Study on Integral Operators by Using Komato Operator on a New Class of Univalent Functions

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Abstract

Let $\mathbb{T}$ be the class of functions $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. By using Komato operator $\mathcal{K}_c^{\delta}(f)$, we introduce a new subclass $\mathbb{T}_c^{\delta}(\alpha,\beta)$, whose elamants satisfying in

$$Re\{\frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'}\} > \alpha|\frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'} - 1| + \beta,$$

and we study linear combination and derive some interesting properties for the class $\mathbb{T}_c^{\delta}(\alpha,\beta)$. Also, we study on some integral operators on $\mathbb{T}_c^{\delta}(\alpha,\beta)$.

Keywords: Univalent, Starlike, Convex, Komato operator, Close-to-convex

1. Introduction

Let $\mathbb{A}$ denote the class of functions analytic in the unit disk $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathbb{T}$ denotes the subclass of $\mathbb{A}$ consisting univalent functions of the from

$$f(z) = \ z - \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk U.

Definition 1.1. A function $f(z)$ in $\mathbb{T}$ is said to be in $\mathbb{T}_c^{\delta}(\alpha,\beta)$ if

$$Re\{\frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'}\} > \alpha|\frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'} - 1| + \beta, \tag{2}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$,$c \geq -1$and $\delta > 0$ and operator $\mathcal{K}_c^{\delta}(f)$ is the Komato operator (Komato, 1990, p141-145) defined by

$$\mathcal{K}_c^{\delta}(f) = \int_0^1 \frac{(c+1)^{\delta}}{\Gamma(\delta)} t^c (\log\frac{1}{t})^{\delta-1} \frac{f(tz)}{t} dt. \tag{3}$$

By applying a simple calculation for $f \in \mathbb{T}$ we get

$$\mathcal{K}_c^{\delta}(f) = z - \sum_{k=2}^{\infty} B_k(c,\delta) a_k z^k, \tag{4}$$

where $B_k(c,\delta) = (\frac{c+1}{c+k})^{\delta}$.

This class $\mathbb{T}_c^{\delta}(\alpha,\beta)$ contains many well-known classes of analytic functions, for example $\mathbb{T}_c^{0}(0,\beta)$ is the class of starlike functions of order at most $\frac{1}{\beta}$, see (Najafzadeh, 2009, p81-89).

Definition 1.2. A function $f(z) \in \mathbb{T}$ is said to be *starlike of order* η $(0 \leq \eta < 1)$ (Kanas, 2000, p647-657) if and only if $Re\{\frac{zf'(z)}{f(z)}\} > \eta$, $z \in U$. We use $S^*(\eta)$ for the class of starlike functions of order η and S^* for the class of starlike functions, $S^*(0) = S^*$.

Definition 1.3. A function $f(z) \in \mathbb{T}$ is said to be *convex of order* η $(0 \leq \eta < 1)$ (Silverman, 1997, p221-227) if and only if $Re\{1 + \frac{zf''(z)}{f'(z)}\} > \eta$, $z \in U$. we use $K(\eta)$ for the class of convex functions of order η.

Definition 1.4. A function $f(z)$ is called *close-to-convex of order* η $(0 \leq \eta < 1)$ (Tehranchi, 2006, p105-118) if and only if there exists $g \in S^*$ satisfying $Re\{e^{i\theta}\frac{zf'(z)}{g(z)}\} > \eta$, $z \in U$, $-\frac{\Pi}{2} \leq \theta < \frac{\Pi}{2}$; we denote this class by $C(\eta)$.

The family $\mathbb{T}_c^{\delta}(\alpha,\beta)$ is of special interest for it contains many well-known as well as new classes of analytic univalent functions. This family is reviewed by S. Najafzadeh, A. Ebadian (Najafzadeh, 2009, p81-89), and in other family with other result, A. Tehranchi, S.R. Kulkarni (Tehranchi, 2006, p105-118).

2. Main Results

We need the following elementary lemmas.

Lemma 2.1. *If $\alpha \geq 0, 0 \leq \beta < 1$ and $\gamma \in \mathbb{R}$, then $Rew > \alpha|w-1| + \beta$ if and only if $Re[w(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}] > \beta$ where w is any complex number.*

Lemma 2.2. *With the same condition in Lemma 2.1, $Rew > \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$.*

The proof of the following result, which is given in (Najafzadeh, 2009, p81-89), needs some corrections and is given for the convenience of the reader.

Theorem 2.3. *Let $f \in \mathbb{T}$, then f is in $\mathbb{T}_c^{\delta}(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha) - k(\alpha+\beta)]}{1-\beta} B_k(c,\delta) a_k < 1, \tag{5}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$,$c \geq -1$ and $\delta > 0$ and $B_k(c,\delta) = (\frac{c+1}{c+k})^{\delta}$.

Proof. Let (5) holds, we will show that (2) is satisfied and so $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$. By Lemma 2.2 it is enough to show that

$$|w - (1 + \alpha|w-1| + \beta)| < |w + (1 - \alpha|w-1| - \beta)|,$$

where $w = \frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_C^{\delta}(f)]'}$, and $B = \frac{z[\mathcal{K}_c^{\delta}(f)]'}{|z[\mathcal{K}_c^{\delta}(f)]'|}$ and by using (4) we may write

$$L < \frac{|z|}{|z[\mathcal{K}_c^{\delta}(f)]'|}[\beta + \sum_{k=2}^{\infty}(k - (1+\alpha) + k(\alpha+\beta))B_k(c,\delta)a_k]$$

$$< \frac{|z|}{|z[\mathcal{K}_c^{\delta}(f)]'|}[2 - \beta - \sum_{k=2}^{\infty}[k + 1 + \alpha - (\alpha+\beta)k]B_k(c,\delta)a_k] < R,$$

where $L = |w - (1 + \alpha|w-1| + \beta)|$, $R = |w + (1 - \alpha|w-1| - \beta)|$ and it is easy to verify that $R - L > 0$. Therefore $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$.

Conversely, suppose that $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$. By Lemma 2.1 and letting $w = \frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_C^{\delta}(f)]'}$ in (2)

we obtain $Re(w(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$

or

$$Re[\frac{z-\sum_{k=2}^{\infty}B_k(c,\delta)a_kz^k}{z(1-\sum_{k=2}^{\infty}kB_k(c,\delta)a_kz^{k-1})}(1+\alpha e^{i\gamma})-\alpha e^{i\gamma}-\beta]>0$$

then

$$Re\{\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)B_k(c,\delta)a_kz^{k-1}-\alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)B_k(c,\delta)a_kz^{k-1}}{(1-\sum_{k=2}^{\infty}kB_k(c,\delta)a_kz^{k-1})}\}>0,$$

for all $z\in U$. Letting $z\longrightarrow 1^-$ yields

$$Re\{\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)B_k(c,\delta)a_k-\alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)B_k(c,\delta)a_k}{(1-\sum_{k=2}^{\infty}kB_k(c,\delta)a_k)}\}>0$$

and so by the mean value theorem we have

$$Re\{1-\beta-\sum_{k=2}^{\infty}[(1-\beta k)+\alpha(1-k)]B_k(c,\delta)a_k\}>0.$$

Thus

$$1-\beta-\sum_{k=2}^{\infty}[(1-\beta k)+\alpha(1-k)]B_k(c,\delta)a_k>0$$

or

$$\sum_{k=2}^{\infty}[(1-\beta k)+\alpha(1-k)]B_k(c,\delta)a_k<1-\beta.$$

Therefore

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}B_k(c,\delta)a_k<1$$

and the proof is complete. □

Corollary 2.4. *Let $f\in\mathbb{T}_c^{\delta}(\alpha,\beta)$, then*

$$a_k<\frac{1-\beta}{[(1+\alpha)-k(\alpha+\beta)]B_k(c,\delta)},\qquad k=2,3,4,...\cdot$$

Definition 2.5. Let $J\in\{1,2,...,m\}$, $f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k\in\mathbb{T}_c^{\delta}(\alpha,\beta)$. Then the linear combination function $F(z)$ is defined by $F(z)=\sum_{j=1}^{m}p_jf_j(z)$ such that $\sum_{j=1}^{\infty}p_j=1$.

Theorem 2.6. *The function $F(z)$ defined upon belongs to $\mathbb{T}_c^{\delta}(\alpha,\beta)$.*
Proof. We have

$$F(z)=\sum_{j=1}^{m}p_j(z-\sum_{k=2}^{\infty}a_{k,j}z^k)=z-\sum_{k=2}^{\infty}(\sum_{j=1}^{m}p_ja_{k,j})z^k.$$

Thus

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}B_k(c,\delta)(\sum_{j=1}^{m}p_ja_{k,j})=$$

$$\sum_{j=1}^{m}p_j[\sum_{k=2}^{\infty}\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)a_{k,j}]\le\sum_{j=1}^{m}p_j=1.$$

This shows that $F(z)\in\mathbb{T}_c^{\delta}(\alpha,\beta)$ and so the proof is completed. □

Theorem 2.7. *Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ belong to $\mathbb{T}_c^{\delta}(\alpha,\beta)$. Then the function $G(z) = z - \sum_{k=2}^{\infty}(a_k^2 + b_k^2)z^k$ is in $\mathbb{T}_{c_1}^{\delta}(\alpha,\beta)$, where*

$$c_1 \le \inf_k\left[\frac{\left(\frac{(1+\alpha)-k(\alpha+\beta)}{2(1-\beta)}\right)^{\frac{1}{\delta}}\left(\frac{c+1}{c+k}\right)^2 - 1}{1 - \left(\frac{(1+\alpha)-k(\alpha+\beta)}{2(1-\beta)}\right)\left(\frac{c+1}{c+k}\right)^2}\right].$$

Proof. Since $f, g \in \mathbb{T}_c^{\delta}(\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty}\left[\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}\right]^2\left(\frac{c+1}{c+k}\right)^{2\delta}a_k^2 \le \sum_{k=2}^{\infty}\left[\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)a_k\right]^2 < 1$$

$$\sum_{k=2}^{\infty}\left[\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}\right]^2\left(\frac{c+1}{c+k}\right)^{2\delta}b_k^2 \le \sum_{k=2}^{\infty}\left[\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)b_k\right]^2 < 1.$$

Thus

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)\right]^2(a_k^2 + b_k^2) < 1.$$

Now we must show

$$\sum_{k=2}^{\infty}\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c_1,\delta)(a_k^2 + b_k^2) < 1.$$

This inequality holds if

$$\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c_1,\delta) \le \frac{1}{2}\left(\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)\right)^2$$

or

$$B_k(c_1,\delta) \le \frac{1}{2}\left(\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}\right)\left(\frac{c+1}{c+k}\right)^{2\delta}.$$

Therefore it is enough

$$c_1 \le \frac{\left(\frac{(1+\alpha)-k(\alpha+\beta)}{2(1-\beta)}\right)^{\frac{1}{\delta}}\left(\frac{c+1}{c+k}\right)^2 - 1}{1 - \left(\frac{(1+\alpha)-k(\alpha+\beta)}{2(1-\beta)}\right)\left(\frac{c+1}{c+k}\right)^2}$$

and this gives the result. □

3. Study on some of Integral operators on $\mathbb{T}_c^{\delta}(\alpha,\beta)$

Definition 3.1. Let $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$, the function $F_\mu(z)$ is defined by

$$F_\mu(z) = (1-\mu)z + \mu\int_0^z \frac{f(t)}{t}dt, \tag{6}$$

that $\mu \ge 0$, $z \in$U if $0 \le \mu \le 2$.

Next we investigate some of properties of the function $F_\mu(z)$ in the class $\mathbb{T}_c^{\delta}(\alpha,\beta)$.

Theorem 3.2. *The function $F_\mu(z)$ defined upon belongs to $\mathbb{T}_c^{\delta}(\alpha,\beta)$.*

Proof. Let $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$, we have

$$F_\mu(z) = (1-\mu)z + \mu\left[\int_0^z dt - \sum_{k=2}^{\infty}\int_0^z a_k t^{k-1}dt\right] = z - \sum_{k=2}^{\infty}\frac{1}{k}\mu a_k z^k.$$

Therefore, since $f(z) \in \mathbb{T}_c^{\delta}(\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}B_k(c,\delta)\frac{1}{k}\mu a_k < \sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}B_k(c,\delta)\frac{\mu}{2} < 1.$$

So the proof is completed. □

Theorem 3.3. *The function $F_\mu(z)$ is starlike of order $\eta(0 \le \eta \le 1)$ in $|z| < r_1(\alpha,\beta,\eta)$, if*

$$r_1(\alpha,\beta,\eta) = \inf_k\{\frac{1-\eta}{\mu(1+k-k\eta-\frac{1}{k})}\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)\}^{\frac{1}{k-1}}.$$

Proof. For $r_1(\alpha,\beta,\eta)$, we must show that $|\frac{zF'_\mu}{F_\mu}-1| < 1-\eta$, or show that

$$|\frac{\sum_{k=2}^\infty z^k a_k(\frac{1}{k}-1)\mu}{z-\sum_{k=2}^\infty \frac{\mu a_k}{k}z^k}| < \frac{\sum_{k=2}^\infty |z|^{k-1}a_k(\frac{1}{k}-1)\mu}{1-\sum_{k=2}^\infty \frac{\mu a_k}{k}|z|^{k-1}} < 1-\eta,$$

$$\sum_{k=2}^\infty |z|^{k-1}a_k\mu\frac{(1+k-k\eta-\frac{1}{k})}{1-\eta} < 1.$$

Therefore it is enough, by Theorem 2.3 and Corollary 2.4, letting

$$|z|^{k-1} < \frac{(1-\eta)[(1+\alpha)-k(\alpha+\beta)]}{\mu(1+k-k\eta-\frac{1}{k})(1-\beta)}B_k(c,\delta).$$

□

Theorem 3.4. *The function $F_\mu(z)$ is convex of order $\eta(0 \le \eta \le 1)$ in $|z| < r_2(\alpha,\beta,\eta)$, if*

$$r_2(\alpha,\beta,\eta) = \inf_k\{\frac{1-\eta}{\mu(k-\eta)}\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)\}^{\frac{1}{k-1}}.$$

Proof. For $r_2(\alpha,\beta,\eta)$, we must show that

$$\left|\frac{zF''_\mu(z)}{F'_\mu(z)}\right| < 1-\eta$$

or

$$\left|\frac{\sum_{k=2}^\infty (k-1)\mu a_k z^{k-1}}{1-\sum_{k=2}^\infty \mu a_k z^{k-1}}\right| \le \frac{\sum_{k=2}^\infty (k-1)\mu a_k|z|^{k-1}}{1-\sum_{k=2}^\infty \mu a_k|z|^{k-1}} < 1-\eta,$$

$$\sum_{k=2}^\infty (k-1)\mu a_k|z|^{k-1} + (1-\eta)\sum_{k=0}^\infty \mu a_k|z|^{k-1} \le 1-\eta,$$

$$\sum_{k=0}^\infty \mu a_k|z|^{k-1}\frac{(k-\eta)}{1-\eta} \le 1.$$

Therefore it is enough, by Theorem 2.3 and Corollary 2.4, that

$$|z|^{k-1} < \frac{1-\eta}{\mu(k-\eta)}\frac{(1+\alpha)-k(\alpha+\beta)}{1-\beta}B_k(c,\delta)\ .$$

□

Theorem 3.5. *If $f \in \mathbb{T}_c^0(0,\beta) = S^*(\beta)$ and $\frac{f(z)}{z} \neq 0$ also $0 \le \mu < 1$, then $F_\mu(z)$ is close-to-convex of order μ.*

Proof. We have $F'_\mu(z) = (1-\mu) + \mu\frac{f(z)}{z}$ so $\frac{zF'_\mu(z)}{f(z)} = \mu + \frac{(1-\mu)z}{f(z)}$. Then

$$Re(\frac{zF'_\mu(z)}{f(z)}) = \mu + (1-\mu)Re(\frac{z}{f(z)}) > \mu.$$

This shows that $F_\mu(z)$ is close-to-convex of order μ. □

Definition 3.6. Let $f \in \mathbb{T}_c^\delta(\alpha,\beta)$, then we define, for every $\gamma(0 \le \gamma < 1)$, the function $H_\gamma(z)$, by

$$H_\gamma(z) = (1-\gamma)f(z) + \gamma\int_0^z \frac{f(t)}{t}dt. \tag{7}$$

Theorem 3.7. *The function $H_\gamma(z)$ defined upon belongs to $\mathbb{T}_c^\delta(\alpha,\beta)$.*
Proof. Let $f(z) \in \mathbb{T}_c^\delta(\alpha,\beta)$, then we have

$$H_\gamma(z) = z - \sum_{k=2}^{\infty}(1 + \frac{\gamma}{k} - \gamma)a_k z^k.$$

Now, since $(1 + \frac{\gamma}{k} - \gamma) < 1, k \geq 2$, therefore by (4), we have

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha) - k(\alpha+\beta)]}{1-\beta} B_k(c,\delta)(1 + \frac{\gamma}{k} - \gamma)a_k < 1.$$

So we obtain $H_\gamma(z) \in \mathbb{T}_c^\delta(\alpha,\beta)$. □

Theorem 3.8. *The function $H_\gamma(z)$ is starlike of order $\gamma(0 \leq \gamma \leq 1)$ in $|z| < r_1(\alpha,\beta,\gamma)$, if*

$$r_1(\alpha,\beta,\gamma) = \inf_k\{\frac{(1-\gamma)[(1+\alpha) - k(\alpha+\beta)]}{(1-\beta)} B_k(c,\delta)\}^{\frac{1}{k-1}}.$$

Proof. For $r_1(\alpha,\beta,\gamma)$ we must show that $|\frac{zH'_\gamma}{H_\gamma} - 1| < 1 - \eta$,
or

$$|\frac{\sum_{k=2}^{\infty} z^k a_k(\frac{\gamma}{k} - \gamma)}{z - \sum_{k=2}^{\infty}(1 + \frac{\gamma}{k} - \gamma)a_k z^k}| < \frac{\sum_{k=2}^{\infty} |z|^{k-1} a_k(\gamma - \frac{\gamma}{k})}{1 - \sum_{k=2}^{\infty}(1 + \frac{\gamma}{k} - \gamma)a_k|z|^{k-1}} < 1 - \gamma,$$

or

$$\sum_{k=2}^{\infty} |z|^{k-1} \frac{a_k}{1-\gamma} < 1.$$

Therefore it is enough, by Theorem 2.3 and Corollary 2.4, letting

$$|z|^{k-1} < \frac{(1-\gamma)[(1+\alpha) - k(\alpha+\beta)]}{(1-\beta)} B_k(c,\delta).$$

□

Theorem 3.9. *The function $H_\gamma(z)$ is convex of order $\gamma(0 \leq \gamma \leq 1)$ in $|z| < r_2(\alpha,\beta,\gamma)$, if*

$$r_2(\alpha,\beta,\gamma) = \inf_k\{\frac{1-\eta}{\mu(k-\eta)} \frac{(1+\alpha) - k(\alpha+\beta)}{1-\beta} B_k(c,\delta)\}^{\frac{1}{k-1}}.$$

Proof. For $r_2(\alpha,\beta,\gamma)$, we must show that

$$\left|\frac{zH''_\gamma(z)}{H'_\gamma(z)}\right| < 1 - \gamma.$$

So

$$\left|\frac{\sum_{k=2}^{\infty} k(k-1)(1 + \frac{\gamma}{k} - \gamma)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k(1 + \frac{\gamma}{k} - \gamma)a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)(1 + \frac{\gamma}{k} - \gamma)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} k(1 + \frac{\gamma}{k} - \gamma)a_k|z|^{k-1}} < 1 - \gamma,$$

$$\sum_{k=2}^{\infty} k^2(1 + \frac{\gamma}{k} - \gamma)a_k|z|^{k-1} \leq 1.$$

Therefore it is enough, by Theorem 2.3 and Corollary 2.4,

$$|z|^{k-1} < \frac{1}{k^2(1 + \frac{\gamma}{k} - \gamma)} \frac{(1+\alpha) - k(\alpha+\beta)}{1-\beta} B_k(c,\delta).$$

□

Theorem 3.10. *If $f \in \mathbb{T}_c^0(0,\beta)$ and $\frac{f(z)}{z} \neq 0, z \in U$ also $0 \leq \gamma < 1$ then $H_\gamma(z)$ is close-to-convex of order γ.*
Proof. We have $H'_\gamma(z) = (1-\gamma)f'(z) + \frac{\gamma f(z)}{z}$, so $\frac{zH'_\gamma(z)}{f(z)} = \gamma + (1-\gamma)\frac{zf'(z)}{f(z)}$. Thus

$$Re(\frac{zH'_\gamma(z)}{f(z)}) = \gamma + (1-\gamma)Re(\frac{zf'(z)}{f(z)}) > \gamma.$$

This shows that $H_\gamma(z)$ is close-to-convex of order γ. $\square$

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