Evolution Mixed Variational Inclusions with Optimal Control

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Abstract

Evolution mixed maximal monotone variational inclusions with optimal control, in reflexive Banach spaces, are analized. Solvability analysis is performed on the basis of composition duality principles. Applications to nonlinear diffusion constrained problems, as well as to quasistatic elastoviscoplastic contact problems exemplify the theory.

Keywords: Evolution variational inclusions, Optimal control, Mixed variational problems, Composition duality methods, Set-valued variational analysis

1. Introduction

The aim of this paper is to study primal and dual evolution mixed maximal monotone inclusions with optimal control, in a variational setting of reflexive Banach spaces. Distributed as well as boundary state constraints are considered through intrinsic control subdifferential mechanisms. For a mixed variational structure of the governing state model, and for computational purposes, imposed constraints are handled via compositional dualization. The state system solvability is analized on the basis of composition duality principles in the sense of (Alduncin, G., 2007a, 2007b, 2011b), which lead to a primal-dual variational analysis.

The solvability analysis is performed by adapting the study of (Akagi, G. & Ôtani, M., 2004), to our primal and dual evolution mixed maximal monotone subdifferential systems. The optimality analysis is given on the basis of Migórski's work on optimal control of evolution hemivariational inequalities (Migórski, S., 2001).

Here, in accordance with (Alduncin, G., 2007a, 2007b, 2011b), we consider primal as well as dual evolution mixed variational inclusions, since both formulations are of relevant importance in computational mechanics. Through the sequel, applications to nonlinear diffusion constrained processes, and quasistatic elastoviscoplastic variational contact problems, exemplify the control theory.

2. Evolution Mixed Variational Inclusions

In this section, we begin introducing the evolution mixed inclusion state problems of the theory. Here, we follow our previous work on evolution mixed variational inclusions (Alduncin, G., 2007b), in an abstract sense of constrained initial boundary value problems.

As a stationary functional framework for the dynamical systems of the theory, we consider primal and dual Ω-field reflexive Banach spaces $V(\Omega)$ and $Y^*(\Omega)$, with respective topological duals denoted by $V^*(\Omega)$ and $Y(\Omega)$, relative to a spatial bounded domain $\Omega \subset \Re^d$, $d \in \{1, 2, 3\}$. Denoting by $\partial\Omega$ the boundary of the domain Ω, assumed to be Lipschitz continuous, let π be the corresponding linear continuous primal trace operator, defined in $V(\Omega)$, with values in a reflexive Banach boundary space $B(\partial\Omega)$, with topological dual $B^*(\partial\Omega)$, and satisfying the compatibility condition

$$(C_\pi)\ \pi \in \mathcal{L}(V(\Omega), B(\partial\Omega)) \text{ is surjective.}$$

Further, let $H(\Omega)$ and $Z^*(\Omega)$ denote the primal and dual Hilbert pivot spaces; i.e., $V(\Omega) \subset H(\Omega) \subset V^*(\Omega)$ and $Y^*(\Omega) \subset Z^*(\Omega) \subset Y(\Omega)$, with continuous and dense embeddings. For the modeling of boundary conditions and constraints, we shall require a primal maximal monotone subdifferential $\partial\Psi : B(\partial\Omega) \to 2^{B^*(\partial\Omega)}$, with proper convex and lower semicontinuous superpotential $\Psi : B(\partial\Omega) \to \Re \cup \{+\infty\}$. Moreover, as a coupling operator of the mixed variational systems, we introduce the linear continuous operator $\Lambda \in \mathcal{L}(V(\Omega), Y(\Omega))$, with transpose $\Lambda^T \in \mathcal{L}(Y^*(\Omega), V^*(\Omega))$. The range of Λ and $-\Lambda^T$ will be denoted by $\mathcal{R}(\Lambda) \subset Y(\Omega)$ and $\mathcal{R}(-\Lambda^T) \subset V^*(\Omega)$.

2.1 Primal evolution mixed state system

For a general evolution mixed variational framework, related to a given time interval $(0, T)$, with $T > 0$ arbitrary and fixed, we introduce the primal and dual evolution reflexive Banach spaces $\mathcal{V} = L^p(0, T; V(\Omega)) \equiv \{v : [0, T] \to V(\Omega) | \, \|v\|_{\mathcal{V}} = [\int_0^T \|v(t)\|^p_{V(\Omega)} dt]^{1/p} < \infty\}$, $2 \leq p < \infty$, and $\mathcal{Y}^* = L^{q^*}(0, T; Y^*(\Omega))$, $q^* = p/(p-1)$, with topological duals $\mathcal{V}^* = L^{q^*}(0, T; V^*(\Omega))$ and $\mathcal{Y} = L^p(0, T; Y(\Omega))$. Further, the solution primal space is given by $\mathcal{W} = \{v : v \in \mathcal{V}, dv/dt \in \mathcal{V}^*\}$, with the operator norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|dv/dt\|_{\mathcal{V}^*}$, being continuous and densely embedded in the Hilbert space $C([0, T]; H(\Omega))$ of time continuous $H(\Omega)$)-functions, and with initial values such that $\{v(0) : v \in \mathcal{W}\} = H(\Omega)$ (Lions, J.-L., 1969). Moreover, we introduce the evolution reflexive Banach boundary space $\mathcal{B}_{\partial\Omega} = L^p(0, T; B(\partial\Omega))$, with dual $\mathcal{B}^*_{\partial\Omega} = L^{q^*}(0, T; B^*(\partial\Omega))$.

Hence, as a general primal evolution mixed state system, of a constrained initial boundary value problem from mechanics, we shall consider the following.

$$
(\widetilde{\mathcal{M}}) \begin{cases} \text{Given } f^* \in \mathcal{V}^*, g \in L^p(0, T; \mathcal{R}(\Lambda)) \text{ and } u_0 \in H(\Omega), \\ \text{find } u \in \mathcal{W} \text{ and } p^* \in \mathcal{Y}^* : \\ \\ -\Lambda^T p^* \in \dfrac{du}{dt} + \partial F(u) + \pi^T \partial\Psi(\pi u) - f^*, \quad \text{in } \mathcal{V}^*, \\ \\ \Lambda u \in \partial G^*(p^*) + g, \qquad\qquad\qquad\qquad \text{in } \mathcal{Y}, \\ \\ u(0) = u_0. \end{cases}
$$

Here, $\partial F : \mathcal{V} \to 2^{\mathcal{V}^*}$ stands for the evolution primal subdifferential of the system model, $\partial G^* : \mathcal{Y}^* \to 2^{\mathcal{Y}}$ corresponds to the evolution subdifferential control mechanism for the distributed constraints of the problem, and $\partial\Psi : \mathcal{B} \to 2^{\mathcal{B}^*}$ is a maximal monotone subdifferential modeling boundary conditions as well as boundary constraints. Also, $\pi^T \in \mathcal{L}(\mathcal{B}^*, \mathcal{V}^*)$ denotes the transpose of the evolution primal trace operator $\pi \in \mathcal{L}(\mathcal{V}, \mathcal{B})$.

From a mechanical point of view, the $\mathcal{V}^*$-primal variational equation of problem $(\widetilde{\mathcal{M}})$ corresponds to the constitutive or balance equation of the system, with an incorporated boundary term upon the application of its Green formula, while the $\mathcal{Y}(\Omega)$-dual subdifferential equation corresponds to a variational model of the imposed interior or distributed constraints, dualized for computational purposes. On the other hand, the essential or primal boundary condition of the problem is modeled variationally by $\partial\Psi_P(\pi_P u) = \partial I_{\{\widehat{u}\}}(\pi_P u)$, and the natural or dual boundary condition by $\partial\Psi_D(\pi_D u) = \{\widehat{s}^*\}$. Here, $I_{\{\widehat{u}\}}$ denotes the indicator functional imposing the primal boundary value $\widehat{u} \in \mathcal{B}_{\partial\Omega_P}$ on the primal part of the boundary $\partial\Omega_P \subset \partial\Omega$, and $\widehat{s}^* \in \mathcal{B}^*_{\partial\Omega_D}$ is the dual boundary value prescribed on the dual part of the boundary $\partial\Omega_D \subset (\partial\Omega \backslash \partial\Omega_P)$. Moreover, we consider a third disjoint and complementary part of the boundary, $\partial\Omega_C = (\partial\Omega \backslash \partial\Omega_P) \backslash \partial\Omega_D$, where the boundary constraints of the problem are assigned, and the corresponding boundary control $\lambda^* \in \mathcal{B}^*_{\partial\Omega_C}$, to be determined, is variationally modeled by the duality relation $\lambda^* \in \partial\Psi_C(\pi_C u) \Leftrightarrow \pi_C u \in \partial\Psi^*_C(\lambda^*)$ (Ekeland, I. & Temam, R.,1974). Thereby, the boundary condition and constraint term of the primal equation of mixed problem $(\widetilde{\mathcal{M}})$ specializes into the specific form

$$
\pi^T \partial\Psi(\pi u) = \pi_P^T \partial I_{\{\widehat{u}\}}(\pi_P u) + \pi_D^T \widehat{s}^* + \pi_C^T \lambda^*, \text{ in } \mathcal{V}^*. \tag{1}
$$

Hence, on the basis of primal boundary compatibility condition $(\boldsymbol{C}_\pi)$, equivalent to the lower boundedness of its transpose operator $\pi^T \in \mathcal{L}(B^*(\partial\Omega), V^*(\Omega))$ (Yosida, K., 1974), the following result is concluded via the injectivity of π^T and compositional dualization (Alduncin, G., 2010).

Lemma 1 *Under compatibility condition $(\boldsymbol{C}_\pi)$, the primal boundary term of* (1) *is such that*

$$
v^* \in \pi_P^T \, \partial I_{\{\widehat{u}\}}(\pi_P u) \iff c^* \in \partial I_{\{\widehat{u}\}}(\pi_P u) \iff v^* \in \partial(I_{\{\widehat{u}\}} \circ \pi_P)(u), \tag{2}
$$

where, for any functional $v^* \in \mathcal{R}(\pi_P^T) \subset \mathcal{V}^*$, $c^* \in \mathcal{B}^*_{\partial\Omega_P}$ is its π_P^T-preimage: $v^* = \pi_P^T c^*$.

Therefore, taking into account compatibility condition $(\mathbf{C}_\pi)$, the boundary conditions and constraint of the problem, can be incorporated variationally to primal evolution mixed problem ($\widetilde{\mathcal{M}}$) as follows.

$$(\mathcal{M})\begin{cases} \text{Given } f^* \in \mathcal{V}^*, g \in L^p(0,T;\mathcal{R}(\Lambda)) \text{ and } u_0 \in H(\Omega), \\ \text{find } u \in \mathcal{W} \text{ and } (p^*, \lambda^*) \in \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C} : \\ \\ -\Lambda^T p^* - \pi_C^T \lambda^* \in \dfrac{du}{dt} + \partial F(u) + \partial(I_{\{\widehat{u}\}} \circ \pi_P)(u) + \pi_D^T \widehat{s}^* - f^*, & \text{in } \mathcal{V}^*, \\ \\ \Lambda u \in \partial G^*(p^*) + g, & \text{in } \mathcal{Y}, \\ \\ \pi_C u \in \partial \Psi_C^*(\lambda^*), & \text{in } \mathcal{B}_{\partial\Omega_C}, \\ \\ u(0) = u_0. \end{cases}$$

This subdifferential form of the evolution mixed problem will be appropriate for performing its qualitative analysis.

Example 2 In order to illustrate the primal theory of evolution mixed variational inclusions, we consider a distributed constrained nonlinear diffusion problem, modeled by the mixed state system of conservation, constitutive and distributed constraint equations

$$\left.\begin{aligned} &\frac{du}{dt} + div\, \boldsymbol{w} = -b^* - p^*, \\ &\boldsymbol{w} = -\boldsymbol{grad}\, u, \\ &u \in \partial\varphi^*(b^*), \\ &u \in \partial\phi^*(p^*), \end{aligned}\right\} \text{ in } \Omega \times (0,T), \tag{3}$$

$$u(0) = u_0, \text{ in } \Omega.$$

Here, u denotes the diffusive field of a transport process evolving in $Q = \Omega \times (0,T)$ (for instance of mass concentration or temperature), with linear flux vector field $\boldsymbol{w}$. Also, b^* corresponds to the divergence of the nonlinear flux vector field denoted by $\widetilde{\boldsymbol{w}}$, and p^* is the distributed control source field governed by a maximal monotone control mechanism $\partial\phi^*$ (Duvaut, G. & Lions, J.-L., 1972). For the nonlinear diffusion constitutive equation, we consider the potential and subdifferential

$$\begin{aligned} &\varphi(u) = \frac{1}{p}\int_\Omega \|\boldsymbol{grad}\, u\|^p d\Omega, \ 2 < p < +\infty, \\ &\partial\varphi(u) = -div\,(\|\boldsymbol{grad}\, u\|^{p-2}\ \boldsymbol{grad}\, u), \end{aligned} \tag{4}$$

the latter in the Gâteaux sense (Vainberg, M. M., 1973). Notice that by convex dualization the nonlinear constitutive equation of the state system (3) is equivalently expressed by $b^* \in \partial\varphi(u)$, φ^* denoting the conjugate of φ, and by definition $b^* = div\ \widetilde{\boldsymbol{w}}$. Hence, the nonlinear flux vector field of the model turns out to be $\widetilde{\boldsymbol{w}} = -\|\boldsymbol{grad}\, u\|^{p-2}\boldsymbol{grad}\, u$, and the total flux vector field of diffusion is given by the sum $\boldsymbol{w} + \widetilde{\boldsymbol{w}}$.

Thereby, an appropriate functional framework of a primal Ω-field space $V(\Omega)$, of u-diffusive functions, and a dual Ω-field space $Y^*(\Omega)$, of $\boldsymbol{w}$-linear flux vector, b^*-nonlinear flux divergence and p^*-distributed source control functions, is provided by the reflexive Banach Sobolev spaces (Adams, R. A., 1975).

$$\begin{aligned} &V(\Omega) = W^{1,p}(\Omega), \ 2 < p < +\infty, \\ &Y^*(\Omega) = \boldsymbol{L}^{q^*}(\Omega) \times (W^{1,p}(\Omega))^* \times (W^{1,p}(\Omega))^*, \ q^* = p/(p-1), \end{aligned} \tag{5}$$

where $(W^{1,p}(\Omega))^*$ denotes the topological dual of primal space $W^{1,p}(\Omega)$, and, in turn, $Y^*(\Omega)$ is the topological dual of the space $Y(\Omega) = \boldsymbol{L}^p(\Omega) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega)$. Further, the primal Dirichlet boundary value trace operator is such that

$$(\boldsymbol{C}_{\boldsymbol{\pi}_{cd}})\ \pi \in \mathcal{L}(W^{1,p}(\Omega), W^{1/q^*,p}(\partial\Omega)) \text{ is surjective.}$$

Then, the primal boundary compatibility condition of the theory, ($\boldsymbol{C}_{\boldsymbol{\pi}}$), is satisfied with boundary space $B(\partial\Omega) = W^{1/q^*,p}(\partial\Omega)$ and its dual $B^*(\partial\Omega) = W^{-1/q^*,q^*}(\partial\Omega)$. Also, the dual Neumann boundary trace operator, of linear flux normal component, $\boldsymbol{\delta}(\cdot) = (\cdot)\boldsymbol{\cdot n}$, $\boldsymbol{\delta} \in \mathcal{L}(\boldsymbol{L}^{q^*}(\Omega), W^{-1/q^*,q^*}(\partial\Omega))$ is surjective (Girault, V. & Raviart, P.-A., 1986); the dual boundary compatibility condition ($\boldsymbol{C}_{\boldsymbol{\delta}_{cd}}$) for dual evolution mixed variational formulations (Alduncin, G., 2011b).

Hence, by integration, and application of the primal variational Green formula of the problem, for $\boldsymbol{w} \in \boldsymbol{L}^{q^*}(\Omega)$,

$$div\, \boldsymbol{w} + grad^T \boldsymbol{w} = \pi^T \delta \boldsymbol{w}, \text{ in } (W^{1,p}(\Omega))^*, \tag{6}$$

to the conservation divergence equation $(3)_1$, the primal evolution mixed variational formulation of constrained nonlinear diffusion problem (3)-(5) turns out to be

$$(\boldsymbol{\mathcal{M}_{cd}}) \begin{cases} \text{Find } u \in \mathcal{W} \text{ and } (\boldsymbol{w}, b^*, p^*) \in \boldsymbol{\mathcal{Y}^*} : \\ -\left(-grad^T \boldsymbol{w} + b^* + p^*\right) \in \dfrac{du}{dt} + \partial(I_{\{\widehat{u}\}} \circ \pi_P)(u) + \pi_D^T \widehat{s}^*, \quad \text{in } \mathcal{V}^*, \\ (-\boldsymbol{grad}\, u, u, u) \in (\boldsymbol{w}, \partial\varphi^*(b^*), \partial\phi^*(p^*)), \quad \text{in } \boldsymbol{\mathcal{Y}}, \\ u(0) = u_0, \end{cases}$$

with natural regularity for the data

$$\widehat{u} \in L^p(0,T; W^{1/q^*,p}(\partial\Omega_P)), \widehat{s}^* \in L^{q^*}(0,T; (W^{1,p}(\Omega_D))^*), \text{ and } u_0 \in L^2(\Omega). \tag{7}$$

2.2 Dual evolution mixed state system

For a general dual evolution mixed inclusion, the solution dual space is given by $\mathcal{X}^* = \{q^* : q^* \in \mathcal{Y}^*, dq^*/dt \in \mathcal{Y}\}$, endowed with the operator norm $\|q^*\|_{\mathcal{X}^*} = \|q^*\|_{\mathcal{Y}^*} + \|dq^*/dt\|_{\mathcal{Y}}$, being continuous and densely embedded in the Hilbert space $C([0,T]; Z^*(\Omega))$ of time continuous $Z^*(\Omega)$-functions, and with initial values such that $\{q^*(0) : q^* \in \mathcal{X}^*\} = Z^*(\Omega)$ (Lions, J.-L., 1969). Then, a general dual evolution mixed state system, of a constrained initial boundary value problem from mechanics, is the following.

$$(\widetilde{\boldsymbol{\mathcal{M}}^*}) \begin{cases} \text{Given } f^* \in L^p(0,T; \mathcal{R}(-\Lambda^T)),\ g \in \mathcal{Y} \text{ and } p_0^* \in Z^*(\Omega), \\ \text{find } u \in \mathcal{V} \text{ and } p^* \in \mathcal{X}^* : \\ -\Lambda^T p^* \in \partial F(u) + \pi^T \partial\Psi(\pi u) - f^*, \quad \text{in } \mathcal{V}^*, \\ \Lambda u \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + g, \quad \text{in } \mathcal{Y}, \\ p^*(0) = p_0^*. \end{cases}$$

In this dual case, a mechanical interpretation of problem $(\widetilde{\boldsymbol{\mathcal{M}}^*})$ is that now the $\mathcal{V}^*$-primal variational equation corresponds to a variational model of the imposed interior or distributed constraints, to be dualized for computational purposes, with an incorporated boundary term upon the application of its Green formula, while the $\mathcal{Y}(\Omega)$-dual subdifferential equation corresponds to the constitutive or balance equation of the system. On the other hand, the essential and natural boundary conditions, as well as the boundary constraints of the problem, are to be variationally modeled as for the primal case of the previous subsection. Hence, the dual evolution mixed problem $(\widetilde{\boldsymbol{\mathcal{M}}^*})$, with boundary conditions primally incorporated and boundary control dualized, takes the subdifferential form

$$(\boldsymbol{\mathcal{M}^*}) \begin{cases} \text{Given } f^* \in L^p(0,T; \mathcal{R}(-\Lambda^T)),\ g \in \mathcal{Y} \text{ and } p_0^* \in Z^*(\Omega), \\ \text{find } u \in \mathcal{V} \text{ and } (p^*, \lambda^*) \in \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C} : \\ -\Lambda^T p^* - \pi_C^T \lambda^* \in \partial F(u) + \partial(I_{\{\widehat{u}\}} \circ \pi_P)(u) + \pi_D^T \widehat{s}^* - f^*, \quad \text{in } \mathcal{V}^*, \\ \Lambda u \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + g, \quad \text{in } \mathcal{Y}, \\ \pi_C u \in \partial\Psi_C^*(\lambda^*), \quad \text{in } \mathcal{B}_{\partial\Omega_C}, \\ p^*(0) = p_0^*, \end{cases}$$

which will be appropriate for qualitative analysis.

Example 3 For the illustration of the dual theory of evolution mixed variational inclusions, we consider a boundary constrained quasistatic evolution elastoviscoplastic problem. Let the spatial domain Ω be occupied by a solid body in its reference configuration, undergoing small strains and displacements, and whose material is characterized by the general elastoviscoplastic constitutive model

$$\boldsymbol{\nabla}_s \boldsymbol{w} - \boldsymbol{C}^{-1}\dot{\boldsymbol{S}} \in \boldsymbol{\partial\varphi}^*(\boldsymbol{S}), \text{ in } \Omega \times (0, T), \tag{8}$$

of a nonlinear Maxwell viscoelastic type (Le Tallec, P., 1990). Here, $\boldsymbol{S}$ denotes the Cauchy stress tensor field, $\boldsymbol{w} = \dot{\boldsymbol{u}}$ stands for the velocity field, $\boldsymbol{C}$ is the elasticity tensor, symmetric and positive definite, with inverse $\boldsymbol{C}^{-1}$, the elastic compliance tensor. Further, $\boldsymbol{\partial\varphi}$ standing for the maximal monotone dissipation subdifferential, its dual $\boldsymbol{\partial\varphi}^*$ corresponds to the subdifferential of a conjugate dissipation superpotential φ^* or yield functional, and $\boldsymbol{\nabla}_s \boldsymbol{w} \equiv 1/2(\boldsymbol{\nabla w} + \boldsymbol{\nabla w}^T)$ is the total small strain rate tensor field, induced by the velocity $\boldsymbol{w}$. To this stress evolution constitutive equation, an initial stress value $\boldsymbol{S}(0) = \boldsymbol{S}_0$ is assigned in the domain Ω.

As a balance equation for the system, we have the quasistatic equilibrium equation

$$-\boldsymbol{divS} = \boldsymbol{b}, \text{ in } \Omega \times (0, T), \tag{9}$$

$\boldsymbol{b}$ denoting the body force data. With respect to the constraints of the problem, we consider on disjoint complementary sub-boundaries of domain Ω, $\partial\Omega = \partial\Omega_C \cup \partial\Omega_{DN}$, of a Lipschitz continuous type, a bilateral contact model of interaction with a rigid fixed foundation on $\partial\Omega_C \times (0, T)$, under the nonseparation condition $w_\nu = 0$, and Tresca's law of dry friction with prescribed shear bound function $f^* > 0$. Then, we associate with equilibrium equation (9) the bilateral contact constraint, with given friction, subdifferentially modeled by the dual inclusions

$$w_\nu \in \partial\psi^*_{C_\nu,0}(s_\nu) = \{0\},$$

$$\boldsymbol{w}_\tau \in \boldsymbol{\partial\psi^*}_{C_\tau, f^*}(\boldsymbol{s}_\tau) = \begin{cases} \{\boldsymbol{0}\}, & \|\boldsymbol{s}_\tau\| < f^*, \\ \{\xi \boldsymbol{s}_\tau : \xi \le 0\}, & \|\boldsymbol{s}_\tau\| = f^*, \\ \boldsymbol{\emptyset}, & \text{otherwise}, \end{cases} \tag{10}$$

where, $\boldsymbol{\nu}$ denotes the unit outward boundary normal vector, $w_\nu = \boldsymbol{w}\cdot\boldsymbol{\nu}$ and $\boldsymbol{w}_\tau = (\boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})\boldsymbol{w}$ are the normal and tangential components of velocity, and $s_\nu = -\boldsymbol{S\nu}\cdot\boldsymbol{\nu}$ and $\boldsymbol{s}_\tau = -(\boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})\boldsymbol{S\nu}$ are the contact pressure and the tangential negative traction. Further, as a generalized maximal monotone velocity-traction boundary condition on the Dirichlet-Neumann boundary $\partial\Omega_{DN} \times (0, T)$, we impose to the problem the subdifferential equation

$$-\boldsymbol{S\nu} \in \boldsymbol{\partial\psi}_{DN}(\boldsymbol{w}). \tag{11}$$

In particular, we shall consider the Dirichlet and Neumann boundary conditions of prescribed velocity $\widehat{\boldsymbol{w}}$, on $\partial\Omega_D \times (0, T) \neq \emptyset$ (nonempty for convenience), and traction $\widehat{\boldsymbol{s}}$, on the complementary sub-boundary $\partial\Omega_N \times (0, T)$, $\partial\Omega_N = \partial\Omega_{DN} \backslash \partial\Omega_D$.

Thereby, from model equations (8)-(11), the dual quasistatic mixed elastoviscoplastic contact physical problem to be considered is the following, for given data $(\boldsymbol{b}, f^*, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{s}}, \boldsymbol{S}_0)$.

Find a primal-dual field $(\boldsymbol{w}, \boldsymbol{S})$, of velocity-stress, such that

$$\begin{aligned}
&\left.\begin{aligned}
&\boldsymbol{divS}(\boldsymbol{x}, t) + \boldsymbol{b}(\boldsymbol{x}, t) \in \boldsymbol{\partial 0}(\boldsymbol{w}(\boldsymbol{x}, t)) = \{\boldsymbol{0}\}, \\
&\boldsymbol{\nabla}_s \boldsymbol{w}(\boldsymbol{x}, t) - \boldsymbol{C}^{-1}(\boldsymbol{x})\frac{\partial \boldsymbol{S}}{\partial t}(\boldsymbol{x}, t) \in \boldsymbol{\partial\varphi}^*(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}, t)),
\end{aligned}\right\} && (\boldsymbol{x}, t) \in \Omega \times (0, T), \\
&\left.\begin{aligned}
&w_\nu(\boldsymbol{x}, t) \in \partial\psi^*_{C_\nu,0}(s_\nu(\boldsymbol{x}, t)), \\
&\boldsymbol{w}_\tau(\boldsymbol{x}, t) \in \boldsymbol{\partial\psi^*}_{C_\tau, f^*(x,t)}(\boldsymbol{s}_\tau(\boldsymbol{x}, t)),
\end{aligned}\right\} && (\boldsymbol{x}, t) \in \partial\Omega_C \times (0, T), \\
&-\boldsymbol{S}(\boldsymbol{x}, t)\boldsymbol{\nu}(\boldsymbol{x}) \in \boldsymbol{\partial\psi}_{DN}(\boldsymbol{w}(\boldsymbol{x}, t)) = \boldsymbol{\partial I}_{\{\widehat{w}(x,t)\}}(\boldsymbol{w}(\boldsymbol{x}, t)), && (\boldsymbol{x}, t) \in \partial\Omega_D \times (0, T), \\
&-\boldsymbol{S}(\boldsymbol{x}, t)\boldsymbol{\nu}(\boldsymbol{x}) \in \boldsymbol{\partial\psi}_{DN}(\boldsymbol{w}(\boldsymbol{x}, t)) = \{\widehat{\boldsymbol{s}}(\boldsymbol{x}, t)\}, && (\boldsymbol{x}, t) \in \partial\Omega_N \times (0, T), \\
&\boldsymbol{S}(\boldsymbol{x}, 0) = \boldsymbol{S}_0(\boldsymbol{x}), && \boldsymbol{x} \in \Omega,
\end{aligned} \tag{12}$$

where $\boldsymbol{\partial 0}$ denotes the zero equilibrium subdifferential, and $\boldsymbol{\partial I}_{\{\widehat{w}(x,t)\}}$ is the subdifferential of the indicator functional $I_{\{\widehat{w}(x,t)\}}$.

For the variational formulation of contact problem (12), let $\boldsymbol{V}(\Omega)$ and $\boldsymbol{Y}(\Omega)$ denote reflexive Banach spaces of velocity and stress Ω-fields, with corresponding trace spaces of boundary velocities and negative tractions $\boldsymbol{B}(\partial\Omega)$ and its dual $\boldsymbol{B}^*(\partial\Omega)$, respectively. Let $\boldsymbol{\pi} \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}(\Omega), \boldsymbol{B}(\partial\Omega))$ and $\boldsymbol{\delta} \in \boldsymbol{\mathcal{L}}(\boldsymbol{Y}(\Omega), \boldsymbol{B}^*(\partial\Omega))$ denote the velocity and negative traction linear continuous trace operators. Further, let $\boldsymbol{H} \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}(\Omega), \boldsymbol{Y}^*(\Omega))$ be the variational symmetric total small strain operator and $\boldsymbol{D} \in \boldsymbol{\mathcal{L}}(\boldsymbol{Y}(\Omega), \boldsymbol{V}^*(\Omega))$ the variational divergence operator, which are related by the primal variational Green formula, for $\boldsymbol{T} \in \boldsymbol{Y}(\Omega)$,

$$\boldsymbol{H}^T\boldsymbol{T} + \boldsymbol{D}\boldsymbol{T} = -\boldsymbol{\pi}^T\boldsymbol{\delta}\boldsymbol{T}, \text{ in} \boldsymbol{V}^*(\Omega), \tag{13}$$

with $\boldsymbol{H}^T \in \boldsymbol{\mathcal{L}}(\boldsymbol{Y}(\Omega), \boldsymbol{V}^*(\Omega))$ and $\boldsymbol{\pi}^T \in \boldsymbol{\mathcal{L}}(\boldsymbol{B}^*(\partial\Omega), \boldsymbol{V}^*(\Omega))$ corresponding to transpose operators. In the application of Green formula (13), the fundamental property for validating the variational incorporation of the essential boundary condition (the primal Dirichlet condition), is the trace compatibility

$$(\boldsymbol{C1}_{\pi_{cp}})\ \boldsymbol{\pi} \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}(\Omega), \boldsymbol{B}(\partial\Omega)) \text{ is surjective,}$$

satisfied in the context of Sobolev spaces (Girault, V. & Raviart, P.-A., 1986), and under which the following result is concluded via compositional dualization (Alduncin, G., 2010), for a. e. $t \in (0, T)$,

$$\begin{aligned} \boldsymbol{v}^* \in \boldsymbol{\pi}_D^T \boldsymbol{\partial I}_{\{\widehat{w}(\cdot,t)\}}(\boldsymbol{\pi}_D \boldsymbol{w}) &\iff \boldsymbol{c}^*_{\boldsymbol{v}^*} \in \boldsymbol{\partial I}_{\{\widehat{w}(\cdot,t)\}}(\boldsymbol{\pi}_D \boldsymbol{w}) \\ &\iff \boldsymbol{v}^* \in \boldsymbol{\partial}(I_{\{\widehat{w}(\cdot,t)\}} \circ \boldsymbol{\pi}_D)(\boldsymbol{w}). \end{aligned} \tag{14}$$

Here, for any functional $\boldsymbol{v}^* \in \mathcal{R}(\boldsymbol{\pi}_D^T) \subset \boldsymbol{V}^*(\Omega)$, $\boldsymbol{c}^*_{\boldsymbol{v}^*} \in \boldsymbol{B}^*(\partial\Omega_D)$ is its $\boldsymbol{\pi}_D^T$-preimage: $\boldsymbol{v}^* = \boldsymbol{\pi}_D^T \boldsymbol{c}^*_{\boldsymbol{v}^*}$.

Completing the set of variational operators for the boundary constrained quasistatic evolution elastoviscoplastic problem, let $\boldsymbol{A} \in \boldsymbol{\mathcal{L}}(\boldsymbol{Y}^*(\Omega), \boldsymbol{Y}^*(\Omega))$ be the elastic compliance operator, symmetric and positive definite, and $\boldsymbol{\partial\Phi}^* : \boldsymbol{Y}(\Omega) \to \boldsymbol{2}^{Y^*(\Omega)}$ the maximal monotone dual dissipation subdifferential of the conjugate superpotential $\Phi^* : \boldsymbol{Y}(\Omega) \to \Re \cup \{\infty\}$, the yield functional. Also, let $\boldsymbol{\partial\Psi}_{DN} : \boldsymbol{B}(\partial\Omega_{DN}) \to \boldsymbol{2}^{B^*(\partial\Omega_{DN})}$ be the boundary maximal monotone subdifferential of the primal superpotential of boundary conditions $\Psi_{DN} : \boldsymbol{B}(\partial\Omega_{DN}) \to \Re \cup \{\infty\}$, defined in its effective domain $\mathcal{D}(\Psi_{DN}) \subset \boldsymbol{B}(\partial\Omega_{DN})$. Similarly, let $\partial\Psi^*_{C_\nu,0} : B^*(\partial\Omega_C) \to \boldsymbol{2}^{B(\partial\Omega_C)}$ and $\boldsymbol{\partial\Psi^*}_{C_\tau,f^*} : \boldsymbol{B}^*(\partial\Omega_C) \to \boldsymbol{2}^{B(\partial\Omega_C)}$ be the boundary maximal monotone subdifferentials of the conjugate superpotentials of bilateral contact, with prescribed friction, $\Psi^*_{C_\nu,0} : B^*(\partial\Omega_C) \to \Re \cup \{\infty\}$ and $\Psi^*_{C_\tau,f^*} : \boldsymbol{B}^*(\partial\Omega_C) \to \Re \cup \{\infty\}$, defined in their effective domains $\mathcal{D}(\Psi^*_{C_\nu,0}) \subset B^*(\partial\Omega_C)$ and $\mathcal{D}(\Psi^*_{C_\tau,f^*}) \subset \boldsymbol{B}^*(\partial\Omega_C)$.

Thereby, from Green formula (13) and compositional relation (14), the dual evolution mixed variational formulation of quasistatic contact problem (12) results to be

$$(\widetilde{\boldsymbol{\mathcal{M}}}^*_{cp}) \begin{cases} \text{Find } \boldsymbol{w} \in \boldsymbol{\mathcal{V}} \text{ and } (\boldsymbol{S}, s_\nu, \boldsymbol{s}_\tau) \in \boldsymbol{\mathcal{X}} \times \mathcal{B}^* \times \boldsymbol{\mathcal{B}}^* : & \\ -\boldsymbol{H}^T\boldsymbol{S} - \boldsymbol{\gamma}^T_{C_\nu} s_\nu - \boldsymbol{\gamma}^T_{C_\tau} \boldsymbol{s}_\tau \in \boldsymbol{\partial}(I_{\{\widehat{w}\}} \circ \gamma_{D_\nu})(\boldsymbol{w}) + \boldsymbol{\gamma}^T_{N_\tau}\widehat{\boldsymbol{s}} - \boldsymbol{b}, & \text{in } \boldsymbol{\mathcal{V}}^*, \\ \boldsymbol{H}\boldsymbol{w} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\Phi}^*(\boldsymbol{S}), & \text{in } \boldsymbol{\mathcal{Y}}^*, \\ \gamma_{C_\nu}\boldsymbol{w} \in \partial\Psi^*_{C_\nu,0}(s_\nu), & \text{in } \mathcal{B}_{\partial\Omega_C}, \\ \boldsymbol{\gamma}_{C_\tau}\boldsymbol{w} \in \boldsymbol{\partial\Psi}^*_{C_\tau,f^*}(\boldsymbol{s}_\tau), & \text{in } \boldsymbol{\mathcal{B}}_{\partial\Omega_C}, \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, & \end{cases}$$

whose data should naturally satisfy the regularity

$$\begin{aligned} \boldsymbol{C}^{-1} \in \boldsymbol{L}^\infty(\Omega), \widehat{\boldsymbol{w}} \in \boldsymbol{L}^{p^*}(0, T; \boldsymbol{B}(\partial\Omega_D)), \widehat{\boldsymbol{s}} \in \boldsymbol{L}^p(0, T; \boldsymbol{B}^*(\partial\Omega_N)), \\ f^* \in L^p(0, T; B^*(\partial\Omega_C)), -\boldsymbol{\gamma}^T_{N_\tau}\widehat{\boldsymbol{s}} + \boldsymbol{b} \in \boldsymbol{L}^p(0, T; \mathcal{R}(-\boldsymbol{H}^T)), \text{ and } \boldsymbol{S}_0 \in \boldsymbol{Y}(\Omega). \end{aligned} \tag{15}$$

3. Composition Duality Principles

For the existence analysis of primal and dual evolution mixed state systems ($\boldsymbol{\mathcal{M}}$) and ($\boldsymbol{\mathcal{M}}^*$), we establish in this section composition duality principles that lead to corresponding primal and dual evolution state inclusions equivalent in a solvability sense. Toward this end, we shall follow the duality approaches of our resent paper (Alduncin, G., 2011b).

3.1 Primal evolution duality principles

For the primal evolution mixed state system ($\boldsymbol{\mathcal{M}}$), we first establish a direct duality principle, following (Le Tallec, P., 1990), in terms of its primal admissibility set, to which any primal component solution belongs, for a.e. $t \in (0, T)$,

$$\begin{aligned} \mathcal{S}_{(p,\lambda)}(t) = \{v \in V(\Omega) : (\Lambda v, \pi_C v) = (p + g(\cdot, t), \lambda), \\ (p, \lambda) \in (\partial G^*(p^*), \partial\Psi^*_C(\lambda^*)) \subset Y(\Omega) \times B(\partial\Omega_C)\}. \end{aligned} \tag{16}$$

Then, introducing the coupling compatibility condition

$$(\boldsymbol{C}_{\Lambda^T,\pi_C^T})\begin{cases}\Lambda^T\in\mathcal{L}(Y^*(\Omega),V^*(\Omega))\text{ has a closed range, }\mathcal{R}(\Lambda^T),\\ \\ \pi_C^T\in\mathcal{L}(B^*(\partial\Omega_C),V^*(\Omega))\text{ has a closed range, }\mathcal{R}(\pi_C^T),\end{cases}$$

the next dual compatibility condition of the problem can be concluded (Alduncin, G., 2011b).

Lemma 4 *Let compatibility condition* $(\boldsymbol{C}_{\Lambda^T,\pi_C^T})$ *be fulfilled. Then the dual admissibility set of primal evolution mixed problem* $(\boldsymbol{\mathcal{M}})$ *is characterized by*

$$\partial I_{\mathcal{S}_{(p,\lambda)}(t)}(v)=\begin{cases}\mathcal{R}(\Lambda^T)+\mathcal{R}(\pi_C^T), & v\in\mathcal{S}_{(p,\lambda)}(t),\\ \\ \emptyset, & \text{otherwise},\end{cases}\tag{17}$$

the subdifferential of the indicator functional $I_{\mathcal{S}_{(p,\lambda)}(t)}$.

We note that due to condition $(\boldsymbol{C}_{\Lambda^T,\pi_C^T})$ and the Close Range Theorem, the sum of the ranges of the transpose coupling operators coincide with the sum of the polar kernels of the own coupling operators; i.e., $\mathcal{R}(\Lambda^T)+\mathcal{R}(\pi_C^T)=\mathcal{N}^\circ(\Lambda)+\mathcal{N}^\circ(\pi_C)$, relation utilized in demonstrating (17).

Thereby, the following primal evolution duality principle for problem $(\boldsymbol{\mathcal{M}})$ is readily concluded.

Theorem 5 *Under compatibility condition* $(\boldsymbol{C}_{\Lambda^T,\pi_C^T})$*, primal evolution mixed problem* $(\boldsymbol{\mathcal{M}})$ *is solvable if, and only if, the primal evolution problem*

$$(\boldsymbol{\mathcal{P}})\begin{cases}\text{Find } u\in\mathcal{W}:\\ \\ 0\in\dfrac{du}{dt}+\partial F(u)+\partial(I_{\{\widehat{u}\}}\circ\pi_P)(u)+\partial I_{\mathcal{S}_{(p,\lambda)}}(u)+\pi_D^T\widehat{s}^*-f^*,\quad\text{in }\mathcal{V}^*,\\ \\ u(0)=u_0,\end{cases}$$

is solvable. That is, if $u\in\mathcal{W}$ *is a solution of primal problem* $(\boldsymbol{\mathcal{P}})$ *then there is an admissible dual function* $(p^*,\lambda^*)\in\mathcal{Y}^*\times\mathcal{B}^*_{\partial\Omega_C}$*, with* $\Lambda^Tp^*+\pi_C^T\lambda^*\in\partial I_{\mathcal{S}_{(p,\lambda)}}(u)$*, such that* u *and* (p^*,λ^*) *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}})$ *and, conversely, if* $u\in\mathcal{W}$ *and* $(p^*,\lambda^*)\in\mathcal{Y}^*\times\mathcal{B}^*_{\partial\Omega_C}$ *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}})$ *then primal admissible function* u *is a solution of primal problem* $(\boldsymbol{\mathcal{P}})$.

For a complementary primal duality principle to evolution mixed state inclusion $(\boldsymbol{\mathcal{M}})$, we next apply the composition duality methodology established in (Alduncin, G., 2007b). Toward this end, we first give a compositional interpretations of dual admissibility characterization (17), which is based on the following compositional result (Alduncin, G., 2011b).

Lemma 6 *The dual inclusions of problem* $(\boldsymbol{\mathcal{M}})$ *necessarily satisfy the compositional dualizations*

$$\begin{aligned}\Lambda u\in\partial G^*(p^*)+g &\implies \Lambda^Tp^*\in\partial(G\circ\Lambda)(u-v_g),\\ \pi_Cu\in\partial\Psi_C^*(\lambda^*) &\implies \pi^T\lambda^*\in\partial(\Psi_C\circ\pi_C)(u),\end{aligned}\tag{18}$$

where $v_g\in\mathcal{V}$ *is a fixed* Λ*-preimage of function* g*:* $\Lambda v_g=g$.

Hence, from Lemma 6 and Theorem 5, and introducing the primal operator and right-hand side term

$$\begin{aligned}\partial\widetilde{F}&=\partial F+\partial(I_{\{\widehat{u}\}}\circ\pi_P):\mathcal{V}\to 2^{\mathcal{V}^*},\\ -\widetilde{f}^*&=\pi_D^T\widehat{s}^*-f^*\in\mathcal{V}^*,\end{aligned}\tag{19}$$

we can conclude the next necessary primal evolution composition duality principle.

Theorem 7 *Primal evolution mixed problem* $(\boldsymbol{\mathcal{M}})$ *is solvable, only if the primal evolution problem*

$$(\widetilde{\boldsymbol{\mathcal{P}}})\begin{cases}\text{Find } u\in\mathcal{W}:\\ \\ 0\in\dfrac{du}{dt}+\partial\widetilde{F}(u)+\partial(G\circ\Lambda)(u-v_g)+\partial(\Psi_C\circ\pi_C)(u)-\widetilde{f}^*,\ \text{in }\mathcal{V}^*,\\ \\ u(0)=u_0,\end{cases}$$

is solvable. That is, if $u \in \mathcal{W}$ *and* $(p^*, \lambda^*) \in \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C}$ *is a solution of mixed problem* $(\mathcal{M})$ *then component u is a primal admissible solution of problem* $(\widetilde{\mathcal{P}})$*. Moreover, compositional primal problem* $(\widetilde{\mathcal{P}})$ *is solvable if primal evolution problem* $(\mathcal{P})$ *is solvable.*

Further, following our previous study (Alduncin, G., 2007b), we introduce the classical primal evolution compatibility conditions

$$(\boldsymbol{C}_{(G,\Lambda),(\Psi_C,\pi_C)}) \begin{cases} \operatorname{int}\mathcal{D}(G) \cap \mathcal{R}(\Lambda) \neq \emptyset, \\ \\ \operatorname{int}\mathcal{D}(\Psi_C) \cap \mathcal{R}(\pi_C) \neq \emptyset, \end{cases}$$

which lead to the compositional operator equalities (Ekeland, I. & Temam, R., 1974)

$$\begin{aligned} \partial(G \circ \Lambda) &= \Lambda^T \partial G \circ \Lambda, \\ \partial(\Psi_C \circ \pi_C) &= \pi_C^T \partial\Psi_C \circ \pi_C. \end{aligned} \tag{20}$$

Hence, by dualization of the dual inclusions of problem $(\mathcal{M})$, and the validity of (18) in its equivalence sense, due to composition duality relations (20), Theorem 7 applies then, additionally, in a sufficiency sense (Alduncin, G., 2007b; Theorem 2.2). That is, the following alternative primal evolution duality principle holds true.

Theorem 8 *Let compatibility condition* $(\boldsymbol{C}_{(G,\Lambda),(\Psi_C,\pi_C)})$ *be satisfied. Then primal evolution mixed problem* $(\mathcal{M})$ *is solvable if, and only if, primal evolution problem* $(\widetilde{\mathcal{P}})$ *is solvable. Moreover, under condition* $(\boldsymbol{C}_{\Lambda^T,\pi_C^T})$*, compositional primal problem* $(\widetilde{\mathcal{P}})$ *is equivalent to primal evolution problem* $(\mathcal{P})$*.*

Example 9 Continuing with the distributed constrained nonlinear diffusion problem of Example 2, for the application of the primal composition duality principles of Theorems 5, 7 and 8 in its existence analysis, as primal evolution compatibility conditions $(\boldsymbol{C}_{\Lambda^T,\pi_C^T})$ and $(\boldsymbol{C}_{(G,\Lambda),(\Psi_C,\pi_C)})$ we have the following (without boundary constraint):

$$(\boldsymbol{C}_{(\Lambda^T)_{cd}})\ grad^T \in \mathcal{L}(\boldsymbol{L}^{p^*}(\Omega), (W^{1,p}(\Omega))^*) \text{ has a closed range, } \mathcal{R}(grad^T),$$

$$(\boldsymbol{C}_{(G,\Lambda)_{cd}})\ \operatorname{int}\mathcal{D}(\phi)\Big(\subset (W^{1,p}(\Omega))^*\Big) \neq \emptyset.$$

This first condition is classically satisfied (Girault, V. & Raviart, P.-A., 1986), and the latter may be consider as a proper condition for control mechanisms (Duvaut, G. & Lions, J.-L., 1972). Then, under condition $(\boldsymbol{C}_{(G,\Lambda)_{cd}})$, in accordance with Theorem 8, primal evolution mixed problem $(\mathcal{M}_{cd})$ is solvable if, and only if, the primal evolution variational problem

$$(\widetilde{\mathcal{P}}_{cd}) \begin{cases} \text{Find } u \in \mathcal{W}: \\ \\ 0 \in \dfrac{du}{dt} + grad^T \boldsymbol{grad}(u) + \partial\varphi(u) + \partial\phi(u), \ \text{ in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

is solvable. Moreover, due to property $(\boldsymbol{C}_{(\Lambda^T)_{cd}})$, compositional primal problem $(\widetilde{\mathcal{P}}_{cd})$ has as an interpretation primal problem $(\mathcal{P}_{cd})$ of Theorem 5, with primal admissibility set (16) characterized by its indicator subdifferential (17).

We note that constitutive subdifferential value $\partial\varphi(u)$, defined by $(4)_2$, as a $\mathcal{V}^*$-functional has the explicit variational expression

$$\partial\varphi(u) = grad^T(\|\boldsymbol{grad}\ u\|^{p-2}\boldsymbol{grad})(u), \tag{21}$$

the primal evolution variational nonlinear flux term.

3.2 Dual evolution duality principles

Considering now the dual evolution mixed state system $(\mathcal{M}^*)$, we first establish as in the previous primal subsection a direct duality principle. In this dual case, we dualized compositionally its dual subdifferential boundary equation, based on primal trace compatibility condition $(\boldsymbol{C}_\pi)$ (Alduncin, G., 2005a); i.e.,

$$\pi_C u \in \partial\Psi_C^*(\lambda^*) \iff \pi_C^T\lambda^* \in \partial(\Psi_C \circ \pi_C)(u). \tag{22}$$

Hence, the primal equation of system $(\mathcal{M}^*)$ becomes

$$-\Lambda^T p^* \in \partial\widetilde{F}(u) + \partial(\Psi_C \circ \pi_C)(u) - \widetilde{f}^*, \text{ in } \mathcal{V}^*, \tag{23}$$

where primal definition (19) is in force. Consequently, the corresponding dual admissibility set of the system, to which any dual component solution belongs, for a.e. $t \in (0,T)$, turns out to be

$$\mathcal{S}_{v^*}(t) = \{q^* \in Y^*(\Omega) : -\Lambda^T q^* = v^* - \widetilde{f}^*(\cdot,t),\ v^* \in \partial\widetilde{F}(u) + \partial(\Psi_C \circ \pi_C)(u) \subset V^*(\Omega)\}. \tag{24}$$

Then, introducing the coupling compatibility condition

$$(\boldsymbol{C}_{-\boldsymbol{\Lambda}})\ -\Lambda \in \mathcal{L}(V(\Omega), Y(\Omega)) \text{ has a closed range, } \mathcal{R}(-\Lambda),$$

the next primal compatibility condition of the problem can be achieved (Alduncin, G., 2011b).

Lemma 4* *Let compatibility condition* $(\boldsymbol{C}_{-\boldsymbol{\Lambda}})$ *be satisfied. Then the primal admissibility set of dual evolution mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *is characterized by*

$$\partial I_{\mathcal{S}_{v^*}(t)}(q^*) = \begin{cases} \mathcal{R}(-\Lambda), & q^* \in \mathcal{S}_{v^*}(t), \\ \\ \emptyset, & \textit{otherwise}, \end{cases} \tag{25}$$

the subdifferential of the indicator functional $I_{\mathcal{S}_{v^*}(t)}$.

Therefore, the following dual evolution duality principle for problem $(\boldsymbol{\mathcal{M}}^*)$ is obtained.

Theorem 5* *Under compatibility condition* $(\boldsymbol{C}_{-\boldsymbol{\Lambda}})$*, dual evolution mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *is solvable if, and only if, the dual evolution problem*

$$(\mathcal{D}) \begin{cases} \textit{Find } p^* \in \mathcal{X}^* : \\ \\ 0 \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + \partial I_{\mathcal{S}_{v^*}(t)}(p^*) + g, \quad \textit{in } \mathcal{Y}, \\ \\ p^*(0) = p_0^*, \end{cases}$$

is solvable. That is, if $p^* \in \mathcal{X}^*$ *is a solution of dual problem* $(\mathcal{D})$ *then there is an admissible primal-dual function* $(u,\lambda^*) \in \mathcal{V} \times \mathcal{B}^*_{\partial\Omega_C}$*, with* $-\Lambda u \in \partial I_{\mathcal{S}_{v^*}}(p^*)$ *and* $\pi_C^T\lambda^* \in \partial(\Psi_C \circ \pi_C)(u)$*, such that* u *and* (p^*,λ^*) *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *and, conversely, if* $u \in \mathcal{V}$ *and* $(p^*,\lambda^*) \in \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C}$ *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *then dual admissible function* p^* *is a solution of dual problem* $(\mathcal{D})$.

As in the primal case, for a complementary dual duality principle to evolution mixed state inclusion $(\boldsymbol{\mathcal{M}}^*)$, we apply the composition duality methodology of (Alduncin, G., 2007b). First, we give a compositional interpretation of primal admissibility characterization (25), based on the next compositional result (Alduncin, G., 2011b).

Lemma 6* *The primal inclusion of problem* $(\boldsymbol{\mathcal{M}}^*)$ *necessarily satisfies the compositional dualization*

$$-\Lambda^T p^* \in \partial\widetilde{F}(u) + \partial(\Psi_C \circ \pi_C)(u) - \widetilde{f}^* \implies -\Lambda u \in \partial((\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T))(p^* + q^*_{\widetilde{f}^*}), \tag{26}$$

where $q^*_{\widetilde{f}^*} \in \mathcal{Y}^*$ *is a fixed* $-\Lambda^T$*-preimage of function* $\widetilde{f}^*$*:* $-\Lambda^T q^*_{\widetilde{f}^*} = \widetilde{f}^*$.

Thereby, the following necessary dual evolution composition duality principle holds true.

Theorem 7* *Dual evolution mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *is solvable, only if the dual evolution problem*

$$(\widetilde{\mathcal{D}}) \begin{cases} \textit{Find } p^* \in \mathcal{X}^* : \\ \\ 0 \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + \partial((\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T))(p^* + q^*_{\widetilde{f}^*}) + g, \quad \textit{in } \mathcal{Y}, \\ \\ p^*(0) = p_0^*, \end{cases}$$

is solvable. That is, if $u \in \mathcal{V}$ *and* $(p^*,\lambda^*) \in \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C}$ *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *then component* p^* *is a dual admissible solution of problem* $(\widetilde{\mathcal{D}})$*. Moreover, compositional dual problem* $(\widetilde{\mathcal{D}})$ *is solvable if dual evolution problem* $(\mathcal{D})$ *is solvable.*

Furthermore, following (Alduncin, G., 2007b), introducing the classical dual evolution compatibility condition

$$(\boldsymbol{C}_{\widetilde{\boldsymbol{F}}^*,-\boldsymbol{\Lambda}^T})\ \mathrm{int}\mathcal{D}((\widetilde{F} + \Psi_C \circ \pi_C)^*) \cap \mathcal{R}(-\Lambda^T) \neq \emptyset,$$

the compositional operator equality (Ekeland, I. & Temam, R., 1974)

$$\partial((\widetilde{F}+\Psi_C\circ\pi_C)^*\circ(-\Lambda^T)) = -\Lambda\partial((\widetilde{F}+\Psi_C\circ\pi_C)^*\circ(-\Lambda^T), \tag{27}$$

holds true. Thus, by dualization of the primal inclusion of problem $(\mathcal{M}^*)$, (23), and the validity of (26), but in its equivalence sense due to compositional dualization (27), Theorem 7* applies now in an equivalence sense (Alduncin, G., 2007b). Therefore, the following alternative dual evolution duality principle is achieved.

Theorem 8* *Let compatibility condition* $(\boldsymbol{C}_{\widetilde{\boldsymbol{F}}^*,-\boldsymbol{\Lambda}^T})$ *be fulfilled. Then dual evolution mixed problem* $(\mathcal{M}^*)$ *is solvable if, and only if, dual evolution problem* $(\widetilde{\mathcal{D}})$ *is solvable. Moreover, under condition* $(\boldsymbol{C}_{-\boldsymbol{\Lambda}})$*, compositional dual problem* $(\widetilde{\mathcal{D}})$ *is equivalent to dual evolution problem* $(\mathcal{D})$*.*

Example 10 In order to relate dual evolution mixed quasistatic contact problem $(\widetilde{\mathcal{M}}^*_{cp})$ of Example 3 with the dual evolution mixed state system of the theory, $(\mathcal{M}^*)$, we will have to dualize its contact subdifferential equations. To this end, taking into account primal trace compatibility condition $(\boldsymbol{C1}_{\pi_{cp}})$, by compositional dualization we have the duality result (Alduncin, G., 2005a)

$$\begin{aligned}\boldsymbol{\gamma}^T_{C_\nu}s_\nu &\in \boldsymbol{\partial\Psi}_{C_\nu,0}(\boldsymbol{w}) = \boldsymbol{\partial}(\Psi_{C_\nu,0}\circ\gamma_{C_\nu})(\boldsymbol{w}),\\ \boldsymbol{\gamma}^T_{C_\tau}\boldsymbol{s}_\tau &\in \boldsymbol{\partial\Psi}_{C_\tau,f^*}(\boldsymbol{w}) = \boldsymbol{\partial}(\Psi_{C_\tau,f^*}\circ\boldsymbol{\gamma}_{C_\tau})(\boldsymbol{w}),\end{aligned} \tag{28}$$

which leads to the dual evolution mixed equivalent version

$$(\mathcal{M}^*_{cp})\begin{cases}\text{Find } \boldsymbol{w}\in\mathcal{V} \text{ and } \boldsymbol{S}\in\mathcal{X}:\\ -\boldsymbol{H}^T\boldsymbol{S}\in\boldsymbol{\partial}\widetilde{\boldsymbol{F}}(\boldsymbol{w})-\widetilde{\boldsymbol{b}}^*, & \text{in } \mathcal{V}^*,\\ \boldsymbol{H}\boldsymbol{w}\in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt}+\boldsymbol{\partial\Phi}^*(\boldsymbol{S}), & \text{in } \mathcal{Y}^*,\\ \boldsymbol{S}(0)=\boldsymbol{S}_0.\end{cases}$$

Here, for a.e. $t\in(0,T)$, the primal subdifferential $\boldsymbol{\partial}\widetilde{\boldsymbol{F}}:\boldsymbol{V}(\Omega)\to \boldsymbol{2}^{V^*(\Omega)}$ is given by

$$\begin{aligned}\boldsymbol{\partial}\widetilde{\boldsymbol{F}} &= \boldsymbol{\partial}(I_{\{\widehat{w}(\cdot,t)\}}\circ\gamma_{D_\nu})+\boldsymbol{\partial}(I_{\{0\}}\circ\gamma_{C_\nu})+\boldsymbol{\partial}(\Psi_{C_\tau,f^*(\cdot,t)}\circ\gamma_{C_\tau}),\\ \widetilde{F} &= I_{\{\widehat{w}(\cdot,t)\}}\circ\gamma_{D_\nu}+I_{\{0\}}\circ\gamma_{C_\nu}+\Psi_{C_\tau,f^*(\cdot,t)}\circ\gamma_{C_\tau},\\ \mathcal{D}(\widetilde{F}) &= \boldsymbol{K}_{\widehat{w}(\cdot,t),0}=\{\boldsymbol{v}\in\boldsymbol{V}(\Omega):\boldsymbol{\gamma}_{C_\nu}\boldsymbol{v}=\widehat{\boldsymbol{w}}(\cdot,t)\text{ in }\boldsymbol{B}(\partial\Omega_D),\\ &\qquad\qquad \gamma_{C_\nu}\boldsymbol{v}=0\text{ in }B(\partial\Omega_C)\},\end{aligned} \tag{29}$$

where the subdifferential sum rule has been used (Ernst, E. & Théra, M., 2009), and the right-hand side term $\widetilde{\boldsymbol{b}}^* = -\boldsymbol{\gamma}^T_{N_\tau}\widehat{\boldsymbol{s}}+\boldsymbol{b}^*\in\boldsymbol{L}^p(0,T;\mathcal{R}(-\boldsymbol{H}^T))$.

Thereby, for this particular dual evolution mixed state system, the corresponding dual duality principle follows, in accordance with the statically admissible set of the mechanical problem, for a.e. $t\in(0,T)$,

$$\mathcal{S}_{w^*}(t)=\{\boldsymbol{T}\in\boldsymbol{Y}(\Omega):-\boldsymbol{H}^T\boldsymbol{T}=\boldsymbol{w}^*-\widetilde{\boldsymbol{b}}^*(\cdot,t),\boldsymbol{w}^*\in\boldsymbol{\partial}\widetilde{\boldsymbol{F}}(\boldsymbol{w})\subset\boldsymbol{V}^*(\Omega)\}, \tag{30}$$

its total small strain compatibility property (Girault, V. & Raviart, P.-A., 1986)

$$(\boldsymbol{C2}_{-\boldsymbol{H}})\ -\boldsymbol{H}\in\mathcal{L}(\boldsymbol{V}(\Omega),\boldsymbol{Y}^*(\Omega))\text{ has a closed range, }\mathcal{R}(-\boldsymbol{H}),$$

and its kinematic compatibility characterization

$$\boldsymbol{\partial I}_{\mathcal{S}_{w*}(t)}(\boldsymbol{T})=\begin{cases}\mathcal{R}(-\boldsymbol{H}), & \boldsymbol{T}\in\mathcal{S}_{w^*}(t),\\ \boldsymbol{\emptyset}, & \text{otherwise},\end{cases} \tag{31}$$

the subdifferential of the indicator functional $I_{\mathcal{S}_{w*}(t)}$. Indeed, the dual evolution duality principle for problem $(\mathcal{M}^*_{cp})$ is given as follows (cf. Theorem 5*): dual evolution mixed contact problem $(\mathcal{M}^*_{cp})$, equivalent to $(\widetilde{\mathcal{M}}^*_{cp})$, is solvable if, and only if, the dual evolution problem

$$(\mathcal{D}_{cp}) \begin{cases} \text{Find } \boldsymbol{S} \in \boldsymbol{\mathcal{X}} : \\ \boldsymbol{0} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\Phi}^*(\boldsymbol{S}) + \boldsymbol{\partial I}_{\boldsymbol{S}_{w*}}(\boldsymbol{S}), \quad \text{in } \boldsymbol{\mathcal{Y}}^*, \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, \end{cases}$$

is solvable. That is, if $\boldsymbol{S} \in \boldsymbol{\mathcal{X}}$ is a solution of dual problem $(\mathcal{D}_{cp})$ then there is a kinematically admissible velocity field $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$, with $-\boldsymbol{Hw} \in \boldsymbol{\partial I}_{\boldsymbol{S}_{w*}}(\boldsymbol{S})$, such that $(\boldsymbol{w}, \boldsymbol{S})$ is a solution of mixed problem $(\boldsymbol{\mathcal{M}}^*_{cp})$ and, conversely, if $(\boldsymbol{w}, \boldsymbol{S}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{X}}$ is a solution of mixed problem $(\boldsymbol{\mathcal{M}}^*_{cp})$ then statically admissible stress field $\boldsymbol{S}$ is a solution of dual problem $(\mathcal{D}_{cp})$.

For a compositional interpretation of kinematic compatibility condition (30), we apply the necessary duality result (cf. Lemma 6*), related to the primal velocity inclusion of mixed problem $(\boldsymbol{\mathcal{M}}^*_{cp})$,

$$-\boldsymbol{H}^T\boldsymbol{S} \in \boldsymbol{\partial F}(\boldsymbol{w}) - \widetilde{\boldsymbol{b}}^* \implies -\boldsymbol{Hw} \in \partial(\widetilde{F}^* \circ (-\boldsymbol{H}^T))(\boldsymbol{S} + \boldsymbol{R}_{\widetilde{b}^*}), \tag{32}$$

where $\boldsymbol{R}_{\widetilde{b}^*} \in \boldsymbol{\mathcal{Y}}$ is a fixed $-\boldsymbol{H}^T$-preimage of function $\widetilde{\boldsymbol{b}}^*$: $-\boldsymbol{H}^T\boldsymbol{R}_{\widetilde{b}^*} = \widetilde{\boldsymbol{b}}^*$. Therefore, in accordance with Theorem 7*, a necessary dual evolution composition duality principle for the elastoviscoplastic contact problem is concluded: dual evolution mixed problem $(\boldsymbol{\mathcal{M}}^*_{cp})$ is solvable only if the dual evolution problem

$$(\widetilde{\mathcal{D}_{cp}}) \begin{cases} \text{Find } \boldsymbol{S} \in \boldsymbol{\mathcal{X}} : \\ \boldsymbol{0} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\Phi}^*(\boldsymbol{S}) + \partial(\widetilde{F}^* \circ (-\boldsymbol{H}^T))(\boldsymbol{S} + \boldsymbol{R}_{\widetilde{b}^*}), \quad \text{in } \boldsymbol{\mathcal{Y}}^*, \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, \end{cases}$$

is solvable. That is, if $(\boldsymbol{w}, \boldsymbol{S}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{X}}$ is a solution of problem $(\boldsymbol{\mathcal{M}}^*_{cp})$ then dual field $\boldsymbol{S}$ is a statically admissible solution of problem $(\widetilde{\mathcal{D}_{cp}})$. Moreover, compositional dual problem $(\widetilde{\mathcal{D}_{cp}})$ is solvable if dual evolution problem $(\mathcal{D}_{cp})$ is solvable.

We should note that for a fully dual compositional duality principle in the sense of (Alduncin, G., 2005a, 2005b), the surjectivity of the total small strain transpose operator $-\boldsymbol{H}^T \in \boldsymbol{\mathcal{L}}(\boldsymbol{Y}(\Omega), \boldsymbol{V}^*(\Omega))$ would be required, guaranteeing the sufficiency of compositional dualization (32) and, consequently, the sufficiency of the above dual duality principle. However, such a total small strain transpose operator surjectivity would require of a primal quotient space $\boldsymbol{V}(\Omega)$ setting (Girault, V. & Raviart, P.-A., 1986). On the other hand, for the validity of the compositional duality principle established by Theorem 8*, the corresponding classical compatibility condition

$$(\boldsymbol{C}_{\widetilde{\boldsymbol{F}}^*, -\boldsymbol{H}^T}) \ \mathrm{int}\mathcal{D}(\widetilde{F}^*) \cap \mathcal{R}(-\boldsymbol{H}^T) \neq \emptyset,$$

should be satisfied in order to apply corresponding compositional operator equality (27).

4. Variational Problems with Optimal Control

In this last section, we state an optimal control problem, governed by primal and dual evolution mixed variational systems $(\boldsymbol{\mathcal{M}})$ and $(\boldsymbol{\mathcal{M}}^*)$, performing the state existence analysis on the basis of (Akagi, G. & Ôtani, M., 2004) work on evolution inclusions. The corresponding optimal control existence is given in accordance with (Migórski, S., 2001) study on optimal control of evolution hemivariational inequalities.

Let C stand for the space of controls, a reflexive Banach space, and let $\mathcal{K}$ be a closed convex subset of admissible controls. As an optimal control problem, we consider the following

$$(\boldsymbol{O}) \begin{cases} \text{Find } \kappa \in \mathcal{K} \subset C : \\ J(\kappa; u_\kappa, p^*_\kappa, \lambda^*_\kappa) \leq J(c; u_\kappa, p^*_\kappa, \lambda^*_\kappa), \ \forall c \in C, \end{cases}$$

where $(u_\kappa, p^*_\kappa, \lambda^*_\kappa) \in \mathcal{W} \times \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C}$ or $(u_\kappa, p^*_\kappa, \lambda^*_\kappa) \in \mathcal{V} \times \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C}$ are corresponding κ-optimal states of the primal or dual evolution mixed systems, respectively. In general, the cost functional $J : C \times (\mathcal{V} \times \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C}) \to \mathfrak{R} \cup \{+\infty\}$ will be of the form

$$J(c; v, q^*, \mu^*) = \int_0^T j(c) + g_1(v) + g_2(q^*) + g_3(\mu^*)\, dt + g_4(v(T)), \tag{33}$$

with appropriate integrand functional components properties.

4.1 Primal evolution mixed control problem

The primal evolution mixed variational inclusion, governing optimal control problem ($\boldsymbol{O}$), is the following.

$$(\boldsymbol{\mathcal{M}_\kappa})\begin{cases} \text{Given } f^* \in \mathcal{V}^*, g \in L^p(0,T;\mathcal{R}(\Lambda)) \text{ and } u_0 \in H(\Omega), \\ \text{find } u \in \mathcal{W} \text{ and } (p^*, \lambda^*) \in \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C}: \\ \\ -\Lambda^T p^* - \pi_C^T \lambda^* \in \dfrac{du}{dt} + \partial\widetilde{F}(u) + B^*\kappa - \widetilde{f}^*, \quad \text{in } \mathcal{V}^*, \\ \\ \Lambda u \in \partial G^*(p^*) + g, \qquad \text{in } \mathcal{Y}, \\ \\ \pi_C u \in \partial\Psi_C^*(\lambda^*), \qquad \text{in } \mathcal{B}_{\partial\Omega_C}, \\ \\ u(0) = u_0, \end{cases}$$

where $B^* \in \mathcal{L}(C, \mathcal{V}^*)$ stands for the coupling optimal control operator, and definitions (19) are in force. Hence, assuming that evolution compatibility conditions ($\boldsymbol{C}_{(G,\Lambda),(\Psi_C,\pi_C)}$), of Subsection 3.1, are satisfied, by the composition duality principle of Theorem 8, evolution mixed control problem ($\boldsymbol{\mathcal{M}_\kappa}$) is solvable if, and only if, primal control problem

$$(\widetilde{\boldsymbol{\mathcal{P}}}_\kappa)\begin{cases} \text{Find } u \in \mathcal{W}: \\ \\ 0 \in \dfrac{du}{dt} + \partial\widetilde{F}(u) + \partial(G\circ\Lambda)(u - v_g) + \partial(\Psi_C \circ \pi_C)(u) + B^*\kappa - \widetilde{f}^*, \text{ in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

is solvable.

Next, in order to apply (Akagi, G. & Ôtani, M., 2004) existence theorem, we express primal problem ($\widetilde{\boldsymbol{\mathcal{P}}}_\kappa$) in a classical subdifferential form. To this end, we introduce the composition primal superpotential

$$\widetilde{G}_g(v) = G \circ \Lambda(v - v_g), \ v \in V(\Omega), \tag{34}$$

v_g being the fixed Λ-preimage of Lemma 6, whose effective domain and subdifferential turn out to be $\mathcal{D}(\widetilde{G}_g) = \mathcal{D}(G \circ \Lambda) + v_g$ and $\partial\widetilde{G}_g(v) = \partial(G\circ\Lambda)(v - v_g)$, respectively. Further, we assume the Moreau-Rockafellar-Robinson conditions (Ernst, E. & Théra, M., 2009)

$$(\boldsymbol{C}_{\widetilde{F},\widetilde{G}_g,\Psi_C\circ\pi_C})\begin{cases} \operatorname{int}\mathcal{D}(\widetilde{F}) \cap \mathcal{D}(\widetilde{G}_g) \neq \emptyset, \\ \\ \operatorname{int}\mathcal{D}(\widetilde{F} + \widetilde{G}_g) \cap \mathcal{D}(\Psi_C \circ \pi_C) \neq \emptyset, \end{cases}$$

under which the primal subdifferential sum

$$\partial\widetilde{\varphi} \equiv \partial(\widetilde{F} + \widetilde{G}_g + \Psi_C \circ \pi_C) = \partial\widetilde{F} + \partial\widetilde{G}_g + \partial(\Psi_C \circ \pi_C) \tag{35}$$

holds true. Thereby, primal evolution problem ($\widetilde{\boldsymbol{\mathcal{P}}}_\kappa$) takes the subdifferential form

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}}_\kappa)\begin{cases} \text{Find } u \in \mathcal{W}: \\ \\ 0 \in \dfrac{du}{dt} + \partial\widetilde{\varphi}(u) + B^*\kappa - f^*, \quad \text{in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

to which the next existence theorem applies in accordance with (Akagi, G. & Ôtani, M., 2004).

Theorem 11 *Let the coercivity and boundedness conditions*

$$(\boldsymbol{C1}_\varphi)\ \|v\|^p_{V(\Omega)} - C_1\|v\|^2_{H(\Omega)} - C_2 \leq C_3\widetilde{\varphi}(v), \forall v \in \mathcal{D}(\widetilde{\varphi}), 2 \leq p < \infty,$$

$$(\boldsymbol{C2}_\varphi)\ \|v^*\|^{p^*}_{V^*(\Omega)} \leq \ell(\|v\|_{H(\Omega)})\{\widetilde{\varphi}(v) + 1\}, \forall v^* \in \partial\widetilde{\varphi}(v),$$

ℓ a non-decreasing real function, be fulfilled. Then primal evolution problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}}_{\boldsymbol{\kappa}})$ has a unique solution.

Therefore, collecting results, we conclude the following mixed solvability result.

Theorem 12 *Under conditions $(\boldsymbol{C}_{\widetilde{F},\widetilde{G}_g,\Psi_C\circ\pi_C})$, $(\boldsymbol{C1}_{\varphi})$ and $(\boldsymbol{C2}_{\varphi})$, primal evolution mixed control problem $(\boldsymbol{\mathcal{M}}_{\boldsymbol{\kappa}})$ possesses a unique solution.*

Example 13 Returning to the distributed constrained nonlinear diffusion problem treated in Examples 2 and 9, we have the primal evolution control problem

$$(\widetilde{\boldsymbol{\mathcal{P}}}_{\boldsymbol{cd,\kappa}})\begin{cases} \text{Find } u \in \mathcal{W} : \\ 0 \in \dfrac{du}{dt} + grad^T\,\boldsymbol{grad}(u) + \partial\varphi(u) + \partial\phi(u) + B^*\kappa, \quad \text{in } \mathcal{V}^*, \\ u(0) = u_0, \end{cases}$$

whose diffusion potential and differential are such that, for $v \in W_0^{1,p}(\Omega)$,

$$\varphi(v) = \frac{1}{2}\int_\Omega \|\boldsymbol{grad}\ v\|^2 d\Omega + \frac{1}{p}\int_\Omega \|\boldsymbol{grad}\ v\|^p d\Omega, \tag{36}$$

$$\partial\varphi(v) = grad^T(1 + \|\boldsymbol{grad}\ v\|^{p-2})\boldsymbol{grad}(v). \tag{37}$$

Notice that the above differential sum rule holds true since the functional $v \mapsto \|\boldsymbol{grad}\ v\|^q_{L^q}$, for $q \geq 2$, is of class C^1. Moreover, it is well known that the nonlinear diffusion variational operator $\partial\varphi = grad^T(1 + \|\boldsymbol{grad}\ v\|^{p-2})\boldsymbol{grad} : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega)$, is a Lipschitz continuous bounded, strongly monotone and coercive operator (Lions, J.-L., 1969). Therefore, since then primal subdifferential sum (35) holds true in this case due to the proper condition $\mathcal{D}(\phi) \neq \emptyset$, of admissible control mechanisms for distributed constraints (with proper lower semicontinuous convex superpotentials ϕ's), assuming the fulfillment of corresponding conditions $(\boldsymbol{C1}_{\varphi})$ and $(\boldsymbol{C2}_{\varphi})$, the variational existence results of Theorems 11 and 12 apply to the primal and corresponding mixed evolution distributed constrained nonlinear diffusion problems $(\widetilde{\boldsymbol{\mathcal{P}}}_{\boldsymbol{cd,\kappa}})$ and

$$(\boldsymbol{\mathcal{M}}_{\boldsymbol{cd,\kappa}})\begin{cases} \text{Find } u \in \mathcal{W} \text{ and } (\boldsymbol{w}, b^*, p^*) \in \boldsymbol{\mathcal{Y}^*} : \\ -\left(-grad^T\boldsymbol{w} + b^* + p^*\right) \in \dfrac{du}{dt} + \partial(I_{\{\widehat{u}\}} \circ \pi_P)(u) + \pi_D^T\widehat{s}^* + B^*\kappa, & \text{in } \mathcal{V}^*, \\ (-\boldsymbol{grad}\ u, u, u) \in (\boldsymbol{w}, \partial\varphi^*(b^*), \partial\phi^*(p^*)), & \text{in } \boldsymbol{\mathcal{Y}}, \\ u(0) = u_0. \end{cases}$$

4.2 Dual evolution mixed control problem

Proceeding next with the dual control theory, the dual evolution mixed variational inclusion, governing optimal control problem $(\boldsymbol{O})$, is given by

$$(\boldsymbol{\mathcal{M}}^*_{\boldsymbol{\kappa}})\begin{cases} \text{Given } f^* \in L^p(0,T;\mathcal{R}(-\Lambda^T)),\ g \in \boldsymbol{\mathcal{Y}} \text{ and } p_0^* \in Z(\Omega), \\ \text{find } u \in \mathcal{V} \text{ and } (p^*, \lambda^*) \in \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C} : \\ -\Lambda^T p^* - \pi_C^T\lambda^* \in \partial\widetilde{F}(u) - \widetilde{f}^*, \quad \text{in } \mathcal{V}^*, \\ \Lambda u \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + B\kappa + g, \quad \text{in } \boldsymbol{\mathcal{Y}}, \\ \pi_C u \in \partial\Psi_C^*(\lambda^*), \quad \text{in } \mathcal{B}_{\partial\Omega_C}, \\ p^*(0) = p_0^*. \end{cases}$$

Here, $B \in \mathcal{L}(C, \boldsymbol{\mathcal{Y}})$ is the coupling optimal control operator in the dual sense, and definitions (19) are in force too. Then, under evolution compatibility condition $(\boldsymbol{C}_{-\boldsymbol{\Lambda}})$ of Subsection 3.2, by the composition duality principle of Theorem 8*, evolution mixed control problem ($\boldsymbol{\mathcal{M}^*_\kappa}$) is solvable if, and only if, dual control problem

$$(\widetilde{\mathcal{D}}_\kappa)\begin{cases} \text{Find } p^* \in \mathcal{X}^* : \\ 0 \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + \partial((\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T))(p^* + q^*_{\widetilde{f^*}}) + B\kappa + g, \quad \text{in } \mathcal{Y}, \\ p^*(0) = p^*_0, \end{cases}$$

is solvable.

Next, in order to apply (Akagi, G. & Ôtani, M., 2004) existence theorem in a dual sense, we express dual problem ($\widetilde{\mathcal{D}}_\kappa$), as for the primal case, in a classical subdifferential form. Thus, we introduce the composition dual superpotential

$$\widetilde{H}_{\widetilde{f^*}}(q^*) = (\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T)(q^* + q^*_{\widetilde{f^*}}),\ q^* \in Y^*(\Omega), \tag{38}$$

where $q^*_{\widetilde{f^*}}$ is the fixed $-\Lambda^T$-preimage of Lemma 6*, with effective domain and subdifferential $\mathcal{D}(\widetilde{H}_{\widetilde{f^*}}) = \mathcal{D}((\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T)) - q^*_{\widetilde{f^*}}$ and $\boldsymbol{\partial}\widetilde{H}_{\widetilde{f^*}}(q^*) = \partial((\widetilde{F} + \Psi_C \circ \pi_C)^* \circ (-\Lambda^T))(q^* + q^*_{\widetilde{f^*}})$, respectively. Also, we assume the corresponding Moreau-Rockafellar-Robinson condition (Ernst, E. & Théra, M., 2009)

$$(\boldsymbol{C}_{\boldsymbol{G^*},\widetilde{\boldsymbol{H}}_{\widetilde{\boldsymbol{f^*}}}})\ \mathrm{int}\mathcal{D}(G^*) \cap \mathcal{D}(\widetilde{H}_{\widetilde{f^*}}) \neq \emptyset,$$

which guarantees the validity of the dual subdifferential sum rule

$$\partial\widetilde{\varphi}^* \equiv \partial(G^* + \widetilde{H}_{\widetilde{f^*}}) = \partial G^* + \partial\widetilde{H}_{\widetilde{f^*}}. \tag{39}$$

Then, dual evolution problem ($\widetilde{\mathcal{D}}_\kappa$) has the subdifferential form

$$(\widetilde{\widetilde{\mathcal{D}}}_\kappa)\begin{cases} \text{Find } p^* \in \mathcal{X}^* : \\ 0 \in \dfrac{dp^*}{dt} + \partial\widetilde{\varphi}^*(p^*) + B\kappa + g, \quad \text{in } \mathcal{Y}, \\ p^*(0) = p^*_0, \end{cases}$$

and in accordance with (Akagi, G. & Ôtani, M., 2004), the following existence theorem applies to the dual control case.

Theorem 11* *Let the coercivity and boundedness dual conditions*

$$(\boldsymbol{C1}_{\boldsymbol{\varphi^*}})\ \|q^*\|^{q^*}_{Y^*(\Omega)} - C^*_1\|q^*\|^2_{Z(\Omega)} - C^*_2 \leq C^*_3\, \widetilde{\varphi}^*(q^*), \forall q^* \in \mathcal{D}(\widetilde{\varphi}^*), 2 \leq q^* < \infty,$$

$$(\boldsymbol{C2}_{\boldsymbol{\varphi^*}})\ \|q\|^p_{Y(\Omega)} \leq \ell^*(\|q^*\|_{Z(\Omega)})\{\widetilde{\varphi}^*(q^*) + 1\}, \forall q \in \partial\widetilde{\varphi}^*(q^*),$$

ℓ^* a non-decreasing real function, be satisfied. Then dual evolution problem ($\widetilde{\widetilde{\mathcal{D}}}_\kappa$) possesses a unique solution.

Therefore, from the above results, the following dual mixed solvability is achieved.

Theorem 12* *Under conditions* ($\boldsymbol{C}_{\boldsymbol{G^*},\widetilde{\boldsymbol{H}}_{\widetilde{\boldsymbol{f^*}}}}$), ($\boldsymbol{C1}_{\boldsymbol{\varphi^*}}$) *and* ($\boldsymbol{C2}_{\boldsymbol{\varphi^*}}$), *dual evolution mixed control problem* ($\boldsymbol{\mathcal{M}^*_\kappa}$) *has a unique solution.*

Example 14 Reconsidering the boundary constrained quasistatic evolution elastoviscoplastic problem of Examples 3 and 10, the corresponding dual evolution mixed control problem is given by

$$(\widetilde{\mathcal{D}}_{cp,\kappa})\begin{cases} \text{Find } \boldsymbol{S} \in \boldsymbol{\mathcal{X}} : \\ \boldsymbol{0} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\Phi^*}(\boldsymbol{S}) + \boldsymbol{\partial}(\widetilde{F}^* \circ (-\boldsymbol{H}^T))(\boldsymbol{S} + \boldsymbol{R}_{\widetilde{b^*}}) + B\kappa, \quad \text{in } \boldsymbol{\mathcal{Y}^*}, \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, \end{cases}$$

which for being expressed as a classical evolution inclusion, we introduce the composition dual superpotential

$$\widetilde{H}_{\widetilde{b^*}}(\boldsymbol{T}) = (\widetilde{F}^* \circ (-\boldsymbol{H}^T))(\boldsymbol{T} + \boldsymbol{R}_{\widetilde{b^*}}),\ \boldsymbol{T} \in \boldsymbol{Y^*}(\Omega), \tag{40}$$

$\boldsymbol{R}_{\widetilde{b}^*}$ denoting a fixed $-\boldsymbol{H}^T$-preimage of $\widetilde{\boldsymbol{b}}^*$, with effective domain and subdifferential $\mathcal{D}(\widetilde{H}_{\widetilde{b}^*}) = \mathcal{D}(\widetilde{F}^* \circ (-\boldsymbol{H}^T)) - \boldsymbol{R}_{\widetilde{b}^*}$ and $\boldsymbol{\partial}\widetilde{\boldsymbol{H}}_{\widetilde{b}^*}(\boldsymbol{T}) = \boldsymbol{\partial}(\widetilde{F}^* \circ (-\boldsymbol{H}^T))(\boldsymbol{T} + \boldsymbol{R}_{\widetilde{b}^*})$, respectively. Further, we assume the Moreau-Rockafellar-Robinson condition

$$(\boldsymbol{C}_{\boldsymbol{\Phi}^*,\widetilde{\boldsymbol{H}}_{\widetilde{b}^*}})\ \mathrm{int}\mathcal{D}(\Phi^*) \cap \mathcal{D}(\widetilde{H}_{\widetilde{b}^*}) \neq \emptyset,$$

under which the dual subdifferential sum rule

$$\boldsymbol{\partial\varphi}^* \equiv \boldsymbol{\partial}(\Phi^* + \widetilde{H}_{\widetilde{b}^*}) = \boldsymbol{\partial\Phi}^* + \boldsymbol{\partial}\widetilde{\boldsymbol{H}}_{\widetilde{f}^*}, \tag{41}$$

holds true. Then, dual evolution problem $(\widetilde{\boldsymbol{\mathcal{D}}}_{cp,\kappa})$ is subdifferentially expressed by

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}}_{cp,\kappa}) \begin{cases} \text{Find } \boldsymbol{S} \in \boldsymbol{\mathcal{X}} : \\ \\ \boldsymbol{0} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\varphi}^*(\boldsymbol{S}) + B\kappa, \quad \text{in } \boldsymbol{\mathcal{Y}}^*, \\ \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, \end{cases}$$

and assuming the corresponding coercivity and boundedness dual conditions of Theorem 11* being satisfied, Theorem 12* applies and the dual evolution mixed control problem

$$(\boldsymbol{\mathcal{M}}^*_{cp,\kappa}) \begin{cases} \text{Find } \boldsymbol{w} \in \boldsymbol{\mathcal{V}} \text{ and } \boldsymbol{S} \in \boldsymbol{\mathcal{X}} : \\ \\ -\boldsymbol{H}^T\boldsymbol{S} \in \boldsymbol{\partial}\widetilde{\boldsymbol{F}}(\boldsymbol{w}) - \widetilde{\boldsymbol{b}}^*, \qquad \text{in } \boldsymbol{\mathcal{V}}^*, \\ \\ \boldsymbol{H}\boldsymbol{w} \in \boldsymbol{A}\dfrac{d\boldsymbol{S}}{dt} + \boldsymbol{\partial\Phi}^*(\boldsymbol{S}) + B\kappa, \quad \text{in } \boldsymbol{\mathcal{Y}}^*, \\ \\ \boldsymbol{S}(0) = \boldsymbol{S}_0, \end{cases}$$

has a unique solution. Some representative elastoviscoplastic constitutive models that further exemplify the theory are: the Norton-Hoff model, for a nonlinear Maxwell viscoelastic material (Le Tallec, P., 1990) (for a generalized Norton-Hoff regularization see (Temam, R., 1986)); the common elastic viscoplastic constitutive (Perzyna, P., 1966) model (see (Rochdi, M., & Sofonea, M., 1997) for the analysis of general elastic viscoplastic contact problems, and for a numerical analysis (Han, W. & Sofonea, M., 2000)); the elastic perfectly plastic model (Sofonea, M., Renon, N. & Shillor, M., 2004); and elastoviscoplastic multi-constitutive models (Alduncin, G., 2011a).

4.3 Evolution optimal control problem

We finally state the solvability of the optimization problem $(\boldsymbol{O})$, governed by primal and dual evolution mixed state systems $(\boldsymbol{\mathcal{M}}_\kappa)$ and $(\boldsymbol{\mathcal{M}}^*_\kappa)$, analyzed in the previous subsections.

Considering the evolution hemivariational control analysis of (Migórski, S., 2001), in its monotone variational sense, let the cost or objective functional of optimization problem $(\boldsymbol{O})$, with general mixed form (33), satisfy the monotone qualifying condition

$$(\boldsymbol{C_J}) \begin{cases} J : C \times (\mathcal{V} \times \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C}) \to \Re \cup \{+\infty\} \text{ is lower semicontinuous,} \\ \text{bounded below, and } C\text{-convex.} \end{cases}$$

Then, in accordance with (Migórski, S., 2001; Theorem 2), the optimization solvability of the control problem is achieved.

Theorem 15 *Under condition $(\boldsymbol{C_J})$, optimization problem $(\boldsymbol{O})$ has a solution.*

Therefore, we can conclude with the following variational results; cf. Theorems 12 and 12*.

Corollary 16 *Under conditions $(\boldsymbol{C_J})$, $(\boldsymbol{C}_{\widetilde{F},\widetilde{G}_g,\Psi_C\circ\pi_C})$, $(\boldsymbol{C1}_{\boldsymbol{\varphi}})$ and $(\boldsymbol{C2}_{\boldsymbol{\varphi}})$, there exists an optimal control pair $(\kappa, (u, p^*, \lambda^*)) \in C \times (\mathcal{W} \times \mathcal{Y}^* \times \mathcal{B}^*_{\partial\Omega_C})$ to problem $(\boldsymbol{O})$-$(\boldsymbol{\mathcal{M}}_\kappa)$.*

and

Corollary 16* *Let conditions $(\boldsymbol{C_J})$, $(\boldsymbol{C}_{G^*,\widetilde{H}_{\widetilde{f}^*}})$, $(\boldsymbol{C1}_{\boldsymbol{\varphi}^*})$ and $(\boldsymbol{C2}_{\boldsymbol{\varphi}^*})$ be satisfied. Then there exists an optimal control pair $(\kappa, (u, p^*, \lambda^*)) \in C \times (\mathcal{V} \times \mathcal{X}^* \times \mathcal{B}^*_{\partial\Omega_C})$ to problem $(\boldsymbol{O})$-$(\boldsymbol{\mathcal{M}}^*_\kappa)$.*

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