A Limit Theorem for Random Allocations

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Abstract

A limit theorem is presented for random allocations. For a fixed period we allocate m balls into N boxes. We repeat the experiment throughout n periods. Let p_q denote the probability that we do not place more than q balls into any of the N boxes during any of the n repetitions. The limit of p_q is determined when $m, n, N \to \infty$.

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1. Introduction and Main Result

Let balls be placed successively and independently into N boxes. At each allocation the ball can fall into each box with probability $1/N$. During a fixed period (for a day, say) we allocate m balls. We repeat the experiment throughout n days. Let p_q denote the probability that we do not place more than q balls into any of the N boxes during any of the n days.

(Avkhadiev & Chuprunov, 2007) proved the following

Theorem A (Avkhadiev & Chuprunov, 2007, Theorem 2) *Let $m \geq 2$. Then*

$$p_1 = \left(1 - \frac{1}{N}\right)^n \left(1 - \frac{2}{N}\right)^n \dots \left(1 - \frac{m-1}{N}\right)^n. \tag{1}$$

If m is fixed and $n, N \to \infty$ such that $n/N \to \alpha$, then $p_1 \to e^{-\frac{m(m-1)}{2}\alpha}$; if $2 \leq q \leq m-1$, then $p_q \to 1$ as $n, N \to \infty$ such that $n/N \leq \alpha' < \infty$.

We extend the above theorem in the following sense: To obtain non-trivial limit for p_q when $q > 1$ we have to consider growing number of balls. We expect that the rate of convergence of m, n, N to ∞ will determine some q such that $\lim p_q$ is non-trivial, but $\lim p_{q-1} = 0$ and $\lim p_{q+1} = 1$. Our main result is the following

Theorem 1 *Let q be a fixed positive integer. Assume that $m, n, N \to \infty$ such that*

$$\frac{n}{N^q}\binom{m}{q+1} \to \alpha \tag{2}$$

where α is a positive finite number and

$$\frac{m^2}{N} \to 0. \tag{3}$$

Then

$$\lim p_l = \begin{cases} 0 & \text{if} \quad 0 \leq l < q, \\ e^{-\alpha} & \text{if} \quad l = q, \\ 1 & \text{if} \quad l > q. \end{cases} \tag{4}$$

Remark 1

$$\left(1-\frac{1}{N^l}\binom{m}{l+1}\right)^n \le p_l \le \left(1-\frac{1}{N^l}\binom{m}{l+1}(1-\varepsilon)\right)^n \tag{5}$$

for $l = 1, 2, \dots, m-1$, where $\varepsilon > 0$ and $\varepsilon \to 0$ if $m \to \infty$ and $N \to \infty$ such that $m^2/N \to 0$.

We want to mention that random allocations have been widely studied. See the classic papers (Weiss, 1958; Rényi, 1962) and (Békéssy, 1963), the traditional monograph (Kolchin, Sevast'yanov & Chistyakov, 1978). For more recent results, the reader can consult (Timashev, 2000) and (Chuprunov & Fazekas, 2005).

2. Proof of Main Result

The proof is based on the following

Theorem B (Avkhadiev & Chuprunov, 2007, Theorem 1) *Let* $m \ge 2$ *and* $1 \le q \le m-1$. *Then*

$$p_q = \left(1-\frac{A_q}{q!N}\right)^n \tag{6}$$

where

$$A_q = \sum_{l=q}^{m-1} \frac{1}{N^{l-1}} \left.\frac{d^l[z^q(f_q(z))^{N-1}]}{dz^l}\right|_{z=0},$$

and

$$f_k(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!}$$

for any non-negative integer k.

We mention that the proof of Theorem B is based on a result by (Timashev, 2000).

Remark 2 First we consider the case $q = 1$ because it is very simple and shows that Condition (2) is natural. Effectively, in virtue of (1),

$$\ln p_1 = n \sum_{i=0}^{m-1} \ln\left(1-\frac{i}{N}\right). \tag{7}$$

Using Taylor's expansion $\ln(1-x) = -x - x^2/(2(1-\vartheta x)^2)$ with $\vartheta \in (0,1)$, we obtain

$$\ln p_1 = -n \sum_{i=0}^{m-1} \frac{i}{N} - \frac{n}{2} \sum_{i=0}^{m-1} \left(\frac{i}{N}\right)^2 \left(1-\vartheta_i \frac{i}{N}\right)^{-2}. \tag{8}$$

For the first addend in (8) we have $-n\sum_{i=0}^{m-1} \frac{i}{N} = -\frac{n}{N}\binom{m}{2} \to -\alpha$. Assuming $m < N$, the second addend in (8) can be handled as

$$\left|-\frac{n}{2} \sum_{i=0}^{m-1} \left(\frac{i}{N}\right)^2 \left(1-\vartheta_i \frac{i}{N}\right)^{-2}\right| \le \frac{n}{2N^2}\left(1-\frac{m}{N}\right)^{-2} \sum_{i=0}^{m-1} i^2$$

$$\le \frac{n}{2N^2}\left(1-\frac{m}{N}\right)^{-2} \frac{(m-1)m(2m-1)}{6} = \frac{n}{N}\binom{m}{2}\frac{2m-1}{6N}\left(1-\frac{m}{N}\right)^{-2} \to 0.$$

Therefore $p_1 \to e^{-\alpha}$, if $\frac{n}{N}\binom{m}{2} \to \alpha$.

Proof of Theorem 1 By the Leibniz formula, we have for $\nu \ge q$

$$\left.\frac{d^\nu[z^q(f_q(z))^{N-1}]}{dz^\nu}\right|_{z=0} = \sum_{k=0}^{\nu} \binom{\nu}{k} \left.\frac{d^k[z^q]}{dz^k}\right|_{z=0} \left.\frac{d^{\nu-k}[(f_q(z))^{N-1}]}{dz^{\nu-k}}\right|_{z=0} = \binom{\nu}{q} q! \left.\frac{d^{\nu-q}[(f_q(z))^{N-1}]}{dz^{\nu-q}}\right|_{z=0}.$$

Therefore

$$A_q = \sum_{l=q}^{m-1} \frac{1}{N^{l-1}} \binom{l}{q} q! \left.\frac{d^{l-q}[(f_q(z))^{N-1}]}{dz^{l-q}}\right|_{z=0}. \tag{9}$$

We see that $\frac{d^l(f_k(z))}{dz^l} = f_{k-l}(z)$, where $f_h(z)$ is defined as 0 for $h < 0$. We have $(f_q(z))^t|_{z=0} = 1 = t^0$ for $q \ge 0$. Now we shall show that for $t \ge k \ge 1$ and $q \ge 1$

$$t_{(k)} \le \left.\frac{d^k[(f_q(z))^t]}{dz^k}\right|_{z=0} \le t^k \tag{10}$$

where $t_{(k)} = t(t-1)\dots(t-k+1)$. We will prove these inequalities by induction. For $k = 1$ we have

$$\left.\frac{d[(f_q(z))^t]}{dz}\right|_{z=0} = t\,(f_q(z))^{t-1}\big|_{z=0}\, f'_q(z)\big|_{z=0} = t\,(f_q(z))^{t-1}\big|_{z=0}\, f_{q-1}(z)\big|_{z=0} = t.$$

By the Leibniz formula,

$$\left.\frac{d^{k+1}[(f_q(z))^t]}{dz^{k+1}}\right|_{z=0} = \left.\frac{d^k[t(f_q(z))^{t-1} f_{q-1}(z)]}{dz^k}\right|_{z=0}$$

$$= t\sum_{l=0}^{k}\binom{k}{l}\left.\frac{d^l[(f_q(z))^{t-1}]}{dz^l}\right|_{z=0} f_{q-1-(k-l)}(z)|_{z=0} = t\sum_{l=k+1-q}^{k}\binom{k}{l}\left.\frac{d^l[(f_q(z))^{t-1}]}{dz^l}\right|_{z=0} = F.$$

Using the induction hypothesis,

$$F \geq t\sum_{l=k+1-q}^{k}\binom{k}{l}(t-1)_{(l)} \geq t(t-1)_{(k)} = (t)_{(k+1)}$$

and

$$F \leq t\sum_{l=k+1-q}^{k}\binom{k}{l}(t-1)^l \leq tt^k = t^{k+1}.$$

Hence (10) holds and can be used as follows

$$\frac{A_q}{q!N} \geq \sum_{k=q}^{m-1}\frac{1}{N^k}\binom{k}{q}(N-1)_{(k-q)} = \frac{1}{N^q}\sum_{k=q}^{m-1}\binom{k}{q}\frac{(N-1)(N-2)\cdots(N-(k-q))}{N^{k-q}} \geq$$

$$\geq \frac{1}{N^q}\sum_{k=q}^{m-1}\binom{k}{q}(1-\varepsilon) = \frac{1}{N^q}\binom{m}{q+1}(1-\varepsilon) \tag{11}$$

where $\varepsilon > 0$ and $\varepsilon \to 0$ as $m, N \to \infty$ such that Condition (3) is satisfied. In (11), we only need prove $(N-1)(N-2)\cdots(N-(k-q))/N^{k-q} \geq 1-\varepsilon$. To do so, given $k = q, q+1, \dots, m-1$ we consider

$$0 \geq \ln\left(\frac{(N-1)(N-2)\cdots(N-(k-q))}{N^{k-q}}\right) \geq \sum_{l=1}^{m-1-q}\ln\frac{N-l}{N} \geq$$

$$\geq \int_1^{m-1-q}\ln\frac{N-x}{N}dx = \left[(a-N)\ln\left(1-\frac{a}{N}\right)-a\right]-\left[(1-N)\ln\left(1-\frac{1}{N}\right)-1\right]$$

where $a = m-1-q$. Note that the second addend in the previous expression tends trivially to 0; whereas the first addend shows the same tendency due to Condition (3) and Taylor's expansion for $\ln(1-a/N)$. This involves that (11) holds.

Again by (10),

$$\frac{A_q}{q!N} \leq \sum_{k=q}^{m-1}\frac{1}{N^k}\binom{k}{q}(N-1)^{k-q} = \frac{1}{N^q}\sum_{k=q}^{m-1}\binom{k}{q}\left(\frac{N-1}{N}\right)^{k-q} \tag{12}$$

$$\leq \frac{1}{N^q}\sum_{k=q}^{m-1}\binom{k}{q} = \frac{1}{N^q}\binom{m}{q+1}. \tag{13}$$

Thus the upper and the lower bounds of $\frac{A_q}{q!N}$ are of the form $\frac{1}{N^q}\binom{m}{q+1}$ and $\frac{1}{N^q}\binom{m}{q+1}(1-\varepsilon)$, respectively, where $\varepsilon > 0$ and $\varepsilon \to 0$ as $m, N \to \infty$ such that Condition (3) is satisfied. In virtue of (6),

$$\left(1-\frac{1}{N^q}\binom{m}{q+1}\right)^n \leq p_q = \left(1-\frac{A_q}{q!N}\right)^n \leq \left(1-\frac{1}{N^q}\binom{m}{q+1}(1-\varepsilon)\right)^n \tag{14}$$

where $\varepsilon > 0$ and $\varepsilon \to 0$ as $m, N \to \infty$ such that Condition (3) is satisfied. Consequently, in virtue of (2),

$$p_q \to e^{-\alpha}.$$

Above we applied only Condition (3) (and we did not apply Condition (2)) to obtain (14). Consequently we have proved Remark 1.

Now consider p_{q+1} and p_{q-1}. Using (5),

$$p_{q+1} \approx \left(1 - \frac{\frac{n}{N^{q+1}}\binom{m}{q+2}}{n}\right)^n \to e^0 = 1,$$

since $\frac{n}{N^{q+1}}\binom{m}{q+2} \to 0$ in virtue of (2) and (3). (Here $a_s \approx b_s$ means that $a_s - b_s \to 0$ as $s \to \infty$). Moreover,

$$p_{q-1} \approx \left(1 - \frac{\frac{n}{N^{q-1}}\binom{m}{q}}{n}\right)^n \to e^{-\infty} = 0$$

because $\frac{n}{N^{q-1}}\binom{m}{q} \to \infty$ and $\frac{n}{N^{q-1}}\binom{m}{q}/n \to 0$ by (2) and (3).

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