# Stable 3-Spheres in $\mathbb{C}^3$

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**Abstract**

By only using spectral theory of the Laplace operator on spheres, we prove that the unit 3-dimensional sphere of a 2-dimensional complex subspace of $\mathbb{C}^3$ is an $\Omega$-stable submanifold with parallel mean curvature, when $\Omega$ is the Kähler calibration of rank 4 of $\mathbb{C}^3$.

**Keywords:** Stability, Parallel Mean Curvature, Cauchy-Riemann inequality, Spheres

## 1. Introduction

In 2000, Frank Morgan introduced the notion of multi-volume for an $m$-dimensional submanifold $M$ of a Euclidean space $\mathbb{R}^{m+n}$, as a volume enclosed by orthogonal projections onto axis $(m+1)$-planes. He characterized stationary submanifolds for the area functional with prescribed multi-volume as submanifolds with mean curvature vector $H$ prescribed by a constant multivector $\xi \in \wedge_{m+1}\mathbb{R}^{m+n}$, namely $H = \xi \lfloor \vec{S}$, where $\vec{S}$ is the unit tangent plane of $M$, and proved the existence of a minimizer among rectifiable currents, as well as their regularity under general conditions of the boundary. In this setting, a question has arisen on conditions for $\|H\|$ to be constant. In (Salavessa, 2010) we extended the variational characterization of hypersurfaces with constant mean curvature $\|H\|$ to submanifolds with higher codimension, when the ambient space is any Riemannian manifold $\bar{M}^{m+n}$, as discovered by Barbosa, do Carmo and Eschenburg (1984, 1988) for the case $n = 1$. This generalization amounts on defining an "enclosed" $(m+1)$-volume of an $m$-dimensional immersed submanifold $F : M^m \to \bar{M}^{m+n}$, $m \geq 2$, as the $\Omega$-volume defined by each one-parameter variation family $F(x,t) = F_t(x)$ of $F(x,0) = F(x)$, where $\Omega$ is a semi-calibration on the ambient space $\bar{M}$, that is, an $(m+1)$-form $\Omega$ which satisfies $|\Omega(e_0, e_1, \ldots, e_m)| \leq 1$, for any orthonormal system $e_i$ of $T\bar{M}$. A submanifold with calibrated extended tangent space $H \oplus TM$ is a critical point of the functional area, for compactly supported $\Omega$-volume preserving variations, if and only if it has constant mean curvature $\|H\|$. In this case we have $H = \|H\|\,\Omega\lfloor\vec{S}$. From a deeper inspection of this proof, one can see that the initial assumption of calibrated extended tangent space can be dropped, since it will appear as a consequence of being a critical point itself. This will be explained in detail in a future paper, and also its relations with Morgan's formalism. Assuming that $M$ has parallel mean curvature $H$, a second variation is then computed, and its non-negativeness defines stability of $M$. This corresponds to the non-negativeness of the quadratic form associated with the $L^2$-self-adjoint $\Omega$-Jacobi operator $\mathcal{J}_\Omega(W) = \mathcal{J}(W) + m\|H\|C_\Omega(W)$, acting on sections in the twisted normal bundle $H^1_{0,T}(NM) = \mathcal{F} \oplus H^1_0(E)$, where the set $\mathcal{F}$ of $H^1_0$-functions with zero mean value is identified with the set of sections of the form $f\nu$, with $f \in \mathcal{F}$ and $\nu = H/\|H\|$, and where $E$ is the orthogonal complement of $\nu$ in the normal bundle. This Jacobi operator is the usual one, but with an extra term, namely a multiple of a first order differential operator $C_\Omega(W)$ that depends on $\Omega$. The twisted normal bundle is the $H^1$-completion of the vector space generated by the set $\mathcal{F}_\Omega$ of compactly supported infinitesimal $\Omega$-volume preserving variations, and, in general, we do not know whether it is larger than $\mathcal{F}_\Omega$ itself. Thus, $\Omega$-stability implies that the area functional of $F_t$ decreases when $t$ approaches $t_0 = 0$, for any family of $\Omega$-volume preserving variations $F_t$ of $F$, but we do not know whether the converse also holds always. In case the ambient space is the Euclidean space $\mathbb{R}^{m+n}$, then a unit $m$-sphere of an $\Omega$-calibrated Euclidean subspace $\mathbb{R}^{m+1}$ of $\mathbb{R}^{m+n}$ is $\Omega$-stable if and only if, for any $(n-1)$-tuple of functions $f_\alpha \in C^\infty(\mathbb{S}^m)$, $2 \leq \alpha \leq n$, the following integral inequality holds:

$$\sum_{\alpha<\beta} -2m \int_{\mathbb{S}^m} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) dM \leq \sum_\alpha \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM, \tag{1}$$

where $W_\alpha$ is a fixed global parallel orthonormal (o.n.) frame of $\mathbb{R}^{n-1}$, the orthogonal complement of $\mathbb{R}^{m+1}$ spanned by $\mathbb{S}^m$, and $\xi$ is the $T^*\mathbb{S}^m$-valued 2-form on $\mathbb{R}^{n-1}_{/\mathbb{S}^m}$

$$\xi(W, W')(X) = \Omega(W, W', *X), \quad W, W' \in \mathbb{R}^{n-1}, X \in T^*\mathbb{S}^m$$

where $* : T\mathbb{S}^m \to \wedge^{m-1}T\mathbb{S}^m$ is the star operator. If (1) holds and

$$\bar{\nabla}_W \Omega(W, e_1, \ldots, e_m) = 0, \quad \forall W \in N\mathbb{S}^m, \tag{2}$$

where $e_i$ is an o.n. frame of $T\mathbb{S}^m$, then in (Salavessa, 2010, Proposition 4.5) we have shown that for each $\alpha < \beta$, $\xi(W_\alpha, W_\beta)$ must be co-exact as a 1-form on $\mathbb{S}^m$, that is,

$$\xi_{\alpha\beta} := \xi(W_\alpha, W_\beta) = \delta\omega_{\alpha\beta},$$

for some globally defined 2-form $\omega_{\alpha\beta}$ on $\mathbb{S}^m$. This is the case when $\Omega$ is a parallel $(m+1)$-form on $\mathbb{R}^{m+n}$. Using these forms $\omega_{\alpha\beta}$, the stability condition (1) is translated into the *long $\Omega$-Cauchy-Riemannian integral inequality:*

$$\sum_{\alpha<\beta} -2m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) dM \leq \sum_\alpha \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM. \tag{3}$$

If we fix $\alpha < \beta$, and set $f = f_\alpha$, $h = f_\beta$, and $f_\gamma = 0 \ \forall \gamma \neq \alpha, \beta$, (1) reduces to

$$-2m \int_{\mathbb{S}^m} f\xi_{\alpha\beta}(\nabla h) dM \leq \int_{\mathbb{S}^m} \|\nabla f\|^2 dM + \int_{\mathbb{S}^m} \|\nabla h\|^2 dM, \tag{4}$$

and if we replace $f$ by $cf$, and $h$ by $c^{-1}h$, where $c^2 = \|\nabla h\|_{L^2}/\|\nabla f\|_{L^2}$, then we obtain the corresponding equivalent *short $\Omega$-Cauchy-Riemannian integral inequality*

$$-m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f, \nabla h) dM \leq \sqrt{\int_{\mathbb{S}^m} \|\nabla f\|^2 dM} \sqrt{\int_{\mathbb{S}^m} \|\nabla h\|^2 dM}, \tag{5}$$

holding for all functions $f, h \in C^\infty(\mathbb{S}^m)$.

The $\Omega$-stability of a submanifold with calibrated extended tangent space and parallel mean curvature depends on the curvature of the ambient space and on the calibration $\Omega$ (Salavessa, 2010). It always holds on Euclidean spheres if $C_\Omega$ vanish. This last condition is equivalent to the condition (2) and $\xi \equiv 0$ ((Salavessa, 2010), Lemma 4.4). In the case $n = 2$ the later condition is satisfied, but for $n \geq 3$ the operator $C_\Omega$ may not vanish for spheres, even if $\Omega$ is parallel. If $C_\Omega$ does not vanish, spheres of calibrated vector subspaces may not be $\Omega$-stable.

We first consider $\Omega$ any parallel $(m+1)$-form on $\mathbb{R}^{m+n}$. Laplace spherical harmonics of $\mathbb{S}^m$ of degree $l$ are the eigenfunctions for the closed eigenvalue problem with respect to the Laplacian operator corresponding to the eigenvalue $\lambda_l = l(l+m-1)$, and they are just the harmonic homogeneous polynomial functions of degree $l$ of $\mathbb{R}^{m+1}$ restricted to $\mathbb{S}^m$. We denote by $E_{\lambda_l}$ the finite-dimensional subspace of $H^1(\mathbb{S}^m)$ spanned by these $\lambda_l$-eigenfunctions. In the first theorem we show how each 1-form $\xi_{\alpha\beta}$ transforms a spherical harmonic $f$ into another spherical harmonic $h$:

**Theorem 1.1** *If $\Omega$ is parallel, then for each $f \in E_{\lambda_l}$, $h = \xi_{\alpha\beta}(\nabla f)$ is also in $E_{\lambda_l}$, and it is $L^2$-orthogonal to $f$.*

In this paper we study the stability of the unit 3-sphere of a 2-dimensional complex subspace of $\mathbb{C}^3$ with respect to the Kähler calibration. In this case $C_\Omega$ does not vanish. Let $\varpi$ be the Kähler form of $\mathbb{C}^3 = \mathbb{R}^6$, and $\Omega$ the Kähler calibration of rank 4,

$$\varpi = dx^{12} + dx^{34} + dx^{56}, \qquad \Omega = \frac{1}{2}\varpi^2.$$

The unit sphere of $\mathbb{R}^4 \times \{0\}$ is immersed into $\mathbb{R}^6 = \mathbb{C}^3$, by the inclusion map $\phi = (\phi_1, \ldots, \phi_4, 0) : \mathbb{S}^3 \to \mathbb{C}^3$. We have only one of those 1-forms

$$\xi := \xi_{56} = *(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \phi^1 d\phi^2 - \phi^2 d\phi^1 + \phi^3 d\phi^4 - \phi^4 d\phi^3,$$

and $\xi = \delta\omega$, with $\omega = \frac{1}{2} * \xi = \frac{1}{2}(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2}\phi^*\varpi$. Our main theorem is the following:

**Theorem 1.2** *Three-dimensional spheres of $\mathbb{C}^2$ are $\Omega$-stable submanifolds of $\mathbb{C}^3$ with parallel mean curvature, where $\Omega = \frac{1}{2}\varpi^2$ is the Kähler calibration of rank* 4.

The Cauchy-Riemann inequality version of the $\Omega$-stability is described in the corollary:

**Corollary 1.1** *The Cauchy-Riemann inequality*

$$-\int_{\mathbb{S}^3}\varpi(\nabla f,\nabla h)dM \leq \frac{2}{3}\sqrt{\int_{\mathbb{S}^3}\|\nabla f\|^2 dM}\sqrt{\int_{\mathbb{S}^3}\|\nabla h\|^2 dM}$$

*holds for any smooth functions $f$ and $h$ of $\mathbb{S}^3$, with equality if and only if $f,h\in E_{\lambda_1}$, with $f=\sum_i\mu_i\phi_i$ and $h=\sum_i\sigma_i\phi_i$, where $\sigma_2=-\mu_1$, $\sigma_1=\mu_2$, $\sigma_4=-\mu_3$, $\sigma_3=\mu_4$.*

Finally, we state that the 3-sphere is the unique smooth closed submanifold that solves the $\Omega$-isoperimetric problem among a certain class of immersed submanifolds:

**Theorem 1.3** *The unit 3-sphere of a complex 2-dimensional subspace of $\mathbb{C}^3$ is the unique closed immersed 3-dimensional submanifold $\phi: M\to\mathbb{C}^3$ with parallel mean curvature, trivial normal bundle, and complex extended tangent space $H\oplus TM$, that is $\Omega$-stable for the Kähler calibration of rank 4, and satisfies the inequality*

$$\int_M S(2+h\|H\|)dM \leq 0,$$

*where $h$ and $S$ are the height functions $h=\langle\phi,\nu\rangle$ and $S=\sum_{ij}\langle\phi,(B(e_i,e_j))^F\rangle B^\nu(e_i,e_j)$.*

***Remark*** On a closed Kähler manifold $(M,J)$ with Kähler form $\varpi(X,Y)=g(JX,Y)$, if $f,h: M\to\mathbb{R}$ are smooth functions, then by the Cauchy-Schwarz inequality,

$$\left|\int_M\varpi(\nabla f,\nabla h)dM\right| \leq \sqrt{\int_M\|\nabla f\|^2 dM}\sqrt{\int_M\|\nabla h\|^2 dM},$$

with equality if and only if $\nabla h=\pm J\nabla f$, or equivalently $f\pm ih: M\to\mathbb{C}$ is a holomorphic function. If this is the case, then $f$ and $h$ are constant functions. On the other hand, globally defined functions, sufficiently close to holomorphic functions defined on a sufficiently large open set, are expected to satisfy an almost equality. This is not the case of $\mathbb{S}^3$, which is not a complex manifold, and somehow explains the coefficient $2/3$ in Corollary 1.1.

***Remark*** In the case of 3-spheres in $\mathbb{C}^3$ we have only one form $\xi_{\alpha\beta}$, that is, the long Cauchy-Riemann inequality is the short one. We wonder if a general proof of short Cauchy-Riemann inequalities can be allways obtained for Euclidean $m$-spheres on $\mathbb{R}^{m+n}$, by using the spectral theory of spheres, when $\Omega$ is any parallel calibration. Note that (4) is immediately satisfied for $f,h\in E_{\lambda_l}$, if $\lambda_l\geq m^2$, that is $l\geq m$, so it remains to consider the cases $l\leq m-1$. For 3-spheres we have to consider polynomial functions up to order $l=2$, while for 2-spheres we have to consider only the case $l=1$. A related remark is given in the end of section 3.

## 2. Preliminaries

We consider an oriented Riemannian manifold $M$ of dimension $m$, with Levi-Civita connection $\nabla$ and Ricci tensor $Ricci^M: TM\to TM$. In what follows $e_1,\ldots,e_m$ denotes a local direct o.n. frame.

**Lemma 2.1** *Let $\xi$ be a co-exact 1-form on a Riemannian manifold $M$, with $\xi=\delta\omega$, where $\omega$ is a 2-form. Then for any function $f\in C^2(M)$,*

$$\xi(\nabla f)=div(\nabla^\omega f),$$

*where $\nabla^\omega f=\sum_i\omega(\nabla f,e_i)e_i$ . Moreover, for any $f,h\in C_0^\infty(M)$*

$$\int_M f\xi(\nabla h)dM=\int_M\omega(\nabla f,\nabla h)dM=-\int_M h\xi(\nabla f)dM.$$

*Proof:* We may assume at a point $x_0$, $\nabla e_i=0$. Then at $x_0$

$$\begin{aligned}\xi(\nabla f) &= \delta\omega(\nabla f)=-\sum_i\nabla_{e_i}\omega(e_i,\nabla f)=\sum_i-\nabla_{e_i}(\omega(e_i,\nabla f))+\omega(e_i,\nabla_{e_i}\nabla f)\\ &= div(\nabla^\omega f)+\sum_{ij}Hessf(e_i,e_j)\omega(e_i,e_j).\end{aligned}$$

The last equality proves the first equality of the lemma, because $Hessf(e_i,e_j)$ is symmetric on $i,j$ and $\omega(e_i,e_j)$ is skew-symmetric. The other equalities of the lemma follow from $div(fX)=\langle\nabla f,X\rangle+fdiv(X)$, holding for any vector field $X$ and function $f$. □

The $\delta$ and star operators acting on $p$-forms on an oriented Riemannian $m$-manifold $M$ satisfy $\delta = (-1)^{mp+m+1} * d*$, $** = (-1)^{p(m-p)}Id$, and for a 1-form $\xi$ the DeRham Laplacian $\Delta$ and the rough Laplacian $\bar{\Delta}$ are related by the following formulas

$$\Delta\xi(X) = (d\delta + \delta d)\xi(X) = -\bar{\Delta}\xi(X) + \xi(Ricci^M(X)),$$
$$\bar{\Delta}\xi(X) = trace\nabla^2\xi(X) = \textstyle\sum_i \nabla_{e_i}\nabla_{e_i}\xi(X) - \nabla_{\nabla_{e_i}e_i}\xi(X).$$

If $\xi = \delta\omega$, then $\delta\xi = 0$, and so $\Delta\xi(X) = \delta d\xi(X) = -\sum_i \nabla_{e_i}(d\xi)(e_i, X)$. We also recall the following well-known formula (see e.g. Salavessa & Pereira do Vale (2006)) for $f \in C^\infty(M)$,

$$(\bar{\Delta}df)(X) = \sum_i \nabla^2_{e_i,e_i} df(X) = g(\nabla(\Delta f), X) + df(Ricci^M(X)).$$

Thus,

$$\begin{aligned} \bar{\Delta}(\nabla f) &= \nabla(\Delta f) + Ricci^M(\nabla f), \\ (\bar{\Delta}\xi)(\nabla f) &= -(\delta d\xi)(\nabla f) + \xi(Ricci^M(\nabla f)). \end{aligned} \tag{6}$$

Now we suppose that $M$ is an immersed oriented hypersurface of a Riemannian manifold $M'$, with Riemannian metric $\langle , \rangle$, defined by an immersion $\phi : M \to M'$ with unit normal $\nu$, second fundamental form $B$ and corresponding Weingarten operator $A$ in the $\nu$ direction, given by

$$B(e_i, e_j) = \langle A(e_i), e_j\rangle = \langle \nabla'_{e_i} e_j, \nu\rangle = -\langle e_j, \nabla'_{e_i}\nu\rangle,$$

where $\nabla'$ denotes the Levi-Civita connection on $M'$. The scalar mean curvature of $M$ is given by

$$H = \frac{1}{m} Trace\, B = \sum_i \frac{1}{m} B(e_i, e_i).$$

The curvature operator of $M'$, $R'(X, Y, Z, W) = \langle -\nabla'_X\nabla'_Y Z + \nabla'_Y\nabla'_X Z + \nabla'_{[X,Y]}Z, W\rangle$, can be seen as a self-adjoint operator of wedge bundles $R' : \wedge^2 TM' \to \wedge^2 TM'$,

$$\langle R'(u \wedge v), z \wedge w\rangle = R'(u, v, z, w),$$

and so $R'(u \wedge v) = \sum_{i<j} R'(u, v, e_i, e_j) e_i \wedge e_j$, where

$$< u \wedge v, z \wedge w >= det \begin{bmatrix} \langle u, z\rangle & \langle u, w\rangle \\ \langle v, z\rangle & \langle v, w\rangle \end{bmatrix}.$$

In what follows, we suppose that $\hat{\xi}$ is a parallel $(m-1)$-form on $M'$, and $\xi$ is given by

$$\xi = *\phi^*\hat{\xi}$$

where $*$ is the star operator on $M$. In this case $\xi$ is obviously co-closed, but not necessarily co-exact. We employ the usual inner products in $p$-forms and morphisms.

**Lemma 2.2** *Assume* $m \geq 3$. *Then for all* $i, j$

$$(\nabla_{e_i}\xi)(e_j) = \textstyle\sum_k -B(e_i, e_k)\hat{\xi}(\nu, *(e_k \wedge e_j)) = -\hat{\xi}(\nu, *(A(e_i) \wedge e_j)),$$
$$\Delta\xi(e_j) = \delta\, d\xi(e_j) = \hat{\xi}\left(\nu, *\gamma(e_j \wedge (m\nabla H - [Ricci^{M'}(\nu)]^T)\gamma) + R'(e_j \wedge \nu)\right) + \xi(\Theta_B(e_j)),$$

*where* $[Ricci^{M'}(\nu)]^T = \sum_k Ricci^{M'}(\nu, e_k)e_k$ *and* $\Theta_B : TM \to TM$ *is the morphism given by,* $\Theta_B = \|B\|^2 Id + mHA - 2A^2$.

*Proof:* We fix a point $x_0 \in M$ and take $e_i$ a local o.n. frame s.t. $\nabla e_i(x_0) = 0$. We will compute $d\xi(e_i, e_j)$, at $x$ on a neigbourhood of $x_0$. Recall that for any $p$-form $\sigma$, we have $*\sigma = \sigma*$, where the star operator on the r.h.s. can be seen as acting on $\wedge^{m-p}TM$, with $*e_i = (-1)^{i-1} e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots e_m$, and for $i < j$, $*(e_i \wedge e_j) = (-1)^{i+j-1} e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_m$. Using the fact that $\hat{\xi}$ is a parallel form on $M'$, we have for $x$ near $x_0$,

$$\begin{aligned} \nabla_{e_i}(\xi(e_j)) &= \textstyle\sum_{k\neq j}(-1)^{j-1}\hat{\xi}(e_1, \ldots, \nabla'_{e_i}e_k, \ldots, \hat{e}_j, \ldots, e_m) \\ &= \textstyle\sum_{k<j}(-1)^{k+j}\hat{\xi}(\nabla'_{e_i}e_k, e_1, \ldots, \hat{e}_k, \ldots, \hat{e}_j, \ldots, e_m) \\ &\quad + \textstyle\sum_{k>j}(-1)^{k+j-1}\hat{\xi}(\nabla'_{e_i}e_k, e_1, \ldots, \hat{e}_j, \ldots, \hat{e}_k, \ldots, e_m) \\ &= \textstyle\sum_{k<j} -\langle\nabla_{e_i}e_k, e_j\rangle\hat{\xi}(*e_k) - B(e_i, e_k)\hat{\xi}(\nu, *(e_k \wedge e_j)) \\ &\quad + \textstyle\sum_{k>j} -\langle\nabla_{e_i}e_k, e_j\rangle\hat{\xi}(*e_k) + B(e_i, e_k)\hat{\xi}(\nu, *(e_j \wedge e_k)) \\ &= \xi(\nabla_{e_i}e_j) + \textstyle\sum_{k\neq j} -B(e_i, e_k)\hat{\xi}(\nu, *(e_k \wedge e_j)). \end{aligned}$$

Hence, $(\nabla_{e_i}\xi)(e_j) = \sum_{k\neq j} -B(e_i,e_k)\hat{\xi}(\nu, *(e_k\wedge e_j))$, which proves the first sequence of equalities of the lemma. Now,

$$\begin{aligned} d\xi(e_i,e_j) &= (\nabla_{e_i}\xi)(e_j) - (\nabla_{e_j}\xi)(e_i) \\ &= \sum_{k\neq j} -B(e_i,e_k)\hat{\xi}(\nu, *(e_k\wedge e_j)) + \sum_{k\neq i} B(e_j,e_k)\hat{\xi}(\nu, *(e_k\wedge e_i)), \end{aligned}$$

and by Codazzi's equation,

$$\begin{aligned} &(\nabla_{e_i}B)(e_j,e_k) = (\nabla_{e_j}B)(e_i,e_k) - R'(e_i,e_j,e_k,\nu) \\ &\textstyle\sum_i (\nabla_{e_i}B)(e_i,e_k) = m\nabla_{e_k}H - Ricci^{M'}(e_k,\nu). \end{aligned}$$

Note that $B_{ik} = (\nabla_{e_j}B)(e_i,e_k)$ is a symmetric matrix, and if we define $A_{ki} = \hat{\xi}(\nu, *(e_k\wedge e_i))$ (valuing zero if $k=i$), then $A_{ik}$ is skew-symmetric. Thus, $\sum_{k\neq i} B_{ik}A_{ki} = \sum_{k,i} B_{ik}A_{ki} = 0$. Furthermore, if we set $C_{ik} = -R'(e_i,e_j,e_k,\nu)$, then $C_{ik} - C_{ki} = R'(e_k,e_i,e_j,\nu)$. Hence,

$$\sum_i\sum_{k\neq i} C_{ik}A_{ki} = \sum_{ik} C_{ik}A_{ki} = \sum_{ik}\frac{1}{2}((C_{ik}+C_{ki}) + (C_{ik}-C_{ki}))A_{ki} = \sum_{ki}\frac{1}{2}R'(e_k,e_i,e_j,\nu)A_{ki}.$$

Therefore, for each $j$, at $x_0$

$$\begin{aligned} -\delta d\xi(e_j) &= \sum_i \nabla_{e_i}(d\xi(e_i,e_j)) \\ &= \sum_{k\neq j}\sum_i -(\nabla_{e_i}B)(e_i,e_k)\hat{\xi}(\nu, *(e_k\wedge e_j)) - B(e_i,e_k)\nabla_{e_i}(\hat{\xi}(\nu, *(e_k\wedge e_j))) \\ &\quad + \sum_{k\neq i}\sum_j (\nabla_{e_i}B)(e_j,e_k)\hat{\xi}(\nu, *e_k\wedge e_i)) + B(e_j,e_k)\nabla_{e_i}(\hat{\xi}(\nu, *(e_k\wedge e_i)) \\ &= \sum_{k\neq j}(-m\nabla_{e_k}H + Ricci^{M'}(e_k,\nu))\hat{\xi}(\nu, *(e_k\wedge e_j)) + \sum_{k,i}\frac{1}{2}R'(e_k,e_i,e_j,\nu)\hat{\xi}(\nu, *(e_k\wedge e_i)) + S \end{aligned}$$

where

$$\begin{aligned} S &= \textstyle\sum_i\sum_{k<j}(-1)^{k+j}B(e_i,e_k)\hat{\xi}(\nabla'_{e_i}\nu, e_1,\ldots,\hat{e}_k,\ldots,\hat{e}_j,\ldots,e_m) \\ &\quad \textstyle+ \sum_i\sum_{k>j}(-1)^{k+j-1}B(e_i,e_k)\hat{\xi}(\nabla'_{e_i}\nu, e_1,\ldots,\hat{e}_j,\ldots,\hat{e}_k,\ldots,e_m) \\ &\quad \textstyle+ \sum_i\sum_{k<i}(-1)^{k+i-1}B(e_j,e_k)\hat{\xi}(\nabla'_{e_i}\nu, e_1,\ldots,\hat{e}_k,\ldots,\hat{e}_i,\ldots,e_m) \\ &\quad \textstyle+ \sum_i\sum_{k>i}(-1)^{k+i}B(e_j,e_k)\hat{\xi}(\nabla'_{e_i}\nu, e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_k,\ldots,e_m) \\ &= \textstyle\sum_i\sum_{k<j} -B(e_i,e_k)B(e_i,e_k)\xi(e_j) + B(e_i,e_j)B(e_i,e_k)\xi(e_k) \\ &\quad \textstyle+ \sum_i\sum_{k>j} B(e_i,e_j)B(e_i,e_k)\xi(e_k) - B(e_i,e_k)B(e_i,e_k)\xi(e_j) \\ &\quad \textstyle+ \sum_i\sum_{k<i} B(e_i,e_k)B(e_j,e_k)\xi(e_i) - B(e_i,e_i)B(e_j,e_k)\xi(e_k) \\ &\quad \textstyle+ \sum_i\sum_{k>i} -B(e_i,e_i)B(e_j,e_k)\xi(e_k) + B(e_i,e_k)B(e_j,e_k)\xi(e_i). \end{aligned}$$

At this point we may assume that at $x_0$ the basis $e_i$ diagonalizes the second fundamental form, that is, $B(e_i,e_j) = \lambda_i\delta_{ij}$. Then,

$$\begin{aligned} S &= \textstyle\sum_i\sum_{k<j} -\delta_{ik}\lambda_i^2\xi(e_j) + \delta_{ij}\delta_{ik}\lambda_i^2\xi(e_k) + \sum_i\sum_{k>j}\delta_{ij}\delta_{ik}\lambda_i^2\xi(e_k) - \delta_{ik}\lambda_i^2\xi(e_j) \\ &\quad \textstyle+ \sum_i\sum_{k<i}\delta_{ik}\delta_{jk}\lambda_k^2\xi(e_i) - \delta_{ii}\delta_{jk}\lambda_i\lambda_j\xi(e_k) + \sum_i\sum_{k>i} -\delta_{ii}\delta_{jk}\lambda_i\lambda_j\xi(e_k) + \delta_{ik}\delta_{jk}\lambda_k^2\xi(e_i) \\ &= \textstyle\sum_{i<j} -\lambda_i^2\xi(e_j) + \sum_{i>j} -\lambda_i^2\xi(e_j) + \sum_{j<i} -\lambda_i\lambda_j\xi(e_j) + \sum_{j>i} -\lambda_i\lambda_j\xi(e_j) \\ &= \textstyle\sum_{i\neq j} -\lambda_i^2\xi(e_j) - \lambda_i\lambda_j\xi(e_j) = \sum_i -\lambda_i^2\xi(e_j) - \lambda_i\lambda_j\xi(e_j) + (\lambda_j^2+\lambda_j^2)\xi(e_j) \\ &= -\|B\|^2\xi(e_j) - mH\xi(A(e_j)) + 2\xi(A^2(e_j)), \end{aligned}$$

and the second sequence of equalities of the lemma is proved. □

If we suppose that $\Theta_B = \mu(x)Id$, taking $e_i$ a diagonalizing o.n. basis of the second fundamental form, $B(e_i,e_j) = \lambda_i\delta_{ij}$, then each $\lambda_i$ satisfies the quadratic equation

$$2\lambda_i^2 - mH\lambda_i + (\mu - \|B\|^2) = 0,$$

which implies that we have at most two distinct possible principal curvatures $\lambda_\pm$. Moreover, from the above equation, summing over $i$, we derive that $\mu(x)$ must satisfy $\mu(x) = \frac{m-2}{m}\|B\|^2 + mH^2$, and so

$$\lambda_\pm = \frac{1}{4}\left(mH \pm \sqrt{\frac{16}{m}\|B\|^2 + m(m-8)H^2}\right).$$

Note that, from $\|B\|^2 \geq m\|H\|^2$, we have $\frac{16}{m}\|B\|^2 + m(m-8)H^2 \geq (m-4)^2H^2$, and so there are one or two distinct principal curvatures. If $M$ is totally umbilical, then $\|B\|^2 = mH^2$ and $\mu = 2(m-1)\|H\|^2$. The previous lemma leads to the following conclusion:

**Lemma 2.3** *Assuming $M' = \mathbb{R}^{m+1}$, $m \geq 3$, and taking $M$ a hypersurface with constant mean curvature, with $\Theta_B = \mu(x)Id$, where $\mu(x)$ is a smooth function on $M$, we get $\mu(x) = \frac{m-2}{m}\|B\|^2 + mH^2$ and*

$$\Delta\xi = \mu\xi.$$

*Furthermore, $\xi$ is an eigenform for the DeRham Laplacian operator, that is $\mu(x)$ is constant, if and only if $\|B\|$ is constant.*

*In case $M$ is a unit m-sphere $\mathbb{S}^m$, then $\Theta_B = \mu Id$, with $\mu = 2(m-1)$, and taking $\nu_x = -x$ as unit normal, then, at each $x \in \mathbb{S}^m$,*

$$\begin{aligned}(\nabla_{e_i}\xi)(e_j) &= \hat{\xi}(x, *(e_i \wedge e_j))\\ d\xi(e_i, e_j) &= 2\hat{\xi}(x, *(e_i \wedge e_j))\\ \Delta\xi &= \delta d\xi = 2(m-1)\xi.\end{aligned}$$

**Lemma 2.4** *If $f \in C^\infty(\mathbb{S}^m)$, then $\Delta(\xi(\nabla f)) = \xi(\nabla\Delta f)$.*

*Proof:* We fix a point $x_0 \in \mathbb{S}^m$ and take $e_i$ a local o.n. frame of the sphere s.t. $\nabla e_i(x_0) = 0$. Let $f \in C^\infty(\mathbb{S}^m)$. The following computations are at $x_0$. Using the above formulas (6) and previous lemma, we have

$$\begin{aligned}\Delta(\xi(\nabla f)) &= \sum_i \nabla_{e_i}(\nabla_{e_i}(\xi(\nabla f))) = \sum_i \nabla_{e_i}\big((\nabla_{e_i}\xi)(\nabla f) + \xi(\nabla_{e_i}\nabla f)\big)\\ &= (\bar{\Delta}\xi)(\nabla f) + 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f) + \xi(\nabla_{e_i}\nabla_{e_i}\nabla f)\\ &= -2(m-1)\xi(\nabla f) + \xi(\nabla\Delta f) + 2(m-1)\xi(\nabla f) + \sum_i 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f).\end{aligned}$$

Since $Hess\, f(e_i, e_j)$ is symmetric in $ij$ and by Lemma 2.3, $(\nabla_{e_i}\xi)(e_j)$ is skew-symmetric, we have

$$\sum_i (\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f) = \sum_{ij} Hess\, f(e_i, e_j)(\nabla_{e_i}\xi)(e_j) = 0,$$

and the lemma is proved. □

## 3. Proof of Theorem 1.1

We denote by $\nabla$ the Levi-Civita connection of $\mathbb{S}^m$ induced by the flat connection $\bar{\nabla}$ of $\mathbb{R}^{m+n}$. We are considering a parallel calibration $\Omega$ on $\mathbb{R}^{m+n}$. We fix $\alpha < \beta$ and define the 1-form on $\mathbb{S}^m$

$$\xi = \xi(W_\alpha, W_\beta) = *\phi^*\hat{\xi} = \delta\omega,$$

where $\hat{\xi} = \hat{\xi}_{\alpha\beta}$ and $\omega = \omega_{\alpha\beta}$.

We recall that the eigenvalues of $\mathbb{S}^m$ for the closed Dirichlet problem are given by $\lambda_l = l(l+m-1)$, with $l = 0, 1, 2 \ldots$. We denote by $E_{\lambda_l}$ the eigenspace of dimension $m_l$ corresponding to the eigenvalue $\lambda_l$, and by $E^+_{\lambda_l}$ the $L^2$-orthogonal complement of the sum of the eigenspaces $E_{\lambda_i}$, $i = 1, \ldots, l-1$, and so it is the sum of all eigenspaces $E_\lambda$ with $\lambda \geq \lambda_l$. If $f \in E_{\lambda_l}$, and $h \in E_{\lambda_s}$, then

$$\int_{\mathbb{S}^m} fh\, dM = 0 \ \text{ if } l \neq s \quad \text{and} \quad \int_{\mathbb{S}^m} \langle\nabla f, \nabla h\rangle\, dM = \delta_{ls}\lambda_l \int_{\mathbb{S}^m} fh\, dM.$$

There exists an $L^2$-orthonormal basis $\psi_{l,\sigma}$ of $L^2(\mathbb{S}^m)$ of eigenfunctions ($1 \leq \sigma \leq m_l$). The Rayleigh characterization of $\lambda_l$ is given by

$$\lambda_l = \inf_{f \in E^+_{\lambda_l}} \frac{\int_{\mathbb{S}^m} \|\nabla f\|^2 dM}{\int_{\mathbb{S}^m} f^2 dM},$$

and the infimum is attained for $f \in E_{\lambda_l}$. Each eigenspace $E_{\lambda_l}$ is exactly composed by the restriction to $\mathbb{S}^m$ of the harmonic homogeneous polynomial functions of degree $l$ of $\mathbb{R}^{m+1}$, and it has dimension $m_l = \binom{m+l}{m} - \binom{m+l-2}{m}$. Thus, each eigenfunction $\psi \in E_{\lambda_l}$ is of the form $\psi = \sum_{|a|=l} \mu_a \phi^a$, where $\mu_a$ are some scalars and $a = (a_1, \ldots, a_{m+1})$ denotes a multi-index of length $|a| = a_1 + \ldots + a_{m+1} = l$ and

$$\phi^a = \phi_1^{a_1} \cdot \ldots \cdot \phi_{m+1}^{a_{m+1}}.$$

From $\nabla\phi_i = \epsilon_i^\top$ and $\sum_i \phi_i^2 = 1$, we see that

$$\begin{cases} \langle \nabla\phi_i, \nabla\phi_j\rangle = \delta_{ij} - \phi_i\phi_j & \|\nabla\phi_i\|^2 = 1 - \phi_i^2 \\ \int_{\mathbb{S}^m} \phi_i^2 dM = \frac{1}{m+1}|\mathbb{S}^m| & \int_{\mathbb{S}^m} \|\nabla\phi_i\|^2 dM = \lambda_1 \int_{\mathbb{S}^2} \phi_i^2 dM = \frac{m}{m+1}|\mathbb{S}^m|. \end{cases} \tag{7}$$

We also denote by $\int_{\mathbb{S}^m} \phi^2 dM$ any of the integrals $\int_{\mathbb{S}^m} \phi_i^2 dM$, $i = 1, \ldots, m+1$. We recall the following:

**Lemma 3.1** *If $P : \mathbb{S}^m \to \mathbb{R}$ is a homogeneous polynomial function of degree $l$, then*

$$\int_{\mathbb{S}^m} P(x)dM = \frac{1}{\lambda_l}\int_{\mathbb{S}^m} \Delta^0 P(x)dM.$$

*In particular,*

$$\int_{\mathbb{S}^m} \phi^a dM = \sum_{1\le i\le m+1} \frac{a_i(a_i-1)}{l(l+m-1)} \int_{\mathbb{S}^m} \phi^{a-2\epsilon_i} dM,$$

*where the terms $a_i < 2$ are considered to vanish. Thus, if some $a_i$ is odd this integral vanishes.*

**Proof of Theorem 1.1** By Lemma 2.4, if $f \in E_{\lambda_k}$ then $\xi(\nabla f) \in E_{\lambda_k}$. From

$$\int_{\mathbb{S}^m} f\xi(\nabla f)dM = \int_{\mathbb{S}^m} \omega(\nabla f, \nabla f)dM = 0$$

we conclude that $f$ and $h = \xi(\nabla f)$ are $L^2$-orthogonal. □

***Remark*** Let us consider $f, h \in E_{\lambda_l}$, and take the globally defined vector field of $\mathbb{S}^m$, $\xi^\sharp = \sum_j \xi(e_j)e_j$. From Lemma 2.2, we have

$$\langle \nabla h, \nabla(\xi(\nabla f))\rangle = -\hat{\xi}(\nu, *(\nabla h \wedge \nabla f)) + Hessf(\nabla h, \xi^\sharp).$$

By Theorem 1.1, $\xi(\nabla f) \in E_{\lambda_l}$ as well. The term $Hessf(\nabla h, \xi^\sharp)$ is a sum of polynomial functions of degree $2l-3+k_\xi$ where $k_\xi$ depends on $\xi^\sharp$, when expressed in terms of $\phi^i$. Let us suppose that all $k_\xi$ are even. Then by Lemma 3.1, $\int_{\mathbb{S}^m} Hessf(\nabla h, \xi^\sharp)dM = 0$. Since $\lambda_l \ge m$, and taking into consideration that $\Omega$ is a semi-calibration,

$$\begin{aligned} -\int_{\mathbb{S}^m} h\xi(\nabla f)dM &= -\frac{1}{\lambda_l}\int_{\mathbb{S}^m}\langle \nabla h, \nabla(\xi(\nabla f))\rangle dM \\ &= \frac{1}{\lambda_l}\int_{\mathbb{S}^m} \hat{\xi}(\nu, *(\nabla h \wedge \nabla f))dM \le \frac{1}{\lambda_l}\int_{\mathbb{S}^m} \|\nabla h\|\,\|\nabla f\| dM \le \frac{1}{m}\|\nabla f\|_{L^2}\|\nabla h\|_{L^2}. \end{aligned}$$

Thus, in this case the short Cauchy-Riemann inequality holds. Inspection of $\xi$ must be required for each case of $\Omega$. A general proof of the short Cauchy-Riemann integral inequality, under appropriate conditions on $\Omega$, will be developed in a future paper.

## 4. 3-Spheres of $\mathbb{C}^2$ in $\mathbb{C}^3$

In this section we specialize the Cauchy-Riemann inequalities for the case $m = n = 3$ and for $\mathbb{R}^6 = \mathbb{C}^3$ we will consider the Kähler calibration $\frac{1}{2}\varpi^2$ that calibrates the complex two-dimensional subspaces, that is,

$$\Omega = dx^{1234} + dx^{1256} + dx^{3456}.$$

Thus, fixing $W_5 = \epsilon_5$ and $W_6 = \epsilon_6$ we have $\hat{\xi} := \hat{\xi}_{56} = dx^{12} + dx^{34}$, and

$$\xi := \xi_{56} = *\phi^*\hat{\xi} = *(d\phi^{12} + d\phi^{34}).$$

The volume element of $\mathbb{S}^m$ is $Vol_{S^m} = \sum_i (-1)^{i-1}\phi_i d\phi^{1\ldots\hat{i}\ldots m}$, and $*\xi$ is the unique 2-form s.t. $\xi \wedge *\xi = \|\xi\|^2 Vol_{S^m}$. Using (7) we see that $\|\xi\| = \|*\xi\| = 1$. Hence

$$\begin{aligned} &\xi = \phi_1 d\phi^2 - \phi_2 d\phi^1 + \phi_3 d\phi^4 - \phi_4 d\phi^3 \\ &*\xi = d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4 = \tfrac{1}{2}d\xi =: d*\omega. \end{aligned}$$

Therefore, we may take $*\omega = \frac{1}{2}\xi$, that is

$$\omega = \frac{1}{2}*\xi = \frac{1}{2}(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2}\phi^*\varpi.$$

Hence, to prove Theorem 1.2 and Corollary 1.1 we have to verify that, for any functions $f, h \in C^\infty(\mathbb{S}^3)$, one of the following equivalent inequalities holds:

$$\int_{\mathbb{S}^3} -3\omega(\nabla f, \nabla h)dM = \int_{\mathbb{S}^3} -3f\xi(\nabla h)dM \le \|\nabla f\|_{L^2}\|\nabla h\|_{L^2} \tag{8}$$
$$\int_{\mathbb{S}^3} -6\omega(\nabla f, \nabla h)dM = \int_{\mathbb{S}^3} -6f\xi(\nabla h)dM \le \|\nabla f\|^2_{L^2} + \|\nabla h\|^2_{L^2}.$$

By Theorem 1.1 we only need to consider both $f, h \in E_{\lambda_l}$, for some $l$. Note that $\lambda_3 = 15$ and since $\Omega$ is a calibration, $\|\xi(X)\| \le \|X\|$.

**Lemma 4.1** *If $f, h \in E^+_{\lambda_3}$ are nonzero, (8) holds, with strict inequality.*

*Proof:* By Schwartz inequality and Rayleigh characterization

$$\int_{\mathbb{S}^3} -3f\xi(\nabla h)dM \le 3\|f\|_{L^2}\|\nabla h\|_{L^2} \le \frac{3}{\sqrt{\lambda_3}}\|\nabla f\|_{L^2}\|\nabla h\|_{L^2} < \|\nabla f\|_{L^2}\|\nabla h\|_{L^2},$$

with strict inequality in the last one, since neither $f$ nor $h$ may be constant. □

We now verify that (8) holds for $f, h \in E_{\lambda_1}$ and $f, h \in E_{\lambda_2}$. From (7) and Lemma 3.1, we have for $i \neq j$

$$\begin{array}{ll}
\int_{\mathbb{S}^3} \phi^2 dM = \frac{1}{4}|\mathbb{S}^3|, & \int_{\mathbb{S}^3} \phi_i^2\phi_j^2 dM = \frac{1}{6}\int_{\mathbb{S}^3}\phi^2 dM \\
\int_{\mathbb{S}^3} \phi^4 dM = \frac{1}{2}\int_{\mathbb{S}^3}\phi^2 dM, & \int_{\mathbb{S}^3} \|\nabla\phi\|^2 dM = 3\int_{\mathbb{S}^3}\phi^2 dM \\
\omega(\nabla\phi_1, \nabla\phi_2) = \frac{1}{2}(1-\phi_1^2-\phi_2^2) & \omega(\nabla\phi_1, \nabla\phi_3) = \frac{1}{2}(-\phi_2\phi_3+\phi_1\phi_4) \\
\omega(\nabla\phi_1, \nabla\phi_4) = \frac{1}{2}(-\phi_2\phi_4-\phi_1\phi_3) & \omega(\nabla\phi_2, \nabla\phi_3) = \frac{1}{2}(\phi_1\phi_3+\phi_4\phi_2) \\
\omega(\nabla\phi_2, \nabla\phi_4) = \frac{1}{2}(\phi_1\phi_4-\phi_2\phi_3) & \omega(\nabla\phi_3, \nabla\phi_4) = \frac{1}{2}(1-\phi_3^2-\phi_4^2).
\end{array} \tag{9}$$

and moreover

**Lemma 4.2**

$$\begin{array}{l}
3\int\omega(\nabla\phi_1, \nabla\phi_2) = 3\int\phi^2 = \|\nabla\phi_1\|_{L^2}\|\nabla\phi_2\|_{L^2} = \|\nabla\phi\|^2_{L^2} \\
3\int\omega(\nabla\phi_3, \nabla\phi_4) = 3\int\phi^2 = \|\nabla\phi_3\|_{L^2}\|\nabla\phi_4\|_{L^2} = \|\nabla\phi\|^2_{L^2} \\
-3\int\omega(\nabla\phi_i, \nabla\phi_j) = 0 \quad \text{for other } ij \\
-3\int\phi_k\omega(\nabla\phi_i, \nabla\phi_j) = 0 \quad \forall i, j, k \\
-3\int\phi_1^2\omega(\nabla\phi_1, \nabla\phi_2) = -3\int\phi_2^2\omega(\nabla\phi_1, \nabla\phi_2) = -\frac{1}{2}\int\phi^2 \\
-3\int\phi_3^2\omega(\nabla\phi_1, \nabla\phi_2) = -3\int\phi_4^2\omega(\nabla\phi_1, \nabla\phi_2) = -\int\phi^2 \\
-3\int\phi_1^2\omega(\nabla\phi_3, \nabla\phi_4) = -3\int\phi_2^2\omega(\nabla\phi_3, \nabla\phi_4) = -\int\phi^2 \\
-3\int\phi_3^2\omega(\nabla\phi_3, \nabla\phi_4) = -3\int\phi_4^2\omega(\nabla\phi_3, \nabla\phi_4) = -\frac{1}{2}\int\phi^2 \\
-3\int\phi_1\phi_4\omega(\nabla\phi_1, \nabla\phi_3) = -3\int\phi_1\phi_3\omega(\nabla\phi_2, \nabla\phi_3) = -\frac{1}{4}\int\phi^2 \\
-3\int\phi_1\phi_3\omega(\nabla\phi_1, \nabla\phi_4) = -3\int\phi_2\phi_3\omega(\nabla\phi_2, \nabla\phi_4) = \frac{1}{4}\int\phi^2 \\
-3\int\phi_2\phi_3\omega(\nabla\phi_1, \nabla\phi_3) = -3\int\phi_2\phi_4\omega(\nabla\phi_1, \nabla\phi_4) = \frac{1}{4}\int\phi^2 \\
-3\int\phi_2\phi_4\omega(\nabla\phi_2, \nabla\phi_3) = -3\int\phi_1\phi_4\omega(\nabla\phi_2, \nabla\phi_4) = -\frac{1}{4}\int\phi^2 \\
-3\int\phi_i\phi_j\omega(\nabla\phi_k, \nabla\phi_s) = 0 \quad \text{for other cases.}
\end{array}$$

**Lemma 4.3** *If $f, h \in E_{\lambda_1}$, that is $f = \sum_i \mu_i\phi_i$, $h = \sum_j \sigma_j\phi_j$, for some constant $\mu_i, \sigma_j$, then (8) holds, with equality if and only if $\sigma_2 = -\mu_1$, $\sigma_1 = \mu_2$, $\sigma_4 = -\mu_3$, $\sigma_3 = \mu_4$.*

*Proof:* Using the previous lemma,

$$\begin{aligned}
-3\int\omega(\nabla f, \nabla h)dM &= (\mu_1\sigma_2 - \mu_2\sigma_1)\int -3\omega(\nabla\phi_1, \nabla\phi_2) + (\mu_3\sigma_4 - \mu_4\sigma_3)\int -3\omega(\nabla\phi_3, \nabla\phi_4) \\
&= -(\mu_1\sigma_2 - \mu_2\sigma_1 + \mu_3\sigma_4 - \mu_4\sigma_3)\|\nabla\phi\|^2_{L^2} \\
&\le \frac{1}{2}\Big(\sum_i \mu_i^2 + \sigma_i^2\Big)\|\nabla\phi\|^2_{L^2} = \frac{1}{2}(\|\nabla f\|^2_{L^2} + \|\nabla h\|^2_{L^2}).
\end{aligned}$$

The equality case follows immediately. □

**Lemma 4.4** *If $f, h \in E_{\lambda_2}$ are nonzero, then (8) holds with strict inequality.*

*Proof:* Set $f = \sum_i \alpha_i\phi_i^2 + \sum_{i<j} A_{ij}\phi_i\phi_j$, and $h = \sum_i \beta_i\phi_i^2 + \sum_{i<j} B_{ij}\phi_i\phi_j$, where $\alpha_i, A_{ij}, \beta_i, B_{ij}$ are constants. Now we compute

$$\begin{aligned}
-3\int\omega(\nabla f,\nabla h) &= -3\int\omega(\nabla\phi_1,\nabla\phi_2)[(2\alpha_1\phi_1 + A_{12}\phi_2 + A_{13}\phi_3 + A_{14}\phi_4)(2\beta_2\phi_2 + B_{12}\phi_1 + B_{23}\phi_3 + B_{24}\phi_4)\\
&\quad -(2\alpha_2\phi_2 + A_{12}\phi_1 + A_{23}\phi_3 + A_{24}\phi_4)(2\beta_1\phi_1 + B_{12}\phi_2 + B_{13}\phi_3 + B_{14}\phi_4)]\\
&\quad -3\int\omega(\nabla\phi_1,\nabla\phi_3)[(2\alpha_1\phi_1 + A_{12}\phi_2 + A_{13}\phi_3 + A_{14}\phi_4)(2\beta_3\phi_3 + B_{13}\phi_1 + B_{23}\phi_2 + B_{34}\phi_4)\\
&\quad -(2\alpha_3\phi_3 + A_{13}\phi_1 + A_{23}\phi_2 + A_{34}\phi_4)(2\beta_1\phi_1 + B_{12}\phi_2 + B_{13}\phi_3 + B_{14}\phi_4)]\\
&\quad -3\int\omega(\nabla\phi_1,\nabla\phi_4)[(2\alpha_1\phi_1 + A_{12}\phi_2 + A_{13}\phi_3 + A_{14}\phi_4)(2\beta_4\phi_4 + B_{14}\phi_1 + B_{24}\phi_2 + B_{34}\phi_3)\\
&\quad -(2\alpha_4\phi_4 + A_{14}\phi_1 + A_{24}\phi_2 + A_{34}\phi_3)(2\beta_1\phi_1 + B_{12}\phi_2 + B_{13}\phi_3 + B_{14}\phi_4)]\\
&\quad -3\int\omega(\nabla\phi_2,\nabla\phi_3)[(2\alpha_2\phi_2 + A_{12}\phi_1 + A_{23}\phi_3 + A_{24}\phi_4)(2\beta_3\phi_3 + B_{13}\phi_1 + B_{23}\phi_2 + B_{34}\phi_4)\\
&\quad -(2\alpha_3\phi_3 + A_{13}\phi_1 + A_{23}\phi_2 + A_{34}\phi_4)(2\beta_2\phi_2 + B_{12}\phi_1 + B_{24}\phi_4 + B_{23}\phi_3)]\\
&\quad -3\int\omega(\nabla\phi_2,\nabla\phi_4)[(2\alpha_2\phi_2 + A_{12}\phi_1 + A_{23}\phi_3 + A_{24}\phi_4)(2\beta_4\phi_4 + B_{14}\phi_1 + B_{24}\phi_2 + B_{34}\phi_3)\\
&\quad -(2\alpha_4\phi_4 + A_{14}\phi_1 + A_{24}\phi_2 + A_{34}\phi_3)(2\beta_2\phi_2 + B_{12}\phi_1 + B_{24}\phi_4 + B_{23}\phi_3)]\\
&\quad -3\int\omega(\nabla\phi_3,\nabla\phi_4)[(2\alpha_3\phi_3 + A_{13}\phi_1 + A_{23}\phi_2 + A_{34}\phi_4)(2\beta_4\phi_4 + B_{14}\phi_1 + B_{24}\phi_2 + B_{34}\phi_3)\\
&\quad -(2\alpha_4\phi_4 + A_{14}\phi_1 + A_{24}\phi_2 + A_{34}\phi_3)(2\beta_3\phi_3 + B_{13}\phi_1 + B_{23}\phi_2 + B_{34}\phi_4)].
\end{aligned}$$

Thus, using Lemma 4.2,

$$\begin{aligned}
-3\int\omega(\nabla f,\nabla h) = &-3\int\omega(\nabla\phi_1,\nabla\phi_2)\quad [2\alpha_1B_{12}\phi_1^2 + 2\beta_2A_{12}\phi_2^2 + A_{13}B_{23}\phi_3^2 + A_{14}B_{24}\phi_4^2\\
&\qquad -2\beta_1A_{12}\phi_1^2 - 2\alpha_2B_{12}\phi_2^2 - A_{23}B_{13}\phi_3^2 - A_{24}B_{14}\phi_4^2]\\
&-3\int\omega(\nabla\phi_3,\nabla\phi_4)\quad [A_{13}B_{14}\phi_1^2 + A_{23}B_{24}\phi_2^2 + 2\alpha_3B_{34}\phi_3^2 + 2\beta_4A_{34}\phi_4^2\\
&\qquad -A_{14}B_{13}\phi_1^2 - A_{24}B_{23}\phi_2^2 - 2\beta_3A_{34}\phi_3^2 - 2\alpha_4B_{34}\phi_4^2]\\
&-3\int\omega(\nabla\phi_1,\nabla\phi_3)\quad [2\alpha_1B_{34}\phi_1\phi_4 + A_{14}B_{13}\phi_1\phi_4 - A_{13}B_{14}\phi_1\phi_4 - 2\beta_1A_{34}\phi_1\phi_4\\
&\qquad +2\beta_3A_{12}\phi_2\phi_3 + A_{13}B_{23}\phi_2\phi_3 - A_{23}B_{13}\phi_2\phi_3 - 2\alpha_3B_{12}\phi_2\phi_3]\\
&-3\int\omega(\nabla\phi_1,\nabla\phi_4)\quad [2\alpha_1B_{34}\phi_1\phi_3 + A_{13}B_{14}\phi_1\phi_3 - A_{14}B_{13}\phi_1\phi_3 - 2\beta_1A_{34}\phi_1\phi_3\\
&\qquad +2\beta_4A_{12}\phi_2\phi_4 + A_{14}B_{24}\phi_2\phi_4 - A_{24}B_{14}\phi_2\phi_4 - 2\alpha_4B_{12}\phi_2\phi_4]\\
&-3\int\omega(\nabla\phi_2,\nabla\phi_3)\quad [2\beta_3A_{12}\phi_1\phi_3 + A_{23}B_{13}\phi_1\phi_3 - A_{13}B_{23}\phi_1\phi_3 - 2\alpha_3B_{12}\phi_1\phi_3\\
&\qquad +2\alpha_2B_{34}\phi_2\phi_4 + A_{24}B_{23}\phi_2\phi_4 - A_{23}B_{24}\phi_2\phi_4 - 2\beta_2A_{34}\phi_2\phi_4]\\
&-3\int\omega(\nabla\phi_2,\nabla\phi_4)\quad [2\beta_4A_{12}\phi_1\phi_4 + A_{24}B_{14}\phi_1\phi_4 - A_{14}B_{24}\phi_1\phi_4 - 2\alpha_4B_{12}\phi_1\phi_4\\
&\qquad +2\alpha_2B_{34}\phi_2\phi_3 + A_{23}B_{24}\phi_2\phi_3 - A_{24}B_{23}\phi_2\phi_3 - 2\beta_2A_{34}\phi_2\phi_3]
\end{aligned}$$

$$\begin{aligned}
= \int\phi^2 \quad \{ & -\tfrac{1}{2}[2\alpha_1B_{12} + 2\beta_2A_{12} - 2\beta_1A_{12} - 2\alpha_2B_{12} + 2\alpha_3B_{34} + 2\beta_4A_{34} - 2\beta_3A_{34} - 2\alpha_4B_{34}]\\
& -[A_{13}B_{23} + A_{14}B_{24} - A_{23}B_{13} - A_{24}B_{14} + A_{13}B_{14} + A_{23}B_{24} - A_{14}B_{13} - A_{24}B_{23}]\\
& +\tfrac{1}{4}\big[ - 2\alpha_1B_{34} - A_{14}B_{13} + A_{13}B_{14} + 2\beta_1A_{34} + 2\beta_3A_{12} + A_{13}B_{23} - A_{23}B_{13} - 2\alpha_3B_{12}\\
&\qquad +2\alpha_1B_{34} + A_{13}B_{14} - A_{14}B_{13} - 2\beta_1A_{34} + 2\beta_4A_{12} + A_{14}B_{24} - A_{24}B_{14} - 2\alpha_4B_{12}\\
&\qquad -2\beta_3A_{12} - A_{23}B_{13} + A_{13}B_{23} + 2\alpha_3B_{12} - 2\alpha_2B_{34} - A_{24}B_{23} + A_{23}B_{24} + 2\beta_2A_{34}\\
&\qquad -2\beta_41A_{12} - A_{24}B_{14} + A_{14}B_{24} + 2\alpha_4B_{12} + 2\alpha_2B_{34} + A_{23}B_{24} - A_{24}B_{23} - 2\beta_2A_{34}\big] \quad \}\\
= \int\phi^2 \quad \{ & -[\alpha_1B_{12} + \beta_2A_{12} - \beta_1A_{12} - \alpha_2B_{12} + \alpha_3B_{34} + \beta_4A_{34} - \beta_3A_{34} - \alpha_4B_{34}]\\
& -[A_{13}B_{23} + A_{14}B_{24} - A_{23}B_{13} - A_{24}B_{14} + A_{13}B_{14} + A_{23}B_{24} - A_{14}B_{13} - A_{24}B_{23}]\\
& +\tfrac{1}{2}\big[ - A_{14}B_{13} + A_{13}B_{14} + A_{13}B_{23} - A_{23}B_{13} + A_{14}B_{24} - A_{24}B_{14} - A_{24}B_{23} + A_{23}B_{24}\big] \quad \}\\
= \int\phi^2 \quad \{ & [-\alpha_1B_{12} - \beta_2A_{12} + \beta_1A_{12} + \alpha_2B_{12} - \alpha_3B_{34} - \beta_4A_{34} + \beta_3A_{34} + \alpha_4B_{34}]\\
& +\tfrac{1}{2}[-A_{13}B_{23} - A_{14}B_{24} + A_{23}B_{13} + A_{24}B_{14} - A_{13}B_{14} - A_{23}B_{24} + A_{14}B_{13} + A_{24}B_{23}] \quad \}
\end{aligned}$$

and applying the same lemmas we see that

$$\|\nabla f\|_{L^2}^2 = \left[ 2(\sum_k \alpha_k^2) - \frac{4}{3}(\sum_{i<j} \alpha_i\alpha_j) + \frac{4}{3}(\sum_{i<j} A_{ij}^2) \right] \int \phi^2.$$

Hence, we have to verify if the following inequality is true:

$$[-\alpha_1 B_{12} - \beta_2 A_{12} + \beta_1 A_{12} + \alpha_2 B_{12} - \alpha_3 B_{34} - \beta_4 A_{34} + \beta_3 A_{34} + \alpha_4 B_{34}] \tag{10}$$

$$+\frac{1}{2}[-A_{13}B_{23} - A_{14}B_{24} + A_{23}B_{13} + A_{24}B_{14} - A_{13}B_{14} - A_{23}B_{24} + A_{14}B_{13} + A_{24}B_{23}] \tag{11}$$

$$+\frac{2}{3}(\sum_{i<j} \alpha_i\alpha_j + \beta_i\beta_j) \tag{12}$$

$$\leq \sum_k (\alpha_k^2 + \beta_k^2) + \frac{2}{3}(\sum_{i<j} A_{ij}^2 + B_{ij}^2). \tag{13}$$

This is equivalent to prove the inequalities

$$(11) \quad \leq \quad \frac{2}{3}(A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2) \tag{14}$$

$$(10) + (12) \quad \leq \quad \sum_k (\alpha_k^2 + \beta_k^2) + \frac{2}{3}(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2). \tag{15}$$

Note that

$$\begin{aligned} 2 \times (11) &\leq (A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2) \\ &\leq \frac{4}{3}(A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2), \end{aligned}$$

and so inequality (14) holds, with equality if and only if

$$A_{13} = A_{14} = A_{23} = A_{24} = B_{13} = B_{14} = B_{23} = B_{24} = 0.$$

Now

$$\begin{aligned} 3 \times (10) &= 3(\alpha_2 - \alpha_1)B_{12} - 3(\beta_2 - \beta_1)A_{12} + 3(\alpha_4 - \alpha_3)B_{34} + 3(-\beta_4 + \beta_3)A_{34} \\ &\leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \\ &\quad + \frac{3}{2}(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2) \\ &\leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \qquad (16) \\ &\quad + 2(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2). \qquad (17) \end{aligned}$$

We will prove that

$$(16) + 3 \times (12) \quad \leq \quad \sum_k 3(\alpha_k^2 + \beta_k^2), \tag{18}$$

with equality iff $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and $\beta_1 = \beta_2 = \beta_3 = \beta_4$, which proves that (15) holds. Furthermore, from (17) we see that equality in (15) is achieved iff

$$A_{12} = A_{34} = B_{12} = B_{34} = 0, \quad \text{and for all } i, j \quad \alpha_i = \alpha_j, \quad \beta_i = \beta_j.$$

In order to prove (18) we only have to show that

$$\frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\alpha_4 - \alpha_3)^2) + 2\sum_{i<j} \alpha_i\alpha_j \leq 3\sum_k \alpha_k^2,$$

or equivalently, that

$$-2\alpha_1\alpha_2 - 2\alpha_3\alpha_4 + 4\alpha_1\alpha_3 + 4\alpha_1\alpha_4 + 4\alpha_2\alpha_3 + 4\alpha_2\alpha_4 \leq 3\sum_k \alpha_k^2.$$

But this is just

$$(\alpha_1-\alpha_3)^2 + (\alpha_3-\alpha_2)^2 + (\alpha_2-\alpha_4)^2 + (\alpha_4-\alpha_1)^2 + (\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2 \geq 0,$$

with equality to zero iff $\alpha_i = \alpha_j \ \forall ij$. We have proved that inequality (8) is satisfied, with equality iff $f = \alpha(\sum_k \phi_k^2) = \alpha$ constant and $h$ constant, and so they must vanish. □

Theorem 1.1, with Lemmas 4.1, 4.3 and 4.4, prove that (8) holds for any pair of functions $(f, h)$, and so Theorem 1.2 is proved. Corollary 1.1 follows from these lemmas.

In (Salavessa, 2010, Theorem 4.2) a uniqueness theorem was obtained, on a class of closed $m$-dimensional submanifolds with parallel mean curvature and calibrated extended tangent in a Euclidean space $\mathbb{R}^{m+n}$, and satisfying an integral height inequality. We will recall such results for the case $\Omega$ parallel. We denote by $B^\nu$ the $\nu$-component of the second fundamental form $B$ and by $B^F$ the $F$-component, $B = B^\nu + B^F$, where $F$ is the orthogonal complement of $\nu$ in the normal bundle.

**Theorem 4.1** *If $\Omega$ is a parallel calibration of rank $(m+1)$ on $\mathbb{R}^{m+n}$, and $\phi : M \to \mathbb{R}^{m+n}$ is an immersed closed $\Omega$-stable $m$-dimensional submanifold with parallel mean curvature and calibrated extended tangent space, and*

$$\int_M S(2 + h\|H\|)dM \leq 0, \tag{19}$$

*where $h = \langle\phi, \nu\rangle$ and $S = \sum_{ij}\langle\phi, (B(e_i, e_j))^F\rangle B^\nu(e_i, e_j)$, then $\phi$ is pseudo-umbilical and $S = 0$. Furthermore, if $NM$ is a trivial bundle, then the minimal calibrated extension of $M$ is a Euclidean space $\mathbb{R}^{m+1}$, and $M$ is a Euclidean $m$-sphere.*

Theorem 1.3 is an immediate consequence of Theorem 1.2 and the above theorem.

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