Relation Between the Smallest and the Greatest Parts of the Partitions of n

I. Ramabhadra Sarma
Department of Mathematics, K L University
Vaddeswaram, A.P-522502, India
E-mail: irbsarma44@yahoo.com

K. Hanuma Reddy
Department of Mathematics, Hindu College
Guntur, A.P-522002, India
E-mail: hanumareddy_k@yahoo.com

S. Rao Gunakala (Corresponding author)
Department of Mathematics and Statistics, The University of the West Indies
St. Augustine, Trinidad and Tobago
Tel: 1-868-662-2002 Ext (84491) E-mail: sreedhara.rao@sta.uwi.edu

D.M.G. Comissiong
Department of Mathematics and Statistics, The University of the West Indies
St. Augustine, Trinidad and Tobago
E-mail: donna.comissiong@sta.uwi.edu

Received: April 12, 2011 Accepted: April 30, 2011 Published: November 1, 2011
doi:10.5539/jmr.v3n4p133 URL: http://dx.doi.org/10.5539/jmr.v3n4p133

Abstract

In this paper the formulae for the number of smallest parts of partitions of $n \in N$ and relations between the *ith* smallest parts and the *ith* greatest parts are obtained.

Keywords: Partition, Conjugate partition, Ferrer diagram, Smallest part of partition, ith smallest part of a partition, Greatest part of a partition, Rank of a partition

1. Introduction

We adopt the common notation on partitions as used in (Andrews, G.E., 1976) and (Andrews, G. E., to appear). A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, ...\lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$ and it is denoted by $n = (\lambda_1, \lambda_2, ...\lambda_r)$. The λ_i are called the parts of the partition. The number of parts of λ is called the length of λ, and is denoted by $l(\lambda)$. $\lambda_1 - I(\lambda)$ is called the rank of the partition. Throughout this paper, λ stands for a partition of n, $\lambda = (\lambda_1, \lambda_2, ...\lambda_r)$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$.

Ferrer (Andrews, G.E., 1976) introduced a representation of a partition by a diagram made of dots " · " or squares "□" as follows. Each λ_i is represented as a row of λ_1 dots (or squares) and these are arranged in parallel rows in the decreasing order of $\lambda' s$. For example, the diagrammatic representation for $11 = (5, 3, 2, 1)$ is given by Figures 1 and 2. These diagrams help us in formulating the definition of the conjugate λ^* of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...\lambda_r)$. The conjugate $\lambda^* = \left(\lambda_1^*, \lambda_2^*, \lambda_3^*, ...\lambda_s^*\right)$ is a *s-partition* of n. Where λ_i^* is the number of dots (squares) in the ith column in the Ferrer diagram for λ. This if $\lambda = (5, 3, 2, 1)$, then $\lambda^* = (4, 3, 2, 1, 1)$.

Let $spt(n)$ denote the number of smallest parts including repetitions in all partitions of n. For $i \geq 1$ let us adopt the following notation $n_s(\lambda)$ = number of smallest parts of λ.

$$spt(n) = \sum_{\lambda \in \xi(n)} n_s(\lambda)$$

If $\langle \mu_1, \mu_2, ..., \mu_k \rangle$ are the distinct parts in λ where $\mu_1 > \mu_2 > ... > \mu_k$, then $\mu_i = g_i = ith$ greatest part of λ, $\mu_{k-i+1} = ith$ smallest parts of λ, and $n_{s_i}(\lambda)$ is the number of *ith* smallest parts in λ, that is, the number of μ_{k-i+1}'s in λ. We write $s_i(\lambda) = \mu_{k-i+1}$ where $t_i = s_i - s_{i-1}$ if s_i and s_{i-1} both exist, $t_i = s_i$ if s_i exists but s_{i-1} does not exist, and $t_i = 0$ if s_i does not exist but s_{i-1} exists. Let $spt_i(n)$ denote the number of *ith* smallest parts in all partitions of n and

$$sum(s_i) = \sum_{\lambda \in \xi(n)} \mu_{k-i+1}$$

Dually, we define the i^{th} greatest part of λ, $gpt(\lambda)$, $n_g(\lambda)$, *ith* greatest part $g_i(\lambda)$ of λ and $sum(g_i)$.

In Table 1, we provide a list of the partitions of 6 with their corresponding $s_1(\lambda)$, $s_2(\lambda)$, $n_{s_1}(\lambda)$, $n_{s_2}(\lambda)$, $g_1(\lambda)$, $g_2(\lambda)$, $n_{g_1}(\lambda)$, $n_{g_2}(\lambda)$. We see that $spt(n) = 26$, $spt_2(n) = 8$, $gpt(n) = 20$, $gpt_2(n) = 14$, $sum(s_2 - s_1) = 14$ and $sum(g_1 - g_2) = 26$.

Let $\xi(n)$ denote the set of all partitions of n and $p(n)$ be the cardinality of $\xi(n)$ for $n \in N$ and $p(0) = 1$. If $1 \leq r \leq n$ we write $p_r(n)$ for the number of partitions of n each consisting of exactly r parts, that is, *r-partitions* of n. If $r \leq 0$ or $r \geq n$, we write $p_r(n) = 0$ and $\xi^*(n)$ denotes the set of all conjugate partitions of n. Also let $p(k, n)$ represent the number of partitions of n using natural numbers at least as large as k only, and let $G(s, n)$ denote the number of partitions of n having greatest part n, as in (Atkin, A.O.L et al, 2003) and (Bringmann, K. et al,to appear). For $m \in z$, $N(m; n)$ = number of $\lambda \in \xi(n)$ such that $r(\lambda) = m$. For $k \in N$,

$$N_k(n) = \sum_{m=-\infty}^{+\infty} m^k N(m; n).$$

Andrews (Andrews, G. E., to appear) proved analytically that

$$spt(n) = np(n) - \frac{1}{2} N_2(n)$$

In this paper, we give a proof of the theorem for the relation between the *ith* smallest parts and *ith* greatest parts of the partitions of the positive integer n. In particular, we show that

$$spt(n) = \sum_{\lambda \in \xi(n)} g_1(\lambda) - \sum_{\lambda \in \xi(n)} g_2(\lambda)$$

Theorem 4. $\xi(n) = \xi^*(n)$.

Proof. The Ferrer diagram for λ consists of r rows of dots, with the *ith* row having λ_i dots. Thus clearly, the columns also have dots in decreasing numbers. Hence the rows in λ^* have dots in decreasing numbers. Since λ, λ^* have the same number of dots, it follows that $\lambda^* = \left(\lambda_1^*, \lambda_2^*, ..., \lambda_s^*\right)$ is a partition of n. Furthermore, $\lambda^{**} = \lambda$. Hence $\xi(n) = \xi^*(n)$. □

Theorem 5. *For* $1 \leq i \leq n$, *let* g_i *be the ith greatest part of* λ. *Then* $\sum_{\lambda \in \xi(n)} g_i = \sum_{\lambda \in \xi(n)} g_i^*$.

Proof. From theorem 1, we have that $\xi(n) = \xi^*(n) \Rightarrow \sum_{\lambda \in \xi(n)} g_i = \sum_{\lambda \in \xi(n)} g_i^*$. □

Theorem 6. *For each* i, *if*

(i) g_i^* *is the greatest part of* λ^*, *then* $n_{s_i}(\lambda) = g_i^* - g_{i+1}^*$.

(ii) the $(i+1)$ *th greatest part does not exist, then* $n_{s_i}(\lambda) = g_i^*$.

(iii) the ith greatest part does not exist, then $n_{s_i}(\lambda) = 0$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \in \xi(n)$ and $\lambda^* = \left(\lambda_1^*, \lambda_2^*, ..., \lambda_r^*\right) \in \xi^*(n)$. The Ferrer diagram of λ and λ^* can be partitioned into rows having equal numbers of dots which can expressed in matrix form as shown in Figure 3. We observe that the number of rows in the *ith* matrix from bottom to top of the diagram for λ is equal to the number for the *ith* smallest parts of λ. Also, the number of columns in the *ith* matrix from top to bottom of the diagram for λ^* is equal to the *ith* greatest part of λ^*. We also observe that

(i) If both the *ith* and the $(i+1)\,th$ greatest parts of λ^* exist, then the difference between the *ith* and the $(i+1)\,th$ greatest parts of λ^* is equal to the number of the *ith* smallest parts of λ.

(ii) If the *ith* greatest part of λ^* exists and the $(i+1)\,th$ greatest parts of λ^* do not exist, then the value of the *ith* greatest part of λ^* is equal to the number of the *ith* smallest part of λ.

(iii) If the *ith* greatest part of λ^* does not exist and the $(i+1)\,th$ greatest part of λ^* exists, then the difference between the *ith* and the $(i+1)\,th$ greatest parts of λ^* is equal to zero.

Hence $n_{s_i}(\lambda) = g_i^* - g_{i+1}^*$ if both g_i^* and g_{i+1}^* exist, $n_{s_i}(\lambda) = g_i^*$ if g_{i+1}^* does not exist, and $n_{s_i}(\lambda) = 0$ if g_i^* does not exist. □

Theorem 7. $spt_i(n) = \sum_{\lambda \in \xi(n)} g_i(\lambda) - \sum_{\lambda \in \xi(n)} g_{i+1}(\lambda)$.

Proof. Since $spt_i(n) = \sum_{\lambda \in \xi(n)} n_{s_i}(\lambda)$ from theorem 3, it follows that

$$n_{s_i}(\lambda) = g_i^*(\lambda) - g_{i+1}^*(\lambda) \quad \text{or} \quad n_{s_i}(\lambda) = g_i^*(\lambda).$$

$$\Rightarrow \sum_{\lambda \in \xi(n)} n_{s_i}(\lambda) = \sum_{\lambda \in \xi(n)} \left[g_i^*(\lambda) - g_{i+1}^*(\lambda)\right]$$

and from theorem 2, we have that

$$\Rightarrow \sum_{\lambda \in \xi(n)} n_{s_i}(\lambda) = \sum_{\lambda \in \xi(n)} \left[g_i(\lambda) - g_{i+1}(\lambda)\right]$$

$$spt_i(n) = \sum_{\lambda \in \xi(n)} g_i(\lambda) - \sum_{\lambda \in \xi(n)} g_{i+1}(\lambda)$$

□

As a consequence of theorem 4, we have the following corollary:

Corollary 1. *Let g_1, g_2 be the first and second greatest parts of λ respectively. Then*

$$spt(n) = \sum_{\lambda \in \xi(n)} g_1(\lambda) - \sum_{\lambda \in \xi(n)} g_2(\lambda)$$

Theorem 8. *If the partition $\lambda \in \xi(n)$ has k distinct parts, then $g_i = s_{k-i+1}$ for $1 \le i \le k$.*

Proof. From theorem 3, the *ith* matrix from top to bottom in the Ferrer diagram is the same as the $(k-i+1)\,th$ matrix from bottom to top. Hence $g_i = s_{k-i+1}$ for $1 \le i \le k$. □

Theorem 9. $gpt_i(n) = \sum_{\lambda \in \xi(n)} t_i(\lambda)$.

Proof. From theorem 4, we have that

$$spt_i(n) = \sum_{\lambda \in \xi(n)} \left[g_i(\lambda) - g_{i+1}(\lambda)\right]$$

also from theorem 5,

$$gpt_{k-i+1}(n) = \sum_{\lambda \in \xi(n)} \left[s_{k-i+1}(\lambda) - s_{k-i-1+1}(\lambda)\right]$$

$$\Rightarrow gpt_i(n) = \sum_{\lambda \in \xi(n)} \left[s_t(\lambda) - s_{t-1}(\lambda)\right], \quad \text{where } t = k-i+1$$

Hence

$$gpt_i(n) = \sum_{\lambda \in \xi(n)} t_i(\lambda).$$

□

As a consequence of theorem 6, we have the following corollary:

Corollary 2. *Let s_1 be the smallest parts of λ. Then*

$$gpt(n) = \sum_{\lambda \in \xi(n)} s_1(\lambda).$$

Remark 2. $\sum_{\lambda \in \xi(n)} [s_t(\lambda) - s_{t-1}(\lambda)] \neq \sum_{\lambda \in \xi(n)} s_t(\lambda) - \sum_{\lambda \in \xi(n)} s_{t-1}(\lambda)$, *since when $s_i(\lambda)$ does not exist, then $s_t(\lambda) - s_{t-1}(\lambda) = 0$.*

Theorem 10. $\sum_{\lambda \in \xi(n)} g_1(\lambda) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_{i-1}}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_i=1}^{g_{i-1}-1} g_i \cdot p_{g_i}(n - k_1g_1 - k_2g_2 \ldots - k_{i-1}g_{i-1})$.

Proof. From (Reddy, K. H., 2010), for every g, we have

$$G(g, n) = p_g(n)$$

Hence

$$\sum_{\lambda \in \xi(n)} g_1(\lambda) = \sum_{g_i=1}^{n} g_1 \cdot p_{g_1}(n).$$

If the greatest parts g_1 appear k_1 times followed by its successor g_2, then

$$G(g_2, n - k_1g_1) = p_{g_2}(n - k_1g_1)$$

The sum of these second greatest parts taken over all partitions is $\sum_{\lambda \in \xi(n)} g_2 p_{g_2}(n - k_1g_1)$, hence

$$\sum_{\lambda \in \xi(n)} g_2(\lambda) = \sum_{k_1=1}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} g_2 \cdot p_{g_2}(n - k_1g_1)$$

The theorem follows by repeating this process

$$\sum_{\lambda \in \xi(n)} g_i(\lambda) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_{i-1}}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_i=1}^{g_{i-1}-1} g_i \cdot p_{g_i}(n - k_1g_1 - k_2g_2 \ldots - k_{i-1}g_{i-1})$$

□

In general we have from theorem 4 the following:

Theorem 11.

$$spt_i(n) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_{i-1}}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_i=1}^{g_{i-1}-1} g_i \cdot p_{g_i}(n - k_1g_1 - k_2g_2 \ldots - k_{i-1}g_{i-1})$$
$$- \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_i}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_{i+1}=1}^{g_i-1} g_{i+1} \cdot p_{g_{i+1}}(n - k_1g_1 - k_2g_2 \ldots - k_ig_i)$$

Proof. From theorem 4, we have

$$spt_i(n) = \sum_{\lambda \in \xi(n)} g_i(\lambda) - \sum_{\lambda \in \xi(n)} g_{i+1}(\lambda)$$

hence

$$spt_i(n) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_{i-1}}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_i=1}^{g_{i-1}-1} g_i \cdot p_{g_i}(n - k_1g_1 - k_2g_2 \ldots - k_{i-1}g_{i-1})$$
$$- \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_i}^{\infty} \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{g_1-1} \ldots \sum_{g_{i+1}=1}^{g_i-1} g_{i+1} \cdot p_{g_{i+1}}(n - k_1g_1 - k_2g_2 \ldots - k_ig_i)$$

□

Corollary 3. $spt(n) = \sum_{g_1=1}^{\infty} g_1 \cdot p_{g_1}(n) - \sum_{k_1=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1} g_2 \cdot p_{g_2}(n - k_1 g_1).$

Proof. Put $i = 1$ into theorem 8. □

Theorem 12. $p_1(n) + p_2(n-k) + p_3(n-2k) + ... = p(k+1, n+k).$

Proof. By induction. □

Theorem 13. *The number of r-partitions of n having k as a smallest part is*

$$j + \sum_{i=0}^{\infty} p_{r-1-i}[n - (k-1)r - 1 - i]$$

where $j = 1$ *if r divides n, otherwise* $j = 0$.

Proof. From (Reddy, K.H., 2010), the number of *r-partitions* of n with smallest part k is

$$p_{r-1}[n - (k-1)r - 1].$$

We fix $k \in \{1, 2, ..., n\}$. For $1 \leq i \leq r$, the number of *r-partitions* of n with $(r-i)$ smallest parts each being k is the number of *i-partitions* of $n - (r-i)k$. Summing over $i = 1$ to r, we get the total number of *r-partitions* of n with smallest parts k. This number is

$$j + \sum_{i=0}^{\infty} p_{r-1-i}[n - (k-1)r - 1 - i]$$

where $j = 1$ if r divides n, otherwise $j = 0$. □

As k varies from 1 to n, we have the following corollaries:

Corollary 4. *The total number of r-partitions is*

$$\sum_{k=1}^{\infty}\left[j + \sum_{i=0}^{\infty} p_{r-1-i}[n - (k-1)r - 1 - i]\right]$$

where $j = 1$ *if r divides n, otherwise* $j = 0$.

Corollary 5. *By taking the sum as r varies, we get*

$$spt(n) = \sum_{r=1}^{\infty}\sum_{k=1}^{\infty}\left[j + \sum_{i=0}^{\infty} p_{r-1-i}[n - (k-1)r - 1 - i]\right]$$

where $j = 1$ *if r divides n, otherwise* $j = 0$.

We now independently derive another formula for $spt(n)$.

Theorem 14. $spt(n) = \sum_{t_1=1}^{\infty}\sum_{s_{i_1}=1}^{\infty} p(s_{i_1}, n - t_1 s_{i_1}) + d(n)$ *where* $d(n)$ *is the number of positive divisors of n.*

Proof. Any partition in $\xi(n)$ has a smallest part which possibly repeats.

(i) If the smallest part d is a divisor of n, then the number of partitions with d as a smallest part is

$$1 + \sum_{t=1}^{[\frac{n}{d}]-1} p(d, n - td)$$

where 1 corresponds to the partition $\left(d, d, ... \, up\ to\ \frac{n}{d}\ times\right)$.

(ii) If d is not a divisor of n there is no partition with equal parts. In this case the total partitions with d as smallest part is

$$\sum_{t=1}^{[\frac{n}{d}]-1} p(d, n-td)$$

which gives

$$spt(n) = \sum_{t=1}^{[\frac{n}{d}]-1} p(d, n-td) + \sum_{\frac{d}{n}} 1$$
$$= \sum_{t=1}^{[\frac{n}{d}]-1} p(d, n-td) + d(n)$$

□

As a consequence of theorem 11, we have $d^1(n-ts)$ = number of divisors of $n-ts$ that are greater than s

$$spt_2(n) = \sum_{s_1=1}^{\infty} \sum_{t_1=1}^{\infty} \left[\sum_{s_2=s_1+1}^{\infty} \sum_{t_2=1}^{\infty} p(s_1, n-t_1s_1-t_2s_2) + d(n-t_1s_1) \right]$$

More generally, if $d^1(n-t_1s_1-t_2s_2-...-t_{i-1}s_{i-1})$ is the number of divisors of $n^1-t_1s_1-t_2s_2-...-t_{i-1}s_{i-1}$ that are greater than s_{i-1}, then

$$spt_i(n) = \sum_{s_1=1}^{\infty} \sum_{s_2=s_1+1}^{\infty} ... \sum_{s_i=s_{i-1}+1}^{\infty} \sum_{t_1=1}^{\infty} \sum_{t_2=1}^{\infty} ... \sum_{t_{i-1}=1}^{\infty} \sum_{t_i=1}^{\infty} p(s_1, n-t_1s_1-t_2s_2-...-t_is_i)$$
$$+ \sum_{s_1=1}^{\infty} \sum_{s_2=s_1+1}^{\infty} ... \sum_{s_{i-1}=s_{i-2}+1}^{\infty} \sum_{t_1=1}^{\infty} \sum_{t_2=1}^{\infty} ... \sum_{t_{i-2}=1}^{\infty} \sum_{t_{i-1}=1}^{\infty} d'(n-t_1s_1-t_2s_2-...-t_{i-1}s_{i-1})$$

Theorem 15. *If* $\{a, m, n, r, S\} \subset N$, $b \in Z$, $r|\left(\frac{n-br}{a}\right)$ *and* $S = \{am+b|m = 1, 2, ..., n\}$, *then*

$$spt(S; n) = \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \left[j + \sum_{i=0}^{\infty} p_{r-1-i} \left[\left(\frac{n-br}{a}\right) - (k-1)r - 1 - i \right] \right]$$

Proof. From (Reddy, K. H., 2010), if $a|n-br$ and $n-br > 0$, then $p_r(\text{"}S\text{"}, n) = p_r\left(\frac{n-br}{a}\right)$, otherwise

$$p_r(\text{"}S\text{"}, n) = 0 \tag{1}$$

From theorem 10,

$$spt(n) = \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \left[j + \sum_{i=0}^{\infty} p_{r-1-i}[n-(k-1)r-1-i] \right] \tag{2}$$

where $j = 1$ if r divides n, otherwise $j = 0$. From equations (1) and (2) we have

$$spt(S; n) = \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \left[j + \sum_{i=0}^{\infty} p_{r-1-i} \left[\left(\frac{n-br}{a}\right) - (k-1)r - 1 - i \right] \right]$$

□

As a consequence of theorem 12, we have the following

$$spt(e; n) = \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \left[j + \sum_{i=0}^{\infty} p_{r-1-i} \left[\left(\frac{n}{2}\right) - (k-1)r - 1 - i \right] \right]$$

$$spt\,(o;n) = \sum_{r=1}^{\infty}\sum_{k=1}^{\infty}\left[j + \sum_{i=0}^{\infty} p_{r-1-i}\left[\left(\frac{n+r}{2}\right) - (k-1)\,r - 1 - i\right]\right]$$

References

Andrews, G. E. (1976). The theory of partitions. *Encyclopedia of Mathematics and its Applications*, Vol. 2, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam.

Andrews, G. E. (to appear). The number of smallest parts in the partitions of n. *J. Reine Angew. Math.*

Atkin, A. O. L. & Garvan, F. G. (2003). Relations between the ranks and cranks of partitions. *Ramanujan J.,* 7, 343-366.

Hanuma Reddy, K. (2007). Note on convex polygons. *Proceedings of the 10th Joint Conference of Information Sciences*, 1691-1697.

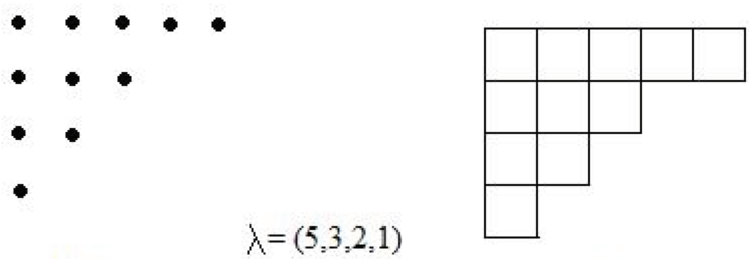
Hanuma Reddy, K. (2010). A note on r-partitions of n in which the least part is k. *International Journal of Computational Mathematical Ideas*, 2 (1), 6-12.

Hanuma Reddy, K. (2010). A note on partitions. *International Journal of Mathematical Sciences*, 9 (3-4), 379-389.

Ramanujan, S. (1988). *The lost notebook and other unpublished papers*. Springer-Verlag, Berlin.

Table 1. The partitions of 6

$\lambda \in \xi(6)$	$s_1(\lambda)$	$s_2(\lambda)$	$s_2 - s_1$	$n_{s_1}(\lambda)$	$n_{s_2}(\lambda)$	$g_1(\lambda)$	$g_2(\lambda)$	$g_1 - g_2$	$n_{g_1}(\lambda)$	$n_{g_2}(\lambda)$
(6)	6	$--$	0	1	0	6	$--$	6	1	0
$(5,1)$	1	5	4	1	1	5	1	4	1	1
$(4,2)$	2	4	2	1	1	4	2	2	1	1
$(3,3)$	3	$--$	0	2	0	3	$--$	3	2	0
$(4,1,1)$	1	4	3	2	1	4	1	3	1	2
$(3,2,1)$	1	2	1	1	1	3	2	1	1	1
$(2,2,2)$	2	$--$	0	3	0	2	$--$	2	3	0
$(3,1,1,1)$	1	3	2	3	1	3	1	2	1	3
$(2,2,1,1,1)$	1	2	1	2	2	2	1	1	2	2
$(2,1,1,1,1)$	1	2	1	4	1	2	1	1	1	4
$(1,1,1,1,1,1)$	1	$--$	0	6	0	1	$--$	1	6	0



$\lambda = (5,3,2,1)$

Fig 1 Fig 2

Figure 1. Diagrammatic representation for $11 = (5,3,2,1)$

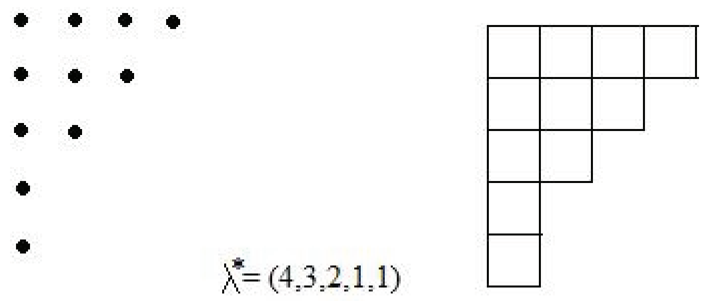


Fig 1 Fig 2

Figure 2. Diagrammatic representation for 11 = (4, 3, 2, 1, 1)

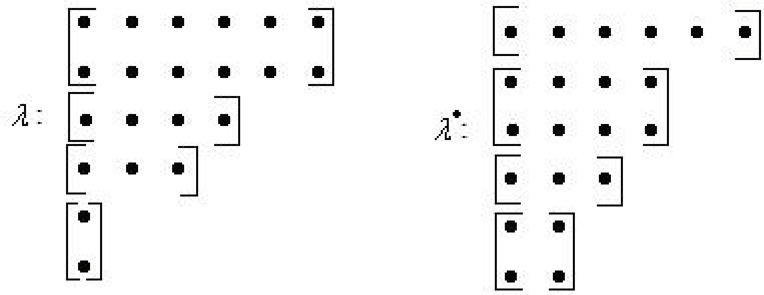


Figure 3. Ferrer diagram for *lambda* and λ^*