# Solution Posedness for a Class of Nonlinear Parabolic Equations with Nonlocal Term

Guilan Liu, Yaping Zhu, Suzhen Huang & Honglei Ji

Department of Basic Teaching

Yancheng Institute Of Technology, Yancheng, Jiangsu Province, 224001

E-mail: yjy2007331@126.com

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**Abstract**

Based on denoising, segmentation and restoration problems of image processing and combined with two-phase flow mathematical theory, this paper proposes a class of nonlinear parabolic equations with nonlocal term. By fixed point theorem, the existence of initial boundary value problem is gotten. And then this paper establishes solution uniqueness and stability about initial value $u_0$ and free term $f$.

**Keywords:** Allen-Cahn equation, Nonlocal term, Initial boundary value problem, Posedness

**Introduction**

Nonlinear parabolic equation is an important class of partial differential equations, which is derived from the widespread nature nonlinear phenomena. Many problems, such as phase transition theory, percolation theory, image processing, biochemistry and other fields can be described by this equation. For example, Cahn-Hilliard equation in fluid mechanics and Allen-Cahn equation in phase transition theory etc. Those equations have clear physical background and extensive application value. Studying them has significant theoretical content for their research and is very necessary.

## 1. A Class of Nonlinear Parabolic Equations with Nonlocal Term

### *1.1 Allen-Cahn Equation*

For describing multidigit flow movement, Allen and Cahn(Allen, S.M. and Cahn, J.W.,1979) introduced the following initial boundary problem of second-order parabolic equation in 1979:

$$\begin{cases} u_t = \varepsilon\Delta u - \frac{1}{\varepsilon}\varphi'(u), t > 0, x \in \Omega \\ \frac{\partial u}{\partial n} = 0, \qquad t > 0, x \in \partial\Omega \\ u|_{t=0} = u_0 \end{cases}, \tag{1}$$

where $\Omega \subseteq R^n$, whose boundary $\partial\Omega$ is smooth, $\overrightarrow{n}$ is the exterior normal vector of region boundary $\Omega$, parameter $\varepsilon > 0$ can be used to indicate the width of two-phase flow transition boundary and $\varphi(u)$ is a double-well potential function, such as

$$\varphi(u) = (u^2 - 1)^2,$$

where $u = \pm 1$ is the equilibrium.

So far, this equation has been many related researches. For any $\varepsilon > 0$, problem (1) is the classic initial value problem of semi-linear parabolic equation and its posedness theory is also very classic, which can be found in papers( Ladyzenskaya,1968-Zhuoqun Wu, 2003) and so on. The first asymptotic analysis for $\varepsilon \to 0$ was proposed by Allen and Cahn(Allen, S.M. and Cahn, J.W,1979), Rubinstein(Rubinstein, 1989) and Fife(Fife, P.C.,1988). Rigorous proof for this asymptotic analysis was given by Bronsard and Kohn(Bronsard, L. and Kohn, R, 1991), De Mottoni and Schatzman(De Mottoni, 1990; De Mottoni, P. and Schatzman, 1990), Chen (Chen, X., 1992), Evans(Evans, L.C., 1992), Ilmanen (Ilmanen, T., 1993) and so on.

### *1.2 Partial Differential Equations Problem in Image Processing*

In recent years, with the rapid development of partial differential equation theory and information science, partial differential equations has a very significant impact on image processing. The thought using partial differential equation in image processing can be traced back to work by Gabor ( Gabor, D.,1965) and the image structure exploration by Koenderink ( Koenderink, J.J.,1984). For image denoising and reconstruction problem, Malik and Perona (Perona, P. ,1990) in 1990 proposed anisotropic diffusion model, whose diffusion coefficient about gradient module $|\nabla u|$ of image gray function is monotonically decreasing. Although in theory the model has made significant improvements, but also improves the filter result. But this model still has shortcomings: If the image is corrupted with noise, such as white noise, $|\nabla u|$ in these noise points may be very large, making the diffusion coefficient small. So when filtering the image, those noise points can be retained, which can lower the denoising performance.

To overcome the shortcomings of Malik-Perona model, Catte, Lions and Morel (Catte, F., Lions, P.L., 1992) proposed the following image selective denoising model in 1992:

$$\begin{cases} u_t = div(F(|\nabla u * G_\sigma|^2)\nabla u), t > 0, x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, \qquad t > 0, x \in \partial\Omega \\ u|_{t=0} = u_0 \end{cases}, \tag{2}$$

where $u$ indicates the grayness of black and white image, $\Omega \subseteq R^2$ is a bounded region(considered image scope), whose boundary $\partial\Omega$ is smooth, $\varphi(u)$ is a double-well potential function and $F$ is monotone decreasing and satisfies

$$F(s) \geq 0, F(0) = 1, \lim_{s\to\infty} F(s) = 0.$$

For example, function $F$ can be taken

$$F(s) = \frac{1}{1 + \frac{s}{k^2}},$$

where $k > 0$ is a parameter. In addition,

$$G_\sigma = \frac{1}{4\pi\sigma} e^{\frac{|x|^2}{4\sigma}}$$

is a Gaussian function and $u_0$ is the original image with noises.

*1.3 Image Denoising and restoration problems based on Allen-Cahn equation*

In recent, people study edge detection, segmentation and restoration in image processing problem using fluid mechanics multidigit flow movement model. Bertozzi, Esedoglu, Gillette(Bertozzi, 2007) proposed image restoration model based on Cahn-Hilliaed equation and achieved good results. But if this model is used to numerical simulate the image, since the fourth-order derivative emerges, the calculation format is more complex and calculation is more time-consuming.

As mentioned earlier, Allen-Cahn equation (1) was proposed by Allen and Cahn in 1979 to describe a class of second order parabolic equation of two-phase transformation boundary movement. Inspired by two-phase movement, this equation can be used in image edge detection and segmentation. Benes, Chalupecky and Mikula (Benes, M, 2004) proposed image segmentation using Allen-Cahn equation. But, as Witkin (Witkin, A.P,1983) said, the Laplace operator also can be used to obfuscation boundary blur of image marginal in edge detection and image denoising process. Combining with Bertozzi's work(Bertozzi, A.L, 2007) etc and starting from the need of image denoising, edge detection and restoration, we propose the following class of initial boundary value problem with nonlocal items on Allen-Cahn equation :

$$\begin{cases} u_t = \varepsilon div(F(|\nabla u * G_\sigma|^2)\nabla u) - \frac{1}{\varepsilon}\varphi'(u) + \lambda \cdot \chi_{\Omega\setminus D}(f - u) \\ \frac{\partial u}{\partial n} = 0, \qquad t > 0, x \in \partial\Omega \\ u|_{t=0} = u_0 \end{cases}, \tag{3}$$

where $\Omega \subseteq R^2$, the boundary $\partial\Omega$ is smooth, $\overrightarrow{n}$ is the exterior normal vector for $\partial\Omega$ and $D \subseteq \Omega$ is the part of losing information. For the right hand selection of equation in problem (3), our motivation is:

The first: $F(\cdot), G_\sigma(\cdot)$ are given in questions (2), where $G_\sigma$ is introduced to denoise image in a certain degree. Selection of $F(\cdot)$ is to achieve anisotropic denoising effect. Exactly as the thought proposed by Catte, Lions and Morel (Catte, F, 1992), it can better maintain image margin in denoising process .

The second: $\varphi(u)$, as in question (1), is a double-well potential function (such as we can select $\varphi(u) = (u^2 - 1)^2$). For convenience of the following discussion, here we assume $\varphi(u)$ satisfies:

$$|\varphi'(u)| \leq C_1|u| + C_2|u|^3, \tag{4}$$

where $C_1, C_2$ are positive constants.

The third is used for image restoration. The restored image agrees to original image as closely as possible in $\Omega \setminus D$, that is, as far as possible without distortion, where $\lambda$ is a constant, $\chi_{\Omega\setminus D}$ is the characteristic function in $\Omega \setminus D$, i.e. $\chi_{\Omega\setminus D} = \begin{cases} 1, x \in \Omega \setminus D, \\ 0, otherwise. \end{cases}$.

Comparing with classical Allen-Cahn model, the biggest difference is that the main item in this equation contains non-local terms:

$$div(F(|\nabla u * G_\sigma|^2)\nabla u).$$

There is some meaningful work about Allen-Cahn equation with non-local term recently. Such as Chen, Hilhorst, and Logak, in order to describe multidigit flow problem, introduced initial boundary problem of nonlinear Allen-Cahn equation with nonlocal term in paper (Chen, X., 1997)(namely, the above double-well potential function with non-local items), and discussed asymptotic condition of the solution when $\varepsilon \to 0$. In paper (Wang, Y.G ,1999) of Yaguang Wang etc., the theory about nonlinear parabolic equation with non-local item was also discussed. But in problem (3), nonlocal term is in the main term of this equation and the above work can not be applied to problem (3), so it is necessary to research the theory of this problem.

## 2. Solution Posedness of Nonlinear Parabolic Equation with Non-local Term

In this section, by energy estimation and Schauder fixed point theorem, solution posedness of nonlinear parabolic equation with non-local term is discussed.

For problem (3), function $\varphi(u)$ satisfies condition (4), we have conclusion:

**Theorem 2.1**(I) For given $\forall\varepsilon > 0, u_0 \in L^2(\Omega), f \in L^2(0, T_0; L^2(\Omega))$, there exists $0 < T \leq T_0$ for which initial boundary problem (3) exists weak solution

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

(II) In bounded solutions, the weak solution of problem (3)

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

is unique, and is stable about initial value $u_0$ and free term $f$.

*2.1 Solution Existence of Initial Boundary Problem (3)*

In this part, we first introduce of the solution existence of initial boundary problem (3).

Introduce space

$$W(0, T) = \{w \in L^2(0, T; H^1(\Omega)), \partial_t w \in L^2(0, T; (H^1(\Omega))'),$$
$$\underset{0\leq t\leq T}{Sup}\|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(0,T;L^2(\Omega))} \leq C_1(T)(\|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))})\},$$

where $(H^1(\Omega))'$ is denoted by the double space of $H^1(\Omega)$) and $C_1(T)$ is a positive constant only depending $T$ and functions $F, \varphi$ in problem (3).

For fixed $\forall w \in W(0, T)$, now define a weak linearization form of $w$ in problem (3). Find solution $u \in W(0, T)$ satisfying

$$(E_w): \begin{cases} (\frac{\partial u}{\partial t}, v) = -\varepsilon \int_\Omega F(|\nabla w * G_\sigma|^2)\nabla u \cdot \nabla v dxdy - \frac{1}{\varepsilon}\int_\Omega \varphi'(w)v dxdy \\ \qquad\qquad\qquad\qquad + \lambda \cdot \int_{\Omega\backslash D}(f-u)v dxdy \\ u(0) = u_0 \end{cases},$$

holds for $\forall v \in H^1(\Omega)$, where $(\cdot, \cdot)$ is denoted by the inner product in $L^2(\Omega)$ and $(\frac{\partial u}{\partial t}, v)$ denotes the dual effect in $((H^1(\Omega))', H^1(\Omega))$.

For $\forall w \in W(0, T)$, the solution $u$ is gotten from the above weak problem $(E_w)$, thus define mapping $u = U(w)$. Here we will use Schauder fixed point theorem to prove: This mapping $u = U(w)$ has a fixed point in $W(0, T)$, thereby problem (3) exists weak solution.

To this end, the following two steps are carried:

(1) Firstly prove $u = U(w)$ is a mapping from $W(0, T)$ to itself;

(2) Prove that mapping $u = U(w)$ is weakly continuous in $W(0, T)$.

The first step: $u = U(w)$ is a mapping from $W(0, T)$ to itself.

In problem $(E_w)$, let $v = u$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 = -\varepsilon\int_\Omega F(|\nabla w * G_\sigma|^2)|\nabla u|^2 - \frac{1}{\varepsilon}\int_\Omega \varphi'(w)u + \lambda \cdot \int_{\Omega\backslash\backslash D}(f-u)u \tag{5}$$

is obtained.

Since

$$\|\nabla w * G_\sigma\|_{L^\infty(\Omega)} \leq \|\nabla G_\sigma\|_{L^\infty(\Omega)} \cdot \|w\|_{L^1(\Omega)}$$

is boundary, there exists some constant $\gamma > 0$ satisfying

$$F(|\nabla w * G_\sigma|^2) \geq \gamma > 0.$$

Substitute it to (5),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \varepsilon\gamma\int_\Omega |\nabla u|^2 \leq -\frac{1}{\varepsilon}\int_\Omega \varphi'(w)u + \lambda \cdot \int_{\Omega\backslash D}(f-u)u. \tag{6}$$

From assumption, for all $w$,

$$|\varphi'(w)| \le C_1|w| + C_2|w|^3.$$

Substitute it to (6),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \varepsilon\gamma\int_\Omega |\nabla u|^2 \le \frac{C_1}{2\varepsilon}\int_\Omega(|w|^2+|u|^2) + \frac{C_2}{2\varepsilon}\int_\Omega(|w|^6+|u|^2)$$
$$+\frac{\lambda}{2}\int_\Omega(|f|^2+|u|^2) - \lambda\cdot\int_{\Omega\setminus D}|u|^2. \tag{7}$$

From this, for arbitrary fixed $\varepsilon > 0$,

$$\frac{d}{dt}||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)} \le C_3(||f||^2_{L^2(\Omega)} + ||w||^2_{L^2(\Omega)} + ||w||^6_{L^6(\Omega)}) + C_4||u||^2_{L^2(\Omega)}, \tag{8}$$

where $C_3, C_4$ are positive constants.

Since $\Omega \subseteq R^2$ is boundary and by Sobolev embedding theorem, for any $\delta > 0$, there exists constant $C_\delta > 0$ for which

$$||w(t,\cdot)||^6_{L^6(\Omega)} \le C_\delta||w(t,\cdot)||^6_{L^2(\Omega)} + \delta||w(t,\cdot)||^6_{H^{\frac{3}{4}}(\Omega)}, \tag{9}$$

Substitute (9) to (8) and use Gronwall inequality,

$$\underset{0\le t\le T}{Sup}||u(t)||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(0,T;L^2(\Omega))}$$
$$\le C(T)(||u_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))} + ||w||^2_{L^2(0,T;L^2(\Omega))}$$
$$+C_\delta||w||^6_{L^6(0,T;L^2(\Omega))} + \delta||w||^6_{L^6(0,T;H^{\frac{3}{4}}(\Omega))}). \tag{10}$$

And by the interpolation inequality (Guilan Zhang, 2010, p.70),

$$\int_0^T ||w(t,\cdot)||^6_{H^{\frac{3}{4}}(\Omega)}dt \le C_0(\int_0^T ||w(t,\cdot)||^{18}_{L^2(\Omega)}dt)^{\frac{1}{12}}(\int_0^T ||w(t,\cdot)||^2_{H^1(\Omega)}dt)^{\frac{9}{4}}. \tag{11}$$

Substitute (11) to (10) and properly select $\delta > 0$ and $T \in (0, T_0]$ as small as possible, for which when $w \in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ satisfies

$$\underset{0\le t\le T}{Sup}||w(t)||^2_{L^2(\Omega)} + ||\nabla w||^2_{L^2(0,T;L^2(\Omega))} \le C_1(T)(||u_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}). \tag{12}$$

We have

$$\underset{0\le t\le T}{Sup}||u(t)||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(0,T;L^2(\Omega))} \le C_1(T)(||u_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}), \tag{13}$$

where $C_1(T)$ is a fixed positive constant only depending $T$ and $F, \varphi$ of problem (5).

From the equation in problem $(E_w)$, $\partial_t u \in L^2(0,T;(H^1(\Omega))')$ is gotten.

Thus, $u = U(w)$ is a mapping from $W(0,T)$ to itself.

The second step: mapping $u = U(w)$ is weakly continuous in $W(0,T)$.

Suppose $\{w_j\}_{j\ge0}$ is a sequence of space $W(0,T)$, and in $W(0,T)$, $w_j \xrightarrow{weak} w(j\to\infty)$, then $u_j = U(w_j)$ is a solution sequence determined by problem $(E_w)$.

By the solution estimation about problem $(E_w)$, we obtain: $u_j$ is boundary in $C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$,

$\frac{\partial}{\partial t}u_j$ is boundary in $L^2(0,T;(H^1(\Omega))')$,

thus $u_j$ is boundary in $H^1(0,T;(H^1(\Omega))')$.

Therefore, $\{u_j\}_{j\ge0}$ contains a sub-sequence (still denoted by $\{u_j\}_{j\ge0}$) such that

$u_j \xrightarrow{weak} u(j\to\infty)$ in $L^2(0,T;H^1(\Omega))$,

$\frac{\partial}{\partial t}u_j \xrightarrow{weak} \frac{\partial}{\partial t}u(j\to\infty)$ in $L^2(0,T;(H^1(\Omega))')$,

and $u_j(0) \to u(0)(j\to\infty)$ in $(H^1(\Omega))'$.

So, by compact embedding theorem:

$u_j \to u(j \to \infty)$ in $L^2(0,T;L^2(\Omega))$ and is almost convergent everywhere in $\Omega \times [0,T]$. And $w_j \xrightarrow{weak} w(j \to \infty)$ in $W(0,T)$. Therefore $w_j \to w(j \to \infty)$ in $L^2(0,T;L^2(\Omega))$ and is almost convergent to $w$ everywhere.

$F(|\nabla w_j * G_\sigma|^2) \to F(|\nabla w_* G_\sigma|^2)(j \to \infty)$ in $L^2(0,T;L^2(\Omega))$.

$\varphi'(w_j) \xrightarrow{weak} \varphi'(w)(j \to \infty)$ in $L^2(0,T;L^2(\Omega))$.

Thus, in problem $(E_w)$, let $j \to \infty$, $u = U(w)$ is obtained, i.e. the mapping $u = U(w)$ is weakly continuous.

Combining with the above proven (1) and (2) and from Schauder fixed point theorem (Gongqing Zhang, 2005), the mapping defined in $(E_w)$ exists a fixed point $u \in W(0,T)$. That is, the weak solution of problem (5) is $u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$

*2.2 Solution Uniqueness of Initial Boundary Problem (3)*

Next, we will prove the second part of theorem 2.1, that is, the bounded weak solution of initial boundary value problem (3) is unique.

Let $\overline{u}$ and $\widehat{u}$ are two bounded solutions of initial boundary value problem (5), and satisfy regularity in the above theorem, then for almost all $t \in [0,T]$,

$$\overline{u_t} = \varepsilon div(F(|\nabla\overline{u} * G_\sigma|^2)\nabla\overline{u}) - \frac{1}{\varepsilon}\varphi'(\overline{u}) + \lambda \cdot \chi_{\Omega\backslash D}(f - \overline{u}) \tag{14}$$

and

$$\widehat{u_t} = \varepsilon div(F(|\nabla\widehat{u} * G_\sigma|^2)\nabla\widehat{u}) - \frac{1}{\varepsilon}\varphi'(\widehat{u}) + \lambda \cdot \chi_{\Omega\backslash D}(f - \widehat{u}). \tag{15}$$

For convenience, denote

$$\overline{\alpha} = F(|\nabla\overline{u} * G_\sigma|^2), \widehat{\alpha} = F(|\nabla\widehat{u} * G_\sigma|^2).$$

From (14) and (15),

$$\begin{aligned}\partial_t(\overline{u} - \widehat{u}) &= \varepsilon div[\overline{\alpha}(\nabla\overline{u} - \nabla\widehat{u}) + (\overline{\alpha} - \widehat{\alpha})\nabla\widehat{u}] \\ &-\frac{1}{\varepsilon}(\varphi'(\overline{u}) - \varphi'(\widehat{u})) - \lambda \cdot \chi_{\Omega\backslash D}(\overline{u} - \widehat{u}).\end{aligned} \tag{16}$$

Multiply both sides of equation (16) by $\overline{u} - \widehat{u}$ , and quadrature it in $\Omega$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\overline{u} - \widehat{u}\|^2_{L^2(\Omega)} + c_1\|\nabla\overline{u} - \nabla\widehat{u}\|^2_{L^2(\Omega)} \\ &\leq \varepsilon\|\overline{\alpha} - \widehat{\alpha}\|_{L^\infty(\Omega)}\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\nabla\overline{u} - \nabla\widehat{u}\|_{L^2(\Omega)} \\ &+\frac{1}{\varepsilon}\int_\Omega(\varphi'(\overline{u}) - \varphi'(\widehat{u}))(\overline{u} - \widehat{u}),\end{aligned} \tag{17}$$

Since $F$ and $G$ are $C^\infty$, it is obvious that

$$\|\overline{\alpha} - \widehat{\alpha}\|_{L^\infty(\Omega)} \leq C\|\overline{u} - \widehat{u}\|_{L^2(\Omega)},$$

where $C > 0$ is a constant and is only relevant to $F$ and $G$.

So, from (17),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\overline{u} - \widehat{u}\|^2_{L^2(\Omega)} + c_1\|\nabla\overline{u} - \nabla\widehat{u}\|^2_{L^2(\Omega)} \\ &\leq \varepsilon C\|\overline{u} - \widehat{u}\|_{L^2(\Omega)}\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\nabla\overline{u} - \nabla\widehat{u}\|_{L^2(\Omega)} \\ &+\frac{1}{\varepsilon}\int_\Omega(\varphi'(\overline{u}) - \varphi'(\widehat{u}))(\overline{u} - \widehat{u}),\end{aligned} \tag{18}$$

Also by the hypothesis $\overline{u}, \widehat{u} \in L^\infty((0,T) \times \Omega)$ and mean value theorem,

$$\int_\Omega(\varphi'(\overline{u}) - \varphi'(\widehat{u}))(\overline{u} - \widehat{u}) \leq C_1\int_\Omega|\overline{u} - \widehat{u}|^2,$$

where $C_1 > 0$ is a constant.

Therefore, from (18).

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}+c_1\|\nabla\overline{u}-\nabla\widehat{u}\|^2_{L^2(\Omega)}$$
$$\leq \varepsilon C\|\overline{u}-\widehat{u}\|_{L^2(\Omega)}\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\nabla\overline{u}-\nabla\widehat{u}\|_{L^2(\Omega)}+\frac{C_1}{\varepsilon}\|\overline{u}-\widehat{u}\|^2_{L^2_\Omega}, \tag{19}$$

And then by Schwarz inequality,

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}+c_1\|\nabla\overline{u}-\nabla\widehat{u}\|^2_{L^2(\Omega)}$$
$$\leq \frac{\varepsilon}{2}\frac{C}{\gamma}\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}\|\nabla\widehat{u}\|^2_{L^2(\Omega)}\|+\frac{C\varepsilon\gamma}{2}\|\nabla\overline{u}-\nabla\widehat{u}\|^2_{L^2(\Omega)}+\frac{C_1}{\varepsilon}\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}. \tag{20}$$

Select appropriate $\gamma > 0$, and from (20),

$$\frac{d}{dt}\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}\leq C_2(\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}\|\nabla\widehat{u}\|^2_{L^2(\Omega)}+\|\overline{u}-\widehat{u}\|^2_{L^2(\Omega)}), \tag{21}$$

where $C_2 > 0$ is a constant.

By hypothesis $\widehat{u}\in L^2(0,T;H^1(\Omega))$ and $\overline{u}(0)=\widehat{u}(0)=u_0$,

Apply Gronwall inequality to (21), we have

$$\overline{u}=\widehat{u}.$$

Remark: 1) Similar with the above proof, it is easy to get the solution of problem (3)

$$u\in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$$

is in a bounded function set, which is continuous and dependent on initial value$u_0$ and free term $f$.

2) Since Allen-Cahn equation with non-local term in problem (3) starts from image denoising, segmentation and restoration, in application background it is reasonable to discuss $u$ in bounded function class about problem (3).

**References**


Allen, S.M. and Cahn, J.W. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta. Metall*, 27, 1084-1095.

Benes, M., Chalupecky, V. and Mikula, K. (2004). Geometrical image segmentation by the Allen-Cahn equation, *Appl. Numer. Math.*, 51, 187-205.

Bertozzi, A.L., Esedoglu, S. and Gillette, A. (2007). Inpainting of binary images using the Cahn-Hilliad equation, *IEEE Trans. Image Proc.*, 16, 285-291.

Bronsard, L. and Kohn, R. (1991). Motion by mean curvature as the singular limit of Ginzburg-Landau model. *J. Diff. Eqs.*, 90, 211-237.

Catte, F., Lions, P.L. and Morel, J.M. (1992). Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J. Numer. Anal.*, 29, 182-193.

Chen, X. (1992). Generation and propagation of interfaces for reaction-diffusion equations. *J. Diff. Eqs.*, 96, 116-141.

Chen, X., Hilhorst, D. and Logak, E. (1997). Asymptotic behavior of solutions of an Allen-Cahn equation with a nonlocal term, *Nonlinear Anal.*, 28, 1283-1298.

De Mottoni, P. and Schatzman, M. (1990). Development of interfaces in . *Proc. R. Soc. Edinb.,* 116A, 207-220.

De Mottoni, P. and Schatzman, M. (1995). Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.*, 347, 1533-1589.

Evans, L.C., Soner, H.M. and Souganidis, P.M. (1992). Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45, 1097-1123.

Fife, P.C. (1988). Dynamics for Internal Layers and Diffusive Interfaces. *CCMS-NSF Reg-ional Conf. Ser. in Appl. Math,* SIAM, Philadelphia .

Friedman, A. (1964). Partial Differential Equations of Parabolic, *Prentice-Hall, Inc.* , Eng-lewood Cliffs, NJ.

Gabor, D. (1965). Information theory in electron microscopy. *Lab. Invest.*, 14, 801 807.

Gongqing Zhang, Maozheng Zhang. (2005). *Functional Analysis lecture*, Peking University Press.

Guilan Liu. (2010). The Existence of Solutions for a Class of Allen-Cahn Equations with Nonlocal Term, *Journal of Sichuan University*, 2.

Ilmanen, T. (1993). Convergence of the Allen-Cahn equation to Brakke' s motion by mean Curvature. *J. Diff. Geom.*, 38, 417-461.

Ladyzenskaya, O.A., Solonnikov, V.A. and Ural'tzeva, N.N. (1968). Linear and Quasilinear Parabolic Equations. *Amer. Math. Soc.*.

Lieberman, GM. (1996). Second Order Parabolic Differential Equations, World Scientific Publishing Co Inc River Edge, NJ .

Lions, J. L. and Magenes, E. (1972). *Non-Homogeneous Boundary Value Problems and Applications.* Vol. II. Springer-Verlag, New York-Heidelberg.

Koenderink, J.J. (1984). The structure of image. *Biol. Cybern*, 50, 363-370.

Perona, P. and Malik, J. (1990). Scale-space and edge detection using anisotropic diffusion, *IEEE Trans. PAMI*, 12, 629-639.

Rubinstein, J., Sternberg, P. and Keller, J.B. (1989). Fast reaction, slow diffusion and curve shortening. *SIAM J. Appl. Math.*, 49, 116-133.

Wang, Y.G . and Oberguggenberger M. (1999). Nonlinear parabolic equations with regularized derivatives. *J. Math. Anal. Appl.*, 233, 644-658.

Witkin, A.P. (1983).Scale-Space Filtering, in Proceedings of IJCAI, Karlsruhe, 1019-1021.

Zhuoqun Wu, Jingxue Wang, Chunpeng Wang. (2003). *Elliptic and Parabolic Equations Introduction*, Science and Technology Publishing House.