frac{t}{2^{n+k-2} q^n})$.

Taking the limit as $n \to \infty$, we get
$M(T(fu, fu, ..., fu), fu, t) \geq 1 \star 1 \star ....1 = 1$. Hence $fu = ffu = T(fu, fu, ..., fu)$.

Theorem 4.10 *Let $(X, M, *)$ be a $DFqM-Space$, k a positive integer, $f : X \longrightarrow X$ and $T : X^k \longrightarrow X$ be mappings, such that*

$$T(X^k) \subseteq f(X) \tag{21}$$

$$M(T(x_1, x_2, ..., x_k), T(x_{k+1}, x_1, x_2, ..., x_{k-1}), qt) \geq Min\{M(fx_1, fx_{k+1}, t), M(fx_2, fx_1, t), ... \\ M(fx_k, fx_{k-1}, t)\} \tag{22}$$

where $x_1, x_2, ..., x_{k+1}$ are arbitrary elemants in X, $0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

$$\text{f(X) is R-complete} \tag{23}$$

$$lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X. \tag{24}$$

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$. Further f and T has a common fixed point provided either f is oci with respect to T and the pair (f, T) are weakly compatible or f is coincidentally idempotent with respect to T and the pair (f, T) are occasionally weakly compatible.

Taking $k = 1$ in the Theorems 4.7 and 4.9, we get the following.

Corollary 4.11 *Let $(X, M, *)$ be a $DFqM-Space$, $f : X \longrightarrow X$ and $T : X \longrightarrow X$ be mappings, such that*

$$T(X) \subseteq f(X) \tag{25}$$

$$M(Tx, Ty, qt) \geq Min\{M(fx, fy, t)\}, \tag{26}$$

for all $x, y \in X$, $0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

$$\text{f(X) is L-complete or R-complete} \tag{27}$$

$$lim_{t\to\infty} M(x, y, t) = 1. \tag{28}$$

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$. Further f and T has a common fixed point provided the pair (f, T) are weakly compatible.

Taking $k = 1$ in the Theorems 4.8 and 4.10, we get the following.

Corollary 4.12 *Let $(X, M, *)$ be a $DFqM-Space$, $f : X \longrightarrow X$ and $T : X \longrightarrow X$ be mappings, such that*

$$T(X) \subseteq f(X) \tag{29}$$

$$M(Tx, Ty, qt) \geq M(fx, fy, t), \tag{30}$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$

$$\text{f(X) is L-complete or R-complete} \tag{31}$$

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$. Further f and T has a common fixed point provided either f is oci with respect to T and the pair (f, T) are weakly compatible or f is coincidentally idempotent with respect to T and the pair (f, T) are occasionally weakly compatible.

If we take f to be the identity mapping in the above corollaries, we get the following.

Corollary 4.13 *Let $(X, M, *)$ be a L-complete or R-complete $DFqM-Space$, $T : X \longrightarrow X$ be mappings, such that*

$$M(Tx, Ty, qt) \geq M(x, y, t), \tag{32}$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$. Then T has a fixed point.

Remark 4.14 Corollary 4.13 is generalised fuzzy version of Banach Contraction Principle proved in (Grabiec, 1988). Corollary 4.11 and 4.12 are generalised and extended version of the result proved in (Reny George & Khan, 2005). Theorems 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10 are generalised and extended fuzzy version of the results proved in (Ciric & Presic, 2007; Dhage, 1987; Presic, 1965).

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