On Regular Co-Medial Algebras

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Abstract

In this paper we define the concepts of co-medial algebra and regular algebra and we will show that for a regular co-medial algebra (A, f, g), there exist a commutative semigroup $(A, +)$, such that the operations f, g have the linear representation on $(A, +)$. As a consequence of above result, we have the linear representation of an $n-$ary medial groupoid with an regular element, which was obtained by (Evans, T., 1963, p. 331-349).

Keywords: Co-medial algebra, Regular algebra, Medial algebra, Regular element

1. Introduction

The medial property was studied initially by (Kolmogorov, A., 1930; Nagumo, M., 1930; Bruck, R. H., 1944; Aczél, J., 1947; Hosszu, M., 1953). An algebra $A = (A, F)$, (without nollary operations) is called medial (entropic, abelian) if it satisfies the identity of mediality:

$$g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1m}, \ldots, x_{nm})) = f(g(x_{11}, \ldots, x_{1m}), \ldots, g(x_{n1}, \ldots, x_{nm})), \tag{1}$$

for every n-ary $f \in F$ and m-ary $g \in F$ (Kurosh, A. G., 1947). The n-ary operation, f, is called idempotent if $f(x, \ldots, x) = x$, for every $x \in A$. The algebra $A = (A, F)$ is called idempotent, if every operation $f \in F$ is idempotent. An idempotent medial algebra is a mode (Romanowska, A. & Smith, J. D. H., 2002). In other words, the algebra, A, is medial if it satisfies the hyperidentity of mediality (Movsisyan, Yu. M., 1986; Movsisyan, Yu. M., 1990; Movsisyan, Yu. M., 1998). Note that a groupoid is medial iff it satisfies the identity of mediality (Ježek, J., Kepka, T., 1983): $xy.uv \approx xu.yv$.

Let g and f be m-ary and n-ary operations on the set, A. We say that the pair of operations, (f, g), is medial (entropic), if the identity (1) holds in the algebra $A = (A, f, g)$. Characterization of medial pair of binary quasigroup operations, obtained in (Movsisyan, Yu. M., 1999) (Also see Movsisyan, Yu. M. & Nazari, E., 2011).

The n-ary groupoid (A, f), is medial if it is satisfies the following identity:

$$f(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{nn})) = f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})).$$

Let $A = (A, F)$ be an algebra and $f \in F$. We say that the element e, is the unit element for the operation f, if:

$$f(x, e, \ldots, e) = f(e, x, e, \ldots, e) = \cdots = f(e, \ldots, e, x) = x,$$

for every $x \in A$. The element e is an unit element for the algebra (A, F), if it is an unit for every operation $f \in F$.

The element e, is an idempotent element for the operation $f \in F$, if: $f(e, \ldots, e) = e$. An element e is idempotent element for the algebra (A, F), if it is an idempotent element for every operation $f \in F$.

Definition 1.1 Let g and f are n-ary operations on the set A. We say that the pair of n-ary operations (f, g), is *co-medial pair operation* if the following identity holds in the algebra $A = (A, f, g)$:

$$g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{nn})) = g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})).$$

The algebra $A = (A, F)$ is called co-medial algebra if, every pair of operations $f, g \in F$ with the same arity is a co-medial pair operation. If $f = g$ then, the co-medial pair operation (f, f) is a medial pair operation.

There exist various algebraic characterizations of different classes of n-ary operations (see for example Dudek, W. A. & Trokhimenko, V. S., 2010). In this article we investigate a generalizatin of algebras with a medial operation, which we called a co-medial algebras.

Definition 1.2 Let (f, g) be a pair of n-ary operations of the algebra, (A, F). For any element, e, of A, let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be mappings of A into A defined by

$$\begin{aligned} \alpha_i &: x \mapsto f(e, \ldots, e, x, e, \ldots, e), \\ \beta_i &: x \mapsto g(e, \ldots, e, x, e, \ldots, e), \end{aligned} \tag{2}$$

with x at the $i-$th place. We call α_i the *$i-$th translation by e with respect to f*. An element e is called *$i-$regular element* with respect to f if α_i is a bijection. The similar definitions go with g. An element e, is called $i-$regular for the pair operation (f, g) if, it is $i-$regular element with respect to the both operations f and g. The element e is called $i-$regular for the algebra (A, F), if it is $i-$regular for every pair operations $f, g \in F$.

2. Preliminary Results

Definition 2.1 Let f be an n-ary operation and J be a non-empty subset of $\{1, 2, \ldots, n\}$, we will say that the element e is *J-regular* with respect to the operation f, if e is a j-regular element with respect to f, for all $j \in J$. The element e is J-regular element for the algebra (A, F), if e is a J-regular element with respect to every $f \in F$.

Definition 2.2 Let $f, g \in F$ be n-ary operations $(2 \leq n)$, $\emptyset \neq J \subseteq \{1, 2, \ldots, n\}$ and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ are J-regular elements of the algebra (A, f, g) (where J contains at leas two elements). The pair operation (f, g) is *(i, J)-regular pair operation* (where $i \in J$), if for every $x \in A$ we have the following equality:

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = g(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \tag{3}$$

The pair operation (f, g) is a *J-regular* if (f, g) is (i, J)-regular for every $i \in J$. The pair operation (f, g) is *regular* if (f, g) is a J-regular pair operation for some $\emptyset \neq J \subseteq \{1, 2, \ldots, n\}$ (where J contains at leas two elements). An algebra (A, F) is called *regular* if every pair operation of (A, F) be a regular pair operation. The equality (3) is a *co-identity* in the sence of (Movsisyan, Yu. M., 1986).

Lemma 2.3 *Let (A, f, g) be a regular co-medial algebra with an idempotent $i-$regular element, then $f = g$.*

Proof: Let (A, f, g) be a regular co-medial algebra with an idempotent $i-$regular element e, then for every $x_1, \ldots, x_n \in A$ we have:

$$\begin{aligned} &\beta_i g(x_1, \ldots, x_n) \\ &= g(e, \ldots, e, g(x_1, \ldots, x_n), e, \ldots, e) \\ &= g(g(e, \ldots, e), \ldots, g(x_1, \ldots, x_n), \ldots, g(e, \ldots, e)) \\ &= g(g(e, \ldots, e, x_1, e, \ldots, e), \ldots, g(e, \ldots, e, x_i, e, \ldots, e), \ldots, g(e, \ldots, e, x_n, e, \ldots, e)) \\ &= g(\beta_i x_1, \ldots, \beta_i x_n). \end{aligned}$$

So, $\beta_i^{-1} g(x_1, \ldots, x_n) = g(\beta_i^{-1} x_1, \ldots, \beta_i^{-1} x_n)$. Also, since (A, f, g) is a regular algebra with the $i-$regular element e, $\alpha_i = \beta_i$. Therefor, $\alpha_i^{-1} = \beta_i^{-1}$ and we have:

$$\begin{aligned} &g(x_1, \ldots, x_n) \\ &= f(e, \ldots, e, \alpha_i^{-1} g(x_1, \ldots, x_n), e, \ldots, e) \\ &= f(e, \ldots, e, \beta_i^{-1} g(x_1, \ldots, x_n), e, \ldots, e) \\ &= f(g(e, \ldots, e), \ldots, g(\beta_i^{-1} x_1, \ldots, \beta_i^{-1} x_n), \ldots, g(e, \ldots, e)) \\ &= f(g(e, \ldots, e, \beta_i^{-1} x_1, e, \ldots, e), \ldots, g(e, \ldots, e, \beta_i^{-1} x_n, e, \ldots, e)) \\ &= f(x_1, \ldots, x_n). \end{aligned}$$

Hence, $f = g$.

The following characterization of a medial n-ary groupoid with an idempotent $J-$regular element, obtained by (Evans, T., 1963).

Theorem 2.4 *Let (A, f) be an $n-$ary medial groupoid with a $J-$regular idempotent element e (where $J \subseteq \{1, 2, \ldots, n\}$ contains at leas two elements), then there exists a commutative semigroup $(A, +)$ with the unit element e, such that the operation f has the following linear representation*

$$f(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where $\alpha_1, \ldots, \alpha_n$, are pairwise commuting endomorphisms of $(A, +)$, $n \geq 2$. If $i, j \in J$, then α_i, α_j are automorphisms of $(A, +)$.

Proof: See, Evans, T., 1963.

Corollary 2.5 *Let (A, f, g) be a regular co-medial $n-$ary algebra with an idempotent $J-$regular element e (where $J \subseteq \{1, 2, \ldots, n\}$ contains at leas two elements), then there exists a commutative semigroup $(A, +)$ with the unit element e, such that*

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where $\alpha_1, \ldots, \alpha_n$, are pairwise commuting endomorphisms of $(A, +)$, $n \geq 2$. If $i, j \in J$, then α_i, α_j are automorphisms of $(A, +)$.

Proof: By Lemma 2.3, $f = g$. Therefor, the algebra (A, f, g) is an $n-$ary medial groupoid, so we can use the Theorem 2.4.

We have described in the Corollary 2.5 structure of the regular co-medial algebra (A, f, g) containing an idempotent $J-$regular element. The purpose of this section is to obtain sufficient properties of finite co-medial algebras to enable us to weaken considerably, in this finite case, the assumptions we need for characterizing regular co-medial algebras which do not contain an idempotent element.

We use continually the following lemma, the proof of which we omit.

Lemma 2.6 *Let (A, F) be a finite algebra and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ are elements of A and $f \in F$, if for all $x, y \in A$,*

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n),$$

implies that $x = y$, then for any $b \in A$ there is a unique element $x \in A$ such that

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = b.$$

Lemma 2.7 *Let (A, f, g) be a finite co-medial algebra with an $J-$regular element e. If $f(a_1, \ldots, a_n) = e$ and $g(c_1, \ldots, c_n) = e$ then, for each i, a_i, is a $J-$regular element with respect to the operation f and c_i is a $J-$regular element with respect to the operation g.*

Proof: Let $x, y \in A$ and $i \in J$, such that

$$f(a_i, \ldots, a_i, x, a_i, \ldots, c_i) = f(a_i, \ldots, a_i, y, a_i, \ldots, a_i),$$

then, we have:

$$g(f(a_1, \ldots, e, \ldots, a_1), \ldots, \overbrace{f(a_i, \ldots, \underbrace{x}_{i-th}, \ldots, a_i)}^{i-th}, \ldots, f(a_n, \ldots, e, \ldots, a_n)) =$$

$$g(f(a_1, \ldots, e, \ldots, a_1), \ldots, \overbrace{f(a_i, \ldots, \underbrace{y}_{i-th}, \ldots, a_i)}^{i-th}, \ldots, f(a_n, \ldots, e, \ldots, a_n)).$$

So, by co-mediality we have:

$$g(f(a_1, \ldots, a_n), \ldots, \overbrace{f(e, \ldots, \underbrace{x}_{i-th}, \ldots, e)}^{i-th}, \ldots, f(a_1, \ldots, a_n)) =$$

$$g(f(a_1, \ldots, a_n), \ldots, \overbrace{f(e, \ldots, \underbrace{y}_{i-th}, \ldots, e)}^{i-th}, \ldots, f(a_1, \ldots, a_n)),$$

$$g(e, \ldots, f(e, \ldots, x, \ldots, e), \ldots, e) = g(e, \ldots, f(e, \ldots, y, \ldots, e), \ldots, e).$$

Thus, by applying regularity of the element e, we have: $x = y$.

Similarly, if

$$g(c_i, \ldots, c_i, x, c_i, \ldots, c_i) = g(c_i, \ldots, c_i, y, c_i, \ldots, c_i),$$

then, $x = y$. By Lemma 2.6, this concludes the proof.

Lemma 2.8 *Let (A, f, g) be a finite co-medial algebra. If e is an i-regular element in (A, f, g), then so are $f(e, \ldots, e)$ and $g(e, \ldots, e)$.*

Proof: Let $x, y \in A$, such that

$$g(f(e,\dots,e),\dots,x,\dots,f(e,\dots,e)) = g(f(e,\dots,e),\dots,y,\dots,f(e,\dots,e)),$$

and t_1 be the element of A satisfying

$$f(e,\dots,t_1,\dots,e) = e,$$

with t_1 at the $i-th$ place. By Lemma 2.7, t_1 is regular element with respect to f. So, if

$$f(t_1,\dots,x_1,\dots,t_1) = x,$$
$$f(t_1,\dots,y_1,\dots,t_1) = y,$$

with x_1, y_1 at the $i-th$ place, then we have:

$$g(f(e,\dots,e),\dots,\overbrace{f(t_1,\dots,x_1,\dots,t_1)}^{i-th},\dots,f(e,\dots,e)) =$$
$$g(f(e,\dots,e),\dots,f(t_1,\dots,y_1,\dots,t_1),\dots,f(e,\dots,e)).$$

So, by co-mediality we have:

$$g(f(e,\dots,t_1,\dots,e),\dots,\overbrace{f(e,\dots,x_1,\dots,e)}^{i-th},\dots,f(e,\dots,t_1,\dots,e)) =$$
$$g(f(e,\dots,t_1,\dots,e),\dots,\overbrace{f(e,\dots,y_1,\dots,e)}^{i-th},\dots,f(e,\dots,t_1,\dots,e)).$$

Since, e is an i-regular element of (A, f, g), $x_1 = y_1$. Hence, $x = y$. By Lemma 2.6, this concludes $f(e,\dots,e)$ is an i-regular element with respect to the operation g. Similarly, $f(e,\dots,e)$ is an i-regular element with respect to the operation f, and $g(e,\dots,e)$ is an i-regular element with respect to the both operations f, g.

Lemma 2.9 *Let (A, f, g) be a finite co-medial algebra with $n-$ary operations, and $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n$ are $J-$regular elements of the algebra (A, f, g) (where $J \subset \{1, 2,\dots,n\}$ contains at leas two elements). Then, for every $b \in A$, there are unique $x_1, x_2 \in A$ such that*

$$f(a_1,\dots,a_{i-1},x_1,a_{i+1},\dots,a_n) = b,$$
$$g(a_1,\dots,a_{i-1},x_2,a_{i+1},\dots,a_n) = b.$$

Proof: Let $x_1, y_1 \in A$, such that

$$f(a_1,\dots,a_{i-1},x_1,a_{i+1},\dots,a_n) = f(a_1,\dots,a_{i-1},y_1,a_{i+1},\dots,a_n),$$

we will prove that, $x_1 = y_1$.

For $k = 1,\dots,i-1,i+1,\dots,n$ let,

$$f(a_k,\dots,t_k,\dots,a_k) = a_1,$$

with t_k at the $j-th$ place. Then, we have:

$$g(f(a_1,\dots,a_{i-1},a_1,a_{i+1},\dots,a_n),\dots,\overbrace{f(a_1,\dots,a_{i-1},x_1,a_{i+1},\dots,a_n)}^{i-th},\dots$$
$$\dots,\overbrace{f(t_1,\dots,t_{i-1},a_1,t_{i+1},\dots,t_n)}^{j-th},\dots,f(a_1,\dots,a_{i-1},a_1,a_{i+1},\dots,a_n)) =$$
$$g(f(a_1,\dots,a_{i-1},a_1,a_{i+1},\dots,a_n),\dots,\overbrace{f(a_1,\dots,a_{i-1},y_1,a_{i+1},\dots,a_n)}^{i-th},\dots$$
$$\dots,\overbrace{f(t_1,\dots,t_{i-1},a_1,t_{i+1},\dots,t_n)}^{j-th},\dots,f(a_1,\dots,a_{i-1},a_1,a_{i+1},\dots,a_n)).$$

So, by co-mediality, we have:

$$g(f(a_1,\dots,t_1,\dots,a_1),\dots,\overbrace{f(a_1,\dots,x_1,\dots,a_1)}^{i-th},\dots,f(a_n,\dots,t_n,\dots,a_n)) =$$
$$g(f(a_1,\dots,t_1,\dots,a_1),\dots,\overbrace{f(a_1,\dots,y_1,\dots,a_1)}^{i-th},\dots,f(a_n,\dots,t_n,\dots,a_n)),$$
$$g(a_1,\dots,\overbrace{f(a_1,\dots,x_1,\dots,a_1)}^{i-th},\dots,a_1) = g(a_1,\dots,\overbrace{f(a_1,\dots,y_1,\dots,a_1)}^{i-th},\dots,a_1),$$

Two applications of the i-regularity of a_1 yield $x_1 = y_1$. Similarly, if

$$g(a_1, \ldots, a_{i-1}, x_2, a_{i+1}, \ldots, a_n) = g(a_1, \ldots, a_{i-1}, y_2, a_{i+1}, \ldots, a_n),$$

then, $x_2 = y_2$.

Specifically, from the above lemmas, we known if the finite co-medial algebra (A, f, g) contains an element e which is $J-$regular element, then

1. $f(e, \ldots, e)$ and $g(e, \ldots, e)$ are also $J-$regular element of (A, f, g),
2. there are unique elements $t_1, t_2 \in A$, which are regular with respect to f and g (respectively), such that (for $i < j$)

$$f(e, \ldots, e, f(e, \ldots, e), e, \ldots, e, t_1, e, \ldots, e) = e,$$
$$g(e, \ldots, e, g(e, \ldots, e), e, \ldots, e, t_2, e, \ldots, e) = e,$$

with $f(e, \ldots, e)$ and $g(e, \ldots, e)$ at the $i - th$ places and t_1, t_2 at the $j - th$ places.

3. for every $b \in A$, there are unique elements $x_1, x_2 \in A$ such that

$$f(e, \ldots, e, x_1, e, \ldots, e, t_1, e, \ldots, e) = b,$$
$$g(e, \ldots, e, x_2, e, \ldots, e, t_2, e, \ldots, e) = b,$$

with x_1, x_2 at the the $i - th$ places and t_1, t_2 at the $j - th$ places, where t_1, t_2 are the elements described in (2).

It is easy to prove that in the finite co-medial algebra (A, f, g) the set of $J-$regular elements is closed under the operations f, g. Thus, if the finite co-medial algebra (A, f, g), contains at least one $J-$regular element, then the algebra (A, f, g), contains an $J-$regular subalgebra, where by $J-$regular subalgebra of a co-medial algebra, finite or infinite, we mean a subalgebra of $J-$regular elements such that if $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ belong to the subalgebra, then there are the unique elements $x_1, x_2 \in A$, for each $b \in A$ such that

$$f(a_1, \ldots, a_{i-1}, x_1, a_{i+1}, \ldots, a_n) = b,$$
$$g(a_1, \ldots, a_{i-1}, x_2, a_{i+1}, \ldots, a_n) = b.$$

Furthermore, if b is in the subalgebra, so are x_1, x_2.

3. The Structure of Regular Co-Medial Algebras

We discuss in this section the structure of a regular co-medial algebra (A, f, g) which does not contain an idempotent element. We construct new operations f^*, g^* on A in terms of f, g, such that (f^*, g^*) is a co-medial pair operation, and the co-medial algebra (A, f^*, g^*), contains an idempotent element. If certain regularity conditions are assumed for (A, f, g), then this idempotent element is also a $J-$regular element in (A, f^*, g^*) and hence we are able to use the corollary 2.5 to describe the structure of the pair operation (f^*, g^*).

Lemma 3.1 *Let (A, f, g) be a co-medial algebra and π, ρ be permutations of $\{1, 2, \ldots, n\}$, then the pair operation (f^*, g^*) is co-medial, where*

$$f^*(x_1, x_2, \ldots, x_n) = f(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n}),$$
$$g^*(x_1, x_2, \ldots, x_n) = g(x_{\rho 1}, x_{\rho 2}, \ldots, x_{\rho n}).$$

Proof: For $x_{ij} \in A$, since (A, f, g) is a co-medial algebra, we have:

$$\begin{aligned}
&g^*(f^*(x_{11}, \ldots, x_{n1}), \ldots, f^*(x_{1n}, \ldots, x_{nn})) \\
&= g(f(x_{\rho 1 \pi 1}, \ldots, x_{\rho n \pi 1}), \ldots, f(x_{\rho 1 \pi n}, \ldots, x_{\rho n \pi n})) \\
&= g(f(x_{\rho 1 \pi 1}, \ldots, x_{\rho 1 \pi n}), \ldots, f(x_{\rho n \pi 1}, \ldots, x_{\rho n \pi n})) \\
&= g^*(f^*(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})).
\end{aligned}$$

Lemma 3.2 *Let (A, f, g) be a finite co-medial algebra and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ are elements of A. Then, the pair operation defined by the following*

$$f^*(x_1, \ldots, x_n) = f(a_1, \ldots, a_{i-1}, f(x_1, \ldots, x_n), a_{i+1}, \ldots, a_n),$$
$$g^*(x_1, \ldots, x_n) = g(a_1, \ldots, a_{i-1}, g(x_1, \ldots, x_n), a_{i+1}, \ldots, a_n),$$

is a co-medial pair operation on A.

Proof: In view of Lemma 3.1, it is sufficient to prove this for $i = 1$.

$$\begin{aligned}&g^*(f^*(x_{11},\dots,x_{1n}),\dots,f^*(x_{n1},\dots,x_{nn}))\\&= g(g(f^*(x_{11},\dots,x_{1n}),\dots,f^*(x_{n1},\dots,x_{nn})),a_2,\dots,a_n).\end{aligned}$$

But,

$$\begin{aligned}&g(f^*(x_{11},\dots,x_{1n}),\dots,f^*(x_{n1},\dots,x_{nn}))\\&= g(f(f(x_{11},\dots,x_{1n}),a_2,\dots,a_n),\dots,f(f(x_{n1},\dots,x_{nn}),a_2,\dots,a_n))\\&= g(f(f(x_{11},\dots,x_{n1}),\dots,f(x_{1n},\dots,x_{nn})),f(a_2,\dots,a_2),\dots,f(a_n,\dots,a_n))\\&= g(f(f(x_{11},\dots,x_{n1}),\dots,f(x_{1n},\dots,x_{nn})),f(a_2,\dots,a_2),\dots,f(a_n,\dots,a_n))\\&= g(f(f(x_{11},\dots,x_{n1}),x_2,\dots,a_n),\dots,f(f(x_{1n},\dots,x_{nn}),a_2,\dots,a_n))\\&= g(f^*(x_{11},\dots,x_{n1}),\dots,f^*(x_{1n},\dots,x_{nn})),\end{aligned}$$

since, (A, f, g) is co-medial.

So,

$$\begin{aligned}&g^*(f^*(x_{11},\dots,x_{1n}),\dots,g(x_{n1},\dots,x_{nn}))\\&= g(g(f^*(x_{11},\dots,x_{n1}),\dots,f^*(x_{1n},\dots,x_{nn})),a_2,\dots,a_n)\\&= g^*(f^*(x_{11},\dots,x_{n1}),\dots,f^*(x_{1n},\dots,x_{nn})).\end{aligned}$$

Hence, (f^*, g^*) is a co-medial pair operation.

Lemma 3.3 *Let (A, f, g) be a co-medial algebra and, $e, t_1, t_2 \in A$ such that*

$$\begin{aligned}f(e,\dots,e,f(e,\dots,e),e,\dots,e,t_1,e,\dots,e) &= e,\\ g(e,\dots,e,g(e,\dots,e),e,\dots,e,t_2,e,\dots,e) &= e.\end{aligned}$$

Then, the pair operation (f^*, g^*) on A, defined by the following

$$\begin{aligned}f^*(x_1,\dots,x_n) &= f(e,\dots,e,f(x_1,\dots,x_n),e,\dots,e,t_1,e,\dots,e),\\ g^*(x_1,\dots,x_n) &= g(e,\dots,e,g(x_1,\dots,x_n),e,\dots,e,t_2,e,\dots,e),\end{aligned}$$

with $f(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n)$ at the $i-th$ place and t_1, t_2 at the $j-th$ place, is a co-medial pair operation with e as an idempotent element.

Proof: This follows immediately by Lemma 3.2 and direct computation of $f^*(e,\dots,e)$ and $g^*(e,\dots,e)$.

Lemma 3.4 *Let (A, f, g) be regular co-medial algebra, then there is a commutative semigroup $(A, +)$, such that*

$$\begin{aligned}f(x_1,\dots,x_n) &= \varphi(\alpha_1 x_1 + \cdots + \alpha_n x_n),\\ g(x_1,\dots,x_n) &= \psi(\alpha_1 x_1 + \cdots + \alpha_n x_n),\end{aligned}$$

where, $\alpha_1,\dots,\alpha_n$, are pairwise commuting endomorphisms of $(A, +)$ and φ, ψ are bijections on A, for $n \geq 2$.

Proof: If $J \subseteq \{1, 2, \dots, n\}$, $i, j \in J$ and e is a $J-$regular element in (A, f, g), then by results of the preceding section, there are $J-$regular elements t_1, t_2, such that

$$\begin{aligned}f(e,\dots,e,f(e,\dots,e),e,\dots,e,t_1,e,\dots,e) &= e,\\ g(e,\dots,e,f(e,\dots,e),e,\dots,e,t_2,e,\dots,e) &= e,\end{aligned}$$

with, $f(e,\dots,e)$ and $g(e,\dots,e)$ at the $i-th$ place and t_1, t_2 at the $j-th$ place.

Furthermore, for k either i or j, and any $b \in A$, the following equations

$$\begin{aligned}f(e,\dots,e,f(e,\dots,e,x_1,e,\dots,e),e,\dots,e,t_1,e,\dots,e) &= b,\\ g(e,\dots,e,g(e,\dots,e,x_2,e,\dots,e),e,\dots,e,t_2,e,\dots,e) &= b,\end{aligned}$$

with x_1, x_2 at the $k-th$ place, have unique solutions. Hence, e is a $J-$regular with respect to the pair operation (f^*, g^*) on A, defined by

$$f^*(x_1, \dots, x_n) = f(e, \dots, e, f(x_1, \dots, x_n), e, \dots, e, t_1, e, \dots, e),$$
$$g^*(x_1, \dots, x_n) = g(e, \dots, e, g(x_1, \dots, x_n), e, \dots, e, t_2, e, \dots, e).$$

So, by lemma 3.3, the pair operation (f^*, g^*) is co-medial with e, as an idempotent element. Thus, by corollary 2.5, there is a commutative semigroup, $(A, +)$, with the unit element e such that

$$f^*(x_1, \dots, x_n) = g^*(x_1, \dots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

where, $\alpha_1, \dots, \alpha_n$ are commuting endomorphisms of $(A, +)$.

Again, by the results of the previous section, the mappings

$$\varphi^{-1} : x \to f(e, \dots, e, x, e, \dots, e, t_1, e, \dots, e),$$
$$\psi^{-1} : x \to g(e, \dots, e, x, e, \dots, e, t_2, e, \dots, e),$$

are bijections on A. Thus,

$$f(x_1, \dots, x_n) = \varphi f^*(x_1, \dots, x_n),$$
$$g(x_1, \dots, x_n) = \psi g^*(x_1, \dots, x_n).$$

Lemma 3.5 *Let $(A, +)$ be a commutative semigroup with a unit element and $\varphi_1, \dots, \varphi_n$ are bijections on A, such that*

$$\varphi_1(x_{11} + \cdots + x_{1n}) + \cdots + \varphi_n(x_{n1} + \cdots + x_{nn}) = \varphi_1(x_{11} + \cdots + x_{n1}) + \cdots + \varphi_n(x_{1n} + \cdots + x_{nn}). \tag{4}$$

Then, there is an automorphism η of $(A, +)$ and fixed elements $c_1, \dots, c_n$ such that for each i, we have:

$$\varphi_i x = \eta x + c_i,$$

for all $x \in A$.

Proof: Let $(A, +)$ be a commutative semigroup with a unit element, e. In the equation (4), for fixed i and all j except $j = 1$, put $x_{ij} = \varphi_i^{-1} e$ and all other x_{pq} be unit element except, x_{1i} and x_{i1}, then we have:

$$\varphi_1 x_{1i} + \varphi_i x_{i1} = \varphi_1 x_{i1} + \varphi_i x_{1i}.$$

So, if $x_{1i} = \varphi_1^{-1} e$, then

$$\varphi_i x_{i1} = \varphi_1 x_{i1} + \varphi_i \varphi_1^{-1} e,$$

for all $x_{i1} \in A$.

Since, φ_1, φ_i are permutations on $(A, +)$, for all $x \in A$ we have:

$$\varphi_i x = \varphi_1 x + k_i,$$

where, k_i is a fixed regular element of $(A, +)$. Substituting for the φ_i in the equation (4) and cancelling the k_i, which we may do since they are regular elements, we get

$$\varphi_1(x_{11} + \cdots + x_{1n}) + \cdots + \varphi_1(x_{n1} + \cdots + x_{nn}) = \varphi_1(x_{11} + \cdots + x_{n1}) + \cdots + \varphi_1(x_{1n} + \cdots + x_{nn}). \tag{5}$$

In the equation (5), let $x_{ii} = \varphi_1^{-1} e$, where, $i \neq 1, 2$, and all other x_{ij} be the unit element, e, except, x_{11}, x_{12}. Then, we have:

$$\varphi_1(x_{11} + x_{12}) + \varphi_1 e = \varphi_1 x_{11} + \varphi_1 x_{12},$$

for all x_{11}, x_{12}.

So, if $x_{11} = x_{12} = \varphi_1^{-1} e$, then

$$\varphi_1(\varphi_1^{-1} e + \varphi_1^{-1} e) + \varphi_1 e = e,$$

it means that, $\varphi_1 e$ has an additive inverse and hence is a regular element.

Now, we define a bijection η on A by the following

$$\varphi_1 x = \eta x + \varphi_1 e,$$

for all $x \in A$. It follows immediately that, η is an automorphism of $(A, +)$.

Hence,

$$\varphi_i x = \varphi_1 x + k_i = \eta x + \varphi_1 e + k_i = \eta x + c_i,$$

where, $c_i = \varphi_1 e + k_i$, as the sum of two regular elements is a regular element.

Theorem 3.6 *Let (A, f, g) be a regular co-medial algebra, then there is a commutative semigroup $(A, +)$ with an unit element, such that*

$$f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d_1,$$
$$g(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2,$$

where, d_1, d_2 are fixed regular elements in $(A, +)$ and $\gamma_1, \ldots, \gamma_n$, $\lambda_1, \ldots, \lambda_n$, are commuting automorphisms of the semigroup $(A, +)$.

Proof: Let (A, f, g) be a regular co-medial algebra, by Lemma 3.4, we know that there is a commutative semigroup with an unit element e, such that

$$f(x_1, \ldots, x_n) = \varphi(\alpha_1 x_1 + \cdots + \alpha_n x_n),$$
$$g(x_1, \ldots, x_n) = \psi(\alpha_1 x_1 + \cdots + \alpha_n x_n),$$

where, $\alpha_1, \ldots, \alpha_n$, are pairwise commuting endomorphisms of $(A, +)$ and φ, ψ are bijections on A.

Since, the operation f is co-medial we have:

$$\varphi(\alpha_1\varphi(\alpha_1 x_{11} + \cdots + \alpha_n x_{1n}) + \cdots + \alpha_n\varphi(\alpha_1 x_{n1} + \cdots + \alpha_n x_{nn})) =$$
$$\varphi(\alpha_1\varphi(\alpha_1 x_{11} + \cdots + \alpha_n x_{n1}) + \cdots + \alpha_n\varphi(\alpha_1 x_{1n} + \cdots + \alpha_n x_{nn})).$$

So,

$$\alpha_1\varphi\alpha_1^{-1}(\alpha_1\alpha_1 x_{11} + \cdots + \alpha_1\alpha_n x_{1n}) + \cdots + \alpha_n\varphi\alpha_n^{-1}(\alpha_n\alpha_1 x_{n1} + \cdots + \alpha_n\alpha_n x_{nn}) =$$
$$\alpha_1\varphi\alpha_1^{-1}(\alpha_1\alpha_1 x_{11} + \cdots + \alpha_1\alpha_n x_{n1}) + \cdots + \alpha_n\varphi\alpha_n^{-1}(\alpha_n\alpha_1 x_{1n} + \cdots + \alpha_n\alpha_n x_{nn}),$$

since φ is bijection.

Let, $\beta_i = \alpha_i\varphi\alpha_i^{-1}$ and $\alpha_i\alpha_j x_{ij} = y_{ij}$, then by substitution and since, $\alpha_1, \ldots, \alpha_n$ are commuting automorphisms of the commutative semigroup $(A, +)$, we have:

$$\beta_1(y_{11} + \cdots + y_{1n}) + \cdots + \beta_n(y_{n1} + \cdots + y_{nn}) =$$
$$\beta_1(y_{11} + \cdots + y_{n1}) + \cdots + \beta_n(y_{1n} + \cdots + y_{nn}).$$

So, by preceding lemma, there is an automorphism η of the semigroup $(A, +)$, and regular elements $c_1, \ldots, c_n$, such that

$$\alpha_i\varphi\alpha_i^{-1} x = \eta x + c_i,$$
$$\varphi x = \alpha_i^{-1}\eta\alpha_i x + \alpha_i^{-1}\alpha_i c_i,$$
$$\varphi x = \sigma x + d_1,$$

where, $\sigma = \alpha_i^{-1}\eta\alpha_i$ is an automorphism of the semigroup and $d_1 = \alpha_i^{-1}\alpha_i c_i$ is a fixed regular element in $(A, +)$.

Hence,

$$f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d_1,$$

where, $\gamma_i = \sigma\alpha_i$ is an automorphism of the semigroup.

Similarly,

$$g(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2.$$

It is easy to check that, $\gamma_1, \ldots, \gamma_n$, $\lambda_1, \ldots, \lambda_n$, are commuting automorphisms of the semigroup $(A, +)$.

The following representation of a medial $n-$ary groupoid was obtained by (Evans, T., 1963).

Corollary 3.7 *Let (A, f) be a medial $n-$ary groupoid with a $i-$ and $j-$regular element, then there exists a commutative semigroup $(A, +)$, such that*

$$f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d,$$

where, d is a fixed regular element in $(A, +)$ and $\gamma_1, \ldots, \gamma_n$ are commuting automorphisms of the semigroup $(A, +)$.

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