Weighted Pseudo-almost Periodic Solutions of Neutral Integral and Differential Equations

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Abstract

In this paper, we study weighted pseudo-almost periodic solutions to general neutral integral equations. By applying the properties of weighted pseudo-almost periodic functions and the fixed point theorem, we establish the conditions for the existence and uniqueness of weighted pseudo-almost periodic solutions of the equations.

Keywords: Weighted pseudo-almost periodic solutions, Neutral integral and differential equations, Existence and uniqueness, Fixed point theorem

1. Introduction

Qualitative analysis such as periodicity, almost periodicity and stability of neutral functional differential equations has been studied extensively by many authors, we can refer to [X. X. Chen, 2007; S. Abbas, 2008; X.X. Chen, 2011; *et al*] and references cited therein. Neutral functional differential equations are not only an extension of functional differential equation but also provide good models in many fields including Mechanics, Biology, Population Ecology, Neural network, and so on. The existence of almost periodic , pseudo-almost periodic ,weighted pseudo-almost periodic solutions is one of the most attractive topic in qualitative theory of differential equations due to their important applications ; see for instance [A.M. Fink, 1974; X.X Chen, 2010; C.Y. He, 1992; *et al*] and references therein. In [2008], Abbas and Bahugura have studied the almost periodic solution of a nonlinear neutral system of the form:

$$u'(t) = A(t)u(t) + F_1(t, u(t - g(t))) + F_2(t, u(t - g(t))) \tag{1.1}$$

In [M.Pinto, 2010], Mannel studied the existence and uniqueness of pseudo-almost periodic and almost periodic solutions to the equation:

$$u(t) = f(t, u(t), u(h_0(t))) + \int_R C(t, s, u(s), u(h(s)))ds, t \in R. \tag{1.2}$$

More specifically, that is the neutral delay integral equations of advanced and delayed decomposition type:

$$u(t) = f(t, u(t), u(h_0(t))) + \int_{-\infty}^{t} C_1(t, s, u(s), u(h_1(s)))ds + \int_{t}^{+\infty} C_2(t, s, u(s), u(h_2(s)))ds,$$

In recent papers, in [2006], Diagana introduced the concept of weighted pseudo-almost periodic functions which is a new generalization of pseudo-almost periodic functions. Motivated by these papers, the purpose of this paper is to investigate the existence and uniqueness of the weighted pseudo-almost periodic solutionsor the following nonlinear neutral integral system:

$$x(t) = F(t, x(t), x(h_0(t))) + \int_{-\infty}^{t} G_1(t, s, x(s), x(h_1(s)))ds + \int_{t}^{+\infty} G_2(t, s, x(s), x(h_2(s)))ds, \tag{1.3}$$

where $h_i \in C(R, R)$ for $i = 0, 1, 2$ and $G_i : R \times R \times R^n \times R^n \to R^n, i = 1, 2$ are continuous.

2. Preliminaries

Let X, Y be complex Banach spaces endowed with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively and $BC((R, Y), \|\cdot\|_\infty)$ be the Banach spaces of bounded continuous functions from R to Y endowed with the supremum norm $\|\phi\|_\infty = \sup_{t\in R} \|\phi(t)\|_Y$. Define a function $\lambda : R^2 \to (0, +\infty)$, $BC_\lambda(R^2 \times X, Y)$will denote the vectorial space of continuous functions $f : R^2 \times X \to Y$ such that f/λ is bounded. Let U be the collection of all functions (weights) $\rho : R \to [0, +\infty)$ which are locally integrable over R such that $\rho(x) > 0$ for almost each $x \in R$. For each $r > 0$ and $\rho \in U$, set

$$m(r, \rho) = \int_{-r}^{r} \rho(x)dx$$

Let $U_\infty = \rho \in U : \lim_{r\to\infty} m(r, p) = \infty; U_B = \{\rho \in U_\infty : \rho \textit{ is bounded and} \inf_{X\in R} \rho(x) > 0\}$. About the definitions of almost periodic and pseudo-almost periodic functions, we refer to [C.Y. Zhang, 1994; Y.H. Wang, 2011; J.K. Hale, 1977; *et al*]. Let $AP(R, X)$ be the set of all almost periodic functions from R to X and $AP(R \times \Omega, X)$ be the set of all periodic functions in $t \in R$ uniformly from $R \times \Omega to X$, where Ω is a closed subset of X. Then $(AP(R, X), \|\cdot\|_\infty)$and $(AP(R \times \Omega, X), \|\cdot\|_\infty)$are all Banach spaces. We denote the set of all pseudo-almost functions by $PAP(R, X)$. Similarly, we can also define the $PAP(P \times X, X)$. For a function $\lambda : R^2 \to (0, +\infty)$ and any closed subset $\Omega \in X$, a function $F \in BC_\lambda(R^2 \times \Omega, Y)$ is called λ-almost periodic in $t, s \in R$ uniformly in any compact subset $K \in \Omega$. We mean that, for each ε, there exists $A_\varepsilon > 0$ such that for every rectangle $R_1 \times R_2 \subset R^2$ of area A_ε and a number $\tau \in R_1 \cap R_2$ with the following property:

$$\|F(t + \tau, s + \tau, x) - F(t, s, x)\|_Y \le \varepsilon\lambda(t, s), \quad (t, s, x) \in R^2 \times \Omega.$$

The number τ above will be called an ε-translation number with respect to λ of F and the class of such functions F will be denoted $AP_\lambda(R^2 \times \Omega, X)$.

Remark1: For $\lambda \equiv 1, AP_1(R^2 \times \Omega, X) = AP(R^2 \times \Omega, X)$.
Next, we need to introduce the weighted ergodic terms i.e.the spaces $PAP_0(R, X, \rho)$ and $PAP_0(R \times X, X, \rho)$. Let$\rho = U_\infty$. Define

$$PAP_0(R, X, \rho) = \{f \in BC(R, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t)\|_X \rho(t)dt = 0\}$$

$$PAP_0(R \times X, X, \rho) = \{f \in BC(R \times X, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t, x)\|_X \rho(t)dt = 0 \; uniformly \; in \; x \in X\}$$

Definition 2.1 For a function $\vartheta : R^2 \to [0, +\infty)$, we say $F(t, s, x) \in PAP^0_\vartheta(R^2 \times X, X, \rho)$ if

$$\|F(t, s, x)\| \le \vartheta(t, s)\widehat{F}(s, x), \; t, s \in R, \; x \in R \tag{2.1}$$

with $0 < \widehat{F}(s, x) \in PAP_0(R \times X, R, \rho)$.

Definition 2.2 Let $f \in BC(R \times X, X)$, f is called weighted pseudo-almost periodic if $f = g + \phi$, where $g \in AP(R \times X, X)$ and $\phi \in PAP_0(R \times X, X, \rho)$. The collection of such functions f will be denoted by $WPAP(R \times X, R, \rho)$.

Now we equip the collection of weighted pseudo-almost periodic functions from $R\times X$ into X, i.e., $WPAP(R\times X, Y, \rho)$,with the supremum norm, then $(WPAP(R \times X, Y, \rho), \|\cdot\|_\infty)$is a Banach space, see details in [S. Abbas, 2008; X.X. Chen, 2011; T. Diagana, 2006; C.Y. Zhang, 1994].

Definition 2.3 Let $\lambda, \vartheta : R^2 \to [0, +\infty)$ be two functions. A function $f : R^2 \times X \to X$ is called (λ, ϑ) weighted pseudo-almost periodic in R^2 uniformly in $x \in X$ if $f = g + \phi$, where $g \in AP_\lambda(R^2 \times X, X)$ and $\phi \in PAP^0_\vartheta(R^2 \times X, X, \rho)$. The collection of such functions will be denoted by $WPAP_{(\lambda,\vartheta)}(R \times X, X, \rho)$.

Remark 2: For $\rho(t) \equiv 1, WPAP_{(\lambda,\vartheta)}(R \times X, X, \rho) = PAP_{(\lambda,\vartheta)}(R \times X, X)$.

Lemma 2.1 ([T. Diagana, 2006]) Fix $\rho \in U_\infty$, the decomposition of a weighted pseudo-almost periodic function f is unique: $f = g + \phi$ where $g \in AP(R \times X, Y)$ and $\phi \in PAP_0(R \times X, Y, \rho)$.

Lemma 2.2 ([T. Diagana, 2006]) Fix $\rho \in U_\infty$, the decomposition of a (λ, ϑ) weighted pseudo-almost periodic function f is unique: $f = g + \phi$ where $g \in AP_\lambda(R^2 \times X, X)$ and $\phi \in PAP^0_\vartheta(R^2 \times X, X, \rho)$.

Lemma 2.3 ([C.Y. Zhang, 1994]) If $f \in WPAP(R \times X, X, \rho)$, then $f(t, x)$ is bounded on $R \times D$, where D is any compact subset of X.

Lemma 2.4 ([T. Diagana, 2006]) If $f \in WPAP(R, X, \rho)$ satisfies the Lipschitz condition:

$$\|f(t, x) - f(t, y)\|_X \leq L_f \|x - y\|_X, \ for\ all\ x, y \in X, t \in R,$$

and $\phi \in WPAP(R, X, \rho)$, then $f(\cdot, \phi(\cdot)) \in WPAP(R, X, \rho)$.

Lemma 2.5 ([X.X. Chen, 2011]) If $f, g \in WPAP(R, X, \rho)$, then $f \pm g \in WPAP(R, X, \rho)$.

Lemma 2.6 Assume that

(i) The continuously differentiable functions $h_i : R \to R$ satisfy the following assumptions:
$h_i^{'}(t) > 0 (i = 1, 2)$ is nondecreasin and $f(h_i(\cdot)) \in AP(R, X)(i = 0, 1, 2)$, where $f \in AP(R, X)$;

(ii) The weight $\rho : R \to (0, +\infty)$ is continuous and is nondecreasing with

$$\lim_{r \to +\infty} \sup [\frac{m(|h(-r)| + |h(r)|, \rho)\rho(r)}{m(r, p)h^{'}(-r)\rho(h^{'}(-r))}] < +\infty,$$

then for any $u \in WPAP(R, X, \rho), u(h_i(\cdot)) \in WPAP(R, X, \rho)(i = 1, 2)$

Proof. Let $h = h_i(i = 1, 2)$. By Lemma 2.1 and consider the decomposition:

$$u(\cdot) = v(\cdot) + w(\cdot), \ where\ v(\cdot) \in AP(R, X)\ and\ w(\cdot) \in PAP_0(R, X, \rho).$$

On one hand, from condition (i), $v(h(t)) \in AP(R, X)$ is obvious. On the other hand, for $r > 0$, we have

$$\begin{aligned}
\frac{1}{m(r,\rho)} \int_{-r}^{r} \|w(h(t))\|\rho(t)dt &= \frac{1}{m(r,\rho)} \int_{-r}^{r} \|w(h(t))\| \frac{\rho(h(t))}{\rho(h(t))} \frac{h^{'}(t)}{h^{'}(t)} \rho(t)dt \\
&\leq \frac{\rho(r)}{h^{'}(-r)\rho(h^{'}(-r))} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|w(h(t))\|\rho(h(t))h^{'}(t)dt \\
&\leq \frac{\rho(r)}{h^{'}(-r)\rho(h^{'}(-r))} \frac{1}{m(r,\rho)} \int_{-|h(-r)|}^{|h(r)|} \|w(h(t))\|\rho(t)dt \\
&\leq \frac{\rho(r)}{h^{'}(-r)\rho(h^{'}(-r))} \frac{m(h|(-r)|+|h(r)|,\rho)}{m(r,\rho)} \\
&\quad \times \frac{1}{m(h|(-r)|+|h(r)|,\rho)} \int_{-(|h(-r)|+|h(r)|)}^{|h(-r)|+|h(r)|} \|w(t)\|\rho(t)dt
\end{aligned}$$

which converges to zero when $r \to +\infty$. Thus $w(h(t)) \in PAP_0(R, X, \rho)$. The proof is complete.

3. Main results

In this section, we require the following assumptions:

(H1) The function $F : R \times R^{2n} \to R^{2n}$ is weighted pseudo-almost periodic satisfying for a constant $L \in (0, 1)$ such that

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq L(|(x_1 - x_2| + |y_1 - y_2|), \ t \in R, \ x_j, y_j \in R^n.$$

(H2) For $i = 1, 2$, there exists $\mu_i = \mu_i(t, s)$ such that for $t, s \in R, x_i, y_i \in R^n, G_i$ satisfies the Lipschitz condition:

$$|G_i(t, s, x_1, y_1) - G_i(t, s, x_2, y_2)| \leq \mu_i(|(x_1 - x_2| + |y_1 - y_2|)$$

where $\int_{-\infty}^{t} \mu_1(t, s)ds + \int_{t}^{+\infty} \mu_2(t, s)ds \leq \mu$, for $t \in R$.

(H3) For $i = 1, 2$, the functions G_i are (λ_i, θ_i) weighted pseudo-almost in $t, s \in R$ uniformly if $(x, y) \in R^{2n}$, that is, we have decomposition: $G_i = Q_1^i + Q_2^i$ with $Q_1^i \in AP_{\lambda_i}(R^2 \times R^{2n}, R^n)$ and $Q_2^i \in PAP_{\theta_i}^0(R^2 \times R^{2n}, R^n, \rho)$ i.e. $|Q_2^i(t, s, x, y)| \leq \theta_i(t, s)\widehat{Q_2^i}(s, x, y)$.

(H4) For some constants $\alpha, \beta > 0, \ i = 1, 2$, the functions $\lambda_i, \vartheta_i : R^2 \to [0, +\infty)$ satisfy:

$$\int_{-\infty}^{t} \lambda_1(t, s)ds + \int_{t}^{+\infty} \lambda_2(t, s) \leq \alpha, \ t \in R; \tag{3.1}$$

$$\int_{s}^{r} \vartheta_1(t, s)dt + \int_{-r}^{s} dt + \int_{-r}^{s} \vartheta_2(t, s)dt \leq \beta, \ for\ |s| \leq r; \tag{3.2}$$

$$\lim_{r \to \infty} \sup_{-r \leq t \leq r} \int_{r}^{+\infty} \vartheta(t, s)\frac{g(s)}{g(t)}ds < \infty, \ where\ 0 < g(x) \in PAP_0(R, \rho). \tag{3.3}$$

(H5) The conditions of the Lemma 2.6 about the functions $h_i(i = 0, 1, 2)$ and the weight function ρ hold.

Define the nonlinear operator:

$$\Phi(u)(t) = F(t, u(t), u(h_0(t))) + \int_{-\infty}^{t} G_1(t, s, u(s), u(h_1(s)))ds + \int_{t}^{+\infty} G_2(t, s, u(s), u(h_1(s)))ds.$$

Lemma 3.1 If the conditions (H1)-(H5) hold, then the operator Φu is weighted pseudo-almost periodic for u weighted pseudo-almost periodic.

Proof. For $u(t)$ being weighted pseudo-almost periodic, from (H1) and (H5), Lemma 2.4 and 2.6, we see that $F(t, u(t), u(h_0(t))), G_i(t, u(t), u(h_i(t))), (i = 1, 2)$ are all weighted pseudo-almost periodic.

Now we only show that $T_2 := \int_t^{+\infty} G_2(t, s, u(s), u(h_2(s)))ds$ is weighted pseudo-almost periodic. For $T_1 := \int_{-\infty}^{t} G_1(t, s, u(s), u(h_1(s))ds$ is weighted pseudo-almost periodic, the proof is similar. Set $G_2 = G$ and $T_2 = T$. From (H1) and (H4), we have the decomposition:

$$G = Q_1 + Q_2, Q_1 \in AP_\lambda(R^2 \times R^{2n}, R^n) \ and \ Q_2 \in PAP_\vartheta^0(R^2 \times R^{2n}, R^n, \rho),$$

where $\lambda, \vartheta : R^2 \to (0, +\infty)$ satisfy (3.1), (3.2) and (3.3). Then

$$M_1(u)(t) := \int_t^{+\infty} Q_1(t, s, u(s), u(h_2(s)))ds$$

is the almost periodic component of $Tu(t)$ and its ergodic component is

$$M_2(u)(t) := \int_t^{+\infty} Q_2(t, s, u(s), u(h_2(s)))ds$$

By [M.Pinto, 2010], $M_1(u) \in AP(R)$. Next, we show that $M_2(u) \in PAP^0(R, \rho)$. From (H3) and (H4), the ergodic component Q_2 satisfies:

$$Q_2(t, s, x, y) \le \vartheta(t, s)\widehat{Q}_2(s, x, y),$$

where

$$\int_{-r}^{s} \vartheta(t, s)dt \le \beta \ for \ |s| \le r,$$

and

$$\lim_{r\to\infty} \sup_{-r\le t\le r} \int_r^{+\infty} \vartheta(t, s)\frac{g(s)}{g(t)}ds < \infty, \ with \ 0 < g(x) \in PAP_0(R, \rho),$$

as well as

$$\widehat{Q}_2(t, u(t), u(h_2(t))) \in PAP^0(R, \rho),$$

hence we have

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \widehat{Q}_2(s, u(s), u(h_2(s)))\rho(s)ds = 0.$$

Finally we show that

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |M_2(u)(t)|\rho(t)dt = 0. \tag{3.4}$$

In fact

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |M_2(u)(t)|\rho(t)dt \le l_1 + l_2$$

where

$$l_1 := \lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt(|\int_t^{r} Q_2(t, s, u(s), u(h_2(s))\rho(t)ds|)$$

and

$$l_2 := \lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt(|\int_r^{+\infty} \vartheta(t, s)\widehat{Q}_2(s, u(s), u(h_2(s))\rho(t)ds|).$$

Since

$$\begin{aligned}
l_1 &\le \lim_{r\to+\infty} \tfrac{1}{m(r,\rho)} \int_{-r}^{r} dt(|\int_t^{r} \vartheta(t, s)\widehat{Q}_2(s, u(s), u(h_2(s)))\rho(t)ds|)\\
&\le \lim_{r\to+\infty} \tfrac{1}{m(r,\rho)} \int_{-r}^{r} dt(|\int_t^{r} \vartheta(t, s)\widehat{Q}_2(s, u(s), u(h_2(s)))\rho(s)ds|)\\
&\le \lim_{r\to+\infty} \tfrac{1}{m(r,\rho)} \int_{-r}^{r} \widehat{Q}_2(s, u(s), u(h_2(s)))\rho(s)ds(\int_{-r}^{s} \vartheta(t, s)dt)\\
&\le \beta \lim_{r\to+\infty} \tfrac{1}{m(r,\rho)} \int_{-r}^{r} \widehat{Q}_2(s, u(s), u(h_2(s)))\rho(s)ds = 0
\end{aligned}$$

and

$$\begin{aligned} l_2 &\leq \lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}dt(|\int_{r}^{+\infty}\vartheta(t,s)\widehat{Q}_2(s,u(s),u(h_2(s)))\rho(t)ds|) \\ &\leq \lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\widehat{Q}_2(t,u(t),u(h_2(t)))\rho(t)dt \\ &\quad \times(|\int_{r}^{+\infty}\vartheta(t,s)\frac{\widehat{Q}_2(s,u(s),u(h_2(s)))}{\widehat{Q}_2(t,u(t),u(h_2(t)))}ds|) = 0, \end{aligned}$$

so (3.4) follows. From Lemma 2.5, we can obtain that $\Phi(u) \in WPAP(R, X, \rho)$ for $u \in WPAP(R, X, \rho)$. Then the proof of Lemma 3.1 is complete.

Theorem 3.2 Under assumptions from (H1) to (H5) with $2(L+\mu) < 1$, then the neutral integral equation (1.3) has a unique weighted pseudo-almost periodic solution.

Proof. For $u \in WPAP(R,\rho)$, from Lemma 3.1, we have $\Phi u(t) \in WPAP(R,\rho)$, now we will obtain that $\Phi : WPAP(R,\rho) \to WPAP(R,\rho)$ has a unique fixed point. For $u, v \in WPAP(R,\rho)$, since

$$\begin{aligned} |\Phi(u)(t) - \Phi(u)(t)| &\leq 2L\|u-v\|_\infty + \int_{-\infty}^{t}|G_1(t,s,u(s),u(h_1(s))) - G_1(t,s,v(s),v(h_1(s)))|ds \\ &\quad + \int_{t}^{+\infty}|G_2(t,s,u(s),u(h_1(s))) - G_2(t,s,v(s),v(h_1(s)))|ds \\ &\leq 2L\|u-v\|_\infty + \int_{-\infty}^{t}\mu_1(t,s)(|u(s)-v(s)| + |u(h_1(s)) - v(h_1(s))|)ds \\ &\quad + \int_{t}^{+\infty}\mu_2(t,s)(|u(s)-v(s)| + |u(h_2(s)) - v(h_2(s))|)ds \\ &\leq 2(L+\mu)\|u-v\|_\infty. \end{aligned}$$

Since $2(L+\mu) < 1$, the operator Φ is a contraction and has a unique fixed point, which is the only weighted pseudo-almost periodic solution to the integral equation (1.3). Then the proof is complete.

Remark 3: For $u \in PAP(R)$ and $\rho \equiv 1$, the integral equation (1.3) has a unique pseudo-almost periodic solution, so the results of this paper extend some present works' results in [M.Pinto, 2010]. Finally, we can consider a general neutral differential equation

$$\frac{dy}{dx} = A(t)y + \frac{d}{dt}F(t,y(t),y(h_0(t))) + Q(t,y(t),y(h_1(t))), \tag{3.5}$$

where $\frac{dx}{dt} = A(t)x$ admits exponential dichotomy and the function $F(t,y(t),y(h_0(t)))$ is differentiable. As we all know, if $X(t)$ is the fundamental matrix of $\frac{dx}{dt} = A(t)x$, any solution of the integral equation

$$y(t) = F(t,y(t),y(h_0(t))) + \int_{-\infty}^{+\infty}G(t,s)(A(s)F(s,y(s),y(h_0(s))) + Q(s,y(s),y(h_1(s)))ds \tag{3.6}$$

is solution of neutral differential equation (3.5), where G is the Green matrix of the linear system $\frac{dx}{dt} = A(t)x$. Considering $A(t) = A$ constant, we have

Theorem 3.3 If the conditions of the Lemma 2.6 about the functions $h_i (i = 0, 1, 2)$ and the weight ρ and the following conditions (C1)-(C2) are held with $2L + (L_A + L_Q)\mu < 1$:

(C1) The eigenvalues λ of the constant A satisfy $Re\lambda \neq 0$ and the Green operator has the norm $\sup_{t\in R}\int_{-\infty}^{+\infty}|G(t,s)|ds = \mu < \infty$.

(C2) $Q = Q(t,u,v)$ and $F_A(t,u,v) = A(t)F(t,u,v)$ are weighted pseudo-almost and for all $t \in R, u_i, v_i \in R^n,\ i = 1,2$ satisfy

$$|Q(t,u_1,v_1) - Q(t,u_2,v_2)| \leq L_Q(|u_1-u_2| + |v_1-v_2|),$$

where $L_Q > 0$ constant,

$$|A(t)(F(t,u_1,v_1) - F(t,u_2,v_2))| \leq L_A(|u_1-u_2| + |v_1-v_2|),$$

where $L_A > 0$ constant.

Then the neutral differential equation (3.5) has a unique weighted pseudo-almost periodic solution.

Proof: From the solution's relation of between equation (3.5) and (3.6), we can easily have the result of the Theorem 3.3 at once from the Theorem 3.2.

Remark 4: In [X.X. Chen, 2011], the workers obtained a weighted pseudo-almost periodic solution of the system£ $\frac{du(t)}{dt} = A(t)u(t) + \frac{d}{dt}F_1(t,u(t-\tau)) + F_2(t,u(t),u(t-\tau))$, which is a special case of the system (3.5). So by the conclusion of Theorem 3.3, the result in [X.X. Chen, 2011] is at once obtained.

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References

A. M. Fink. (1974). Almost periodic differential equations, *Lecture Notes on Mathematics,* vol. 377, Spinger-Verlag, Berlin.

C. Y. He. (1992). *Almost Periodic Differential Equations*, Higher Education Publishing House, Beijing, (in Chinese).

C. Y. Zhang. (1994). Pseudo almost periodic solutions of some differential equations, *J.Math Appl.,* 181(1), 62-67.

G. Tr. Stamov & J. O. Alzabut. (2010). Almost periodic solutions for abstract impulsive differential equations, *Nonlinear Analysis,* 72, 2475-2464.

H.-X. Li, F.-L.Huang & J.-Y. Li. (2001). Composition of pseudo almost-periodic functions and semilinear differential equations, *J.Math.Anal.,* 255(2), 436-446. http://dx.doi.org/10.1006/jmaa.2000.7225

J. K. Hale. (1977). *Theory of Functional Differential Equations*, Springer-Verlag, New York.

J. O. Alzabut, G. T. Stamov & E. Sermutlu. (2011). Positive almost periodic solutions for a delay logarithmic population model, *Mathematical and Computer Modeling,* 53, 161-167. http://dx.doi.org/10.1016/j.mcm.2010.07.029

L. Maniar & R. Schnaubelt. (2003). Almost periodicity of inhomogeneous parabolic evolution equations, *Lecture Notes in Pure and Appl.Math,* vol234, Dekker, New York, pp.299-318.

M. Pinto. (2010). Pseudo-almost periodic solutions of neutral integral and differential equations with applications, *Nonlinear Analysis,* 72, 4377-4383. http://dx.doi.org/10.1016/j.na.2009.12.042

S. Abbas & D. Ba. (2008). Almost periodic solutions of neutral functional differential equations. *Comput.Math.Appl.,* 55, 2539-2601. http://dx.doi.org/10.1016/j.camwa.2007.10.011

T. Diagana. (2006). Weighted pseudo almost periodic functions and applications, *C.R.Acad.Sci.Paris,* 343(10), 643-646. http://dx.doi.org/10.1016/j.crma.2006.10.008

T. Diagana. (2009). Existence of weighted pseudo almost periodic solutions to some classes of hyperbolic evolution equations, *J.Math.Anal.Appl.,* 350, 18-28. http://dx.doi.org/10.1016/j.jmaa.2008.09.041

X. X Chen & F. X Lin. (2010). Almost periodic solutions of neutral differential equations, *Nonlinear Analysis,* 11, 1182-1189. http://dx.doi.org/10.1016/j.nonrwa.2009.02.010

X. X. Chen & Xiangyang Hu. (2011). Weighted pseudo almost periodic solutions of neutral functional differential equations, *Nonlinear Analysis Real World Appl.,* 12, 601-610. http://dx.doi.org/10.1016/j.nonrwa.2010.07.003

X. X. Chen. (2007). Periodicity in a nonlinear discrete predator-prey system with state dependent delays.*Nonlinear Analysis Real World Appl.,* 8, 435-446. http://dx.doi.org/10.1016/j.nonrwa.2005.12.005

Y. H. Wang & Y. H. Xia. (2011). The existence of almost periodic solutions of a certain nonlinear system, *Commun Nonlinear Sci Numer Simulat,* 16, 1060-1072. http://dx.doi.org/10.1016/j.cnsns.2010.05.003

Z.-H. Zhao, Y. K. Chang & W. S. Li. (2010). Asymptotically almost periodic, almost periodic and pseudo-almost periodic mild solutions for neutral differential equations, *Nonlinear Analysis,* 11, 3037-3044. http://dx.doi.org/10.1016/j.nonrwa.2009.10.024

Z. Li, F. Chen. (2009). Almost periodic solutions of a discrete almost periodic logistic equation, *Math .Comput. Modelling,* 50, 254-259. http://dx.doi.org/10.1016/j.mcm.2008.12.017