Generalized Fibonacci Sequences Generated from a k–Fibonacci Sequence

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Received: January 29, 2012 Accepted: February 15, 2012 Published: April 1, 2012
doi:10.5539/jmr.v4n2p97 URL: http://dx.doi.org/10.5539/jmr.v4n2p97

The research is financed in part by CICYT Project number MTM2008-05866-C03-02 from Ministerio de Educación y Ciencia of Spain

Abstract

In this paper we will prove that all k–Fibonacci sequence contains generalized Fibonacci sequences and we will indicate the form of obtain them.

Keywords: k–Fibonacci sequences, Recurrence relations, k–Lucas sequences

1. Introduction

k–Fibonacci numbers are defined (Falcon, S., 2007 (32) and (38) & 2009) by the recurrence equation $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

The k–Fibonacci sequences, for $k = 1, 2, \dots, 100$ are listed in *The Online Encyclopedia of Integer Sequences* (Sloane, N. J. A., 2006), from now on OEIS.

Characteristic equation of the initial recurrence relation is $r^2 - k\,r - 1 = 0$ and its solutions are $\sigma_1 = \dfrac{k + \sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k - \sqrt{k^2+4}}{2}$. As particular cases (Spinadel, V. W., 2002),

1. If $k = 1$, $\sigma_1 = \dfrac{1+\sqrt{5}}{2}$ is called *the Golden Ratio*, Φ
2. If $k = 2$, $\sigma_1 = 1 + \sqrt{2}$ is called *the Silver Ratio*
3. If $k = 3$, $\sigma_1 = \dfrac{3+\sqrt{13}}{2}$ is called *the Bronze Ratio*

Characteristic solutions verify these properties: $\sigma_1\sigma_2 = 1,\ \sigma_1 + \sigma_2 = k,\ \sigma_1 - \sigma_2 = \sqrt{k^2+4},\ \sigma^2 = k\sigma + 1$.

Among other properties, we have proved Binet's Identity (Falcon, S., 2007 (38)): $F_{k,n} = \dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$ and the formula of Honsberger (Honsberger, R. A., 1985) or Convolution formula (Falcon, S., 2007 (32) and (38) & 2009) for the k–Fibonacci numbers: $F_{k,m+r} = F_{k,r}F_{k,m+1} + F_{k,r-1}F_{k,m}$.

Finally, by summing up two alternate terms of the k–Fibonacci sequence, we obtain the corresponding k–Lucas sequence: $\{L_{k,n}\} = \{F_{k,n-1} + F_{k,n+1}\}$ (Falcon, S., 2011).

k–Lucas numbers are defined (Falcon, S. 2011) by the recurrence equation $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ with the initial conditions $L_{k,0} = 2$, $L_{k,1} = k$.

2. On the k, l–Fibonacci Sequences Contained in the k–Fibonacci Sequences

A generalized Fibonacci sequence (k, l–Fibonacci sequence) is a recurrent sequence $F(k,l) = \{F_n(k,l)\}$ such that $F_{n+1}(k,l) = kF_n(k,l) + lF_{n-1}(k,l)$, with the initial conditions $F_0(k,l) = 0$ and $F_1(k,l) = 1$. If $l = 1$, then we obtain the k–Fibonacci sequence.

In (Falcon, S., 2009) we have proved that for any $r, n \in N$ the k–Fibonacci number $F_{k,rn}$ is multiple of $F_{k,n}$. Then, for a fixed value of r, we can find out the sequence $\dfrac{F_{k,rn}}{F_{k,n}} = \{0, 1, \dfrac{F_{k,2r}}{F_{k,r}}, \ldots\}$. With object to simplify the writing, we will put

$$G_{k,n}(r) = \frac{F_{k,rn}}{F_{k,r}} \tag{1}$$

and the sequence generated is $G_k(r) = \{G_{k,n}(r)\}$.

For instance: for $k = 3$ and $r = 2$, it is $F_{3,2} = 11$, $F_{3,4} = 33, F_{3,6} = 360, etc$. Then, $G_{3,2}(2) = \dfrac{F_{3,4}}{F_{3,2}} = \dfrac{33}{3} = 11$, $G_{3,3}(2) = \dfrac{F_{3,6}}{F_{3,2}} = \dfrac{360}{3} = 120$, etc. and we obtain the sequence $\{0, 1, 11, 120, 1309, 14279, \ldots\}$. But this sequence is a $(11, -1)$ generalized Fibonacci sequence wich terms verify the recurrence relation $G_{3,n+1}(2) = 11G_{3,n}(2) - G_{3,n-1}(2)$.

In Corollary 1, we will prove that if r is an odd number, we obtain a new k–Fibonacci sequence, so and as they have been studied in (Falcon, S., 2007 (32), (38)). For instance, if $r = 3$ and $k = 4$, the $G_4(3)$ sequence is $\{0, 1, 76, 5777, 439128, \ldots\}$ listed in OEIS as A049669. The elements of this sequence verify the recurrence relation $G_{4,n+1}(3) = 76G_{4,n}(3) + G_{4,n-1}(3)$ so this is the 76-Fibonacci sequence F_{76}.

Lemma 2.1 *The elements of the sequence* $\{G_{k,n}(r)\} = \{\frac{F_{k,rn}}{F_{k,r}}\}$ *verify the recurrence relation* $G_{k,n+1}(r) = F_{k,r-1}G_{k,n}(r) + F_{k,rn+1}$.

Proof: From Honsberger's identity, if $m = rn$, then $F_{k,rn+r} = F_{k,r}F_{k,rn+1} + F_{k,r-1}F_{k,rn}$. So, $G_{k,n+1}(r) = \dfrac{F_{k,r(n+1)}}{F_{k,r}} = F_{k,rn+1} + F_{k,r-1}\dfrac{F_{k,rn}}{F_{k,r}} = F_{k,rn+1} + F_{k,r-1}G_{k,n}(r)$

With the help of this Lemma, we will prove the main theorem of this paper.

Theorem 2.2 *The elements of the sequence* $G_k(r)$ *verify the recurrence relation*

$$G_{k,n+1}(r) = G_{k,2}(r)G_{k,n}(r) - (-1)^r G_{k,n-1}(r) \tag{2}$$

for $n \geq 1$, *with the initial conditions* $G_{k,0}(r) = 0$ *and* $G_{k,1}(r) = 1$.

Proof: Initial conditions are obvious.

By applying the Binet's identity to the Right Hand Side (RHS) of Equation (2), and taking into account $\sigma_1\sigma_2 = -1 \rightarrow \frac{\sigma_2}{\sigma_1} = -\sigma_2^2$, it is

$$\begin{aligned}
(RHS) &= \frac{F_{k,2r}}{F_{k,r}} \cdot \frac{F_{k,rn}}{F_{k,r}} - (-1)^r \frac{F_{k,r(n-1)}}{F_{k,r}} \\
&= \frac{1}{F_{k,r}}\left(\frac{\sigma_1^{2r} - \sigma_2^{2r}}{\sigma_1^r - \sigma_2^r} \cdot \frac{\sigma_1^{rn} - \sigma_2^{rn}}{\sigma_1 - \sigma_2} - (-1)^r \frac{\sigma_1^{rn-r} - \sigma_2^{rn-r}}{\sigma_1 - \sigma_2}\right) \\
&= \frac{1}{F_{k,r}} \frac{1}{\sigma_1^r - \sigma_2^r} \frac{1}{\sigma_1 - \sigma_2}\left[\sigma_1^{rn+2r} + \sigma_2^{rn+2r} - \sigma_1^{rn}\sigma_2^{2r} - \sigma_1^{2r}\sigma_2^{rn}\right. \\
&\quad \left. -(-1)^r\left(\sigma_1^{rn} + \sigma_2^{rn} - (-1)^r\sigma_1^{rn}\sigma_2^{2r} - (-1)^r\sigma_1^{2r}\sigma_2^{rn}\right)\right] \\
&= \frac{1}{F_{k,r}} \frac{1}{\sigma_1 - \sigma_2} \frac{1}{\sigma_1^r - \sigma_2^r}\left[\sigma_1^{rn+2r} - (-1)^r\sigma_1^{rn} + \sigma_2^{rn+2r} - (-1)^r\sigma_2^{rn}\right] \\
&= \frac{1}{F_{k,r}} \frac{1}{\sigma_1 - \sigma_2} \frac{1}{\sigma_1^r - \sigma_2^r}\left[\sigma_1^{rn+r}(\sigma_1^r - \sigma_2^r) - \sigma_2^{rn+r}(\sigma_1^r - \sigma_2^r)\right] \\
&= \frac{1}{F_{k,r}} \frac{\sigma_1^{rn+r} - \sigma_2^{rn+r}}{\sigma_1 - \sigma_2} = \frac{F_{k,r(n+1)}}{F_{k,r}} = G_{k,n+1}(r)
\end{aligned}$$

because $(-1)^r\sigma_1^{rn} = (\sigma_1\sigma_2)^r\sigma_1^{rn} = \sigma_1^{rn+r}\sigma_2^r$

Corollary *If* r *is an odd number, the sequence* $F_k(r)$ *is a* k*–Fibonacci sequence.*

On the other hand, as a consequence of the Honsberger's formula, it is $F_{k,2r} = F_{k,r}(F_{k,r-1} + F_{k,r+1}) = F_{k,r}L_{k,r}$ and the constant of these k–Fibonacci sequences takes the form $G_{k,2}(r) = \dfrac{F_{k,2r}}{F_{k,r}} = F_{k,r-1} + F_{k,r+1}$, being r an odd number: all subsequence obtained from the k–Fibonacci sequence when r is odd is a bisection of the corresponding k–Lucas sequence.

This circumstance does more easy to find the k–Fibonacci sequences that we can obtain of this form. For instance, for $k = 2$ (Pell sequence) and $r = 5$, it is $F_{2,4} = 12$ and $F_{2,6} = 70$ so we get the 82–Fibonacci sequence.

2.3 Table of the Generalized Fibonacci Sequences Extracted from the k–Fibonacci Sequence

2.3.1 First case

Let us suppose $r = 3, 5, 7, 9, \ldots$. The second term of the generalized Fibonacci sequence obtained ($n = 2$), determines the order of the k–Fibonacci sequence according to the following relation proved in (Falcon, S. 2011):

$$G_{k,2}(2p+1) = \frac{F_{k,2(2p+1)}}{F_{k,2p+1}} = L_{k,2p+1}$$

Finally, for $k = 1, 2, 3, 4, 5 \ldots$, we find the generalized Fibonacci sequences $G_{k,n}(r)$ contained in the k–Fibonacci sequence $F_{k,n}$. In the Table 1, the first generalized Fibonacci sequences obtained of this form are related, where the inner number indicates the obtained k–Fibonacci sequence:

<Table 1>

It is interesting to indicate that all the row sequences of this table are bisections of the corresponding k–Lucas sequence.

For instance, the sequence of the firs row, $\{1, 4, 11, 29, 76, \ldots\}$ is the bisection of the classical Lucas sequence, L_{2n+1}, while the sequence of the second row $\{2, 14, 82, 478, \ldots\}$ is the bisection PL_{2n+1} of the Pell-Lucas sequence.

2.3.2 Lemma

For $n \geq 2$: $L_{k,2p+1} = (k^2 + 2)L_{k,2p-1} - L_{k,2p-3}$

Proof: From the definition of the k-Lucas numbers:

$$\begin{aligned} L_{k,2p+1} &= kL_{k,2p} + L_{k,2p-1} = (k^2+1)L_{l,2p-1} + kL_{k,2p-2} \\ &= (k^2+1)L_{k,2p-1} + k\frac{L_{k,2p-1} - L_{k,2p3}}{k} = (k^2+2)L_{k,2p-1} - L_{k,2p-3} \end{aligned}$$

Then, the row sequences of this last table verify the recurrence relation
$G_{k,n+1} = (k^2+2)G_{k,n} - G_{k,n-1} = L_{k,2}G_{k,n} - G_{k,n-1}$ for $n \geq 1$ with the initial conditions $G_{k,1} = k = L_{k,1}$ and $G_{k,2} = k^3+3k = L_{k,3}$ for $k \in \mathcal{N}$.

With respect to the classical Fibonacci sequence, in OEIS it is indicated that $F_4 = \{\frac{F(3n)}{2}\}$, $F_{11} = \{\frac{F(5n)}{5}\}$, $F_{29} = \{\frac{F(7n)}{13}\}$, $F_{76} = \{\frac{F(9n)}{34}\}$, etc. where the denominators are the terms of the bisection classical Fibonacci sequence $\{F_{2n+1}\}$.

2.3.3 Second case

Let us suppose $r = 2, 4, 6, 8, 10, \ldots$. The second term of the generalized Fibonacci subsequence obtained ($n = 2$), determines the order of the k-Fibonacci sequence according to the preceding formula $G_{k,mr} = F_{k,mr+1} + F_{k,mr-1} = L_{k,mr}$ and taking into account $F_{k,-1} = F_{k,1} = 1$:

Then, for $k = 1, 2, 3, 4, 5 \ldots$, we find the generalized Fibonacci subsequences $G_{k,n}(r)$ contained in the k–Fibonacci sequence $F_{k,n}$. The following table shows the first of these sequences:

<Table 2>

All the row sequences are listed in OEIS and are the even bisection of the k–Lucas sequence, $L_{k,2n}$.

These row sequences verify the recurrence relation $G_{k,n+1} = (k^2 + 2)G_{k,n} - G_{k,n-1} = L_{k,2}G_{k,n} - G_{k,n-1}$ for $n \geq 1$ with the initial conditions $G_{k,1} = 2 = L_{k,0}$ and $G_{k,2} = k^2 + 2 = L_{k,2}$ with $k \in \mathcal{N}$.

3. Conclusion

Here we have proved that any k–Fibonacci sequence contains infinite generalized Fibonacci sequences and we have indicated a form to obtain them.

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Table 1. Generalized Fibonacci sequences for *r* odd

k\r	**1**	**3**	**5**	**7**	**9**
1	1	4	11	29	76
2	2	14	82	478	2786
3	3	36	393	4287	46764
4	4	76	1364	24476	439204
5	5	140	3775	101785	2744420

Table 2. Generalized Fibonacci subsequences for *r* even

k\r	**0**	**2**	**4**	**6**	**8**
1	2	3	7	18	47
2	2	6	34	198	1154
3	2	11	119	1298	14159
4	2	18	322	2778	103682
5	2	27	727	19602	528527