Representational Consistency of Group Rings

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Abstract

It is shown that if a group ring satisfies the idempotent conjecture and some other restrictions it will be representational consistent. As an application we show the real numbers as a field are representational consistent. This leads to the problem of whether a circle can be given a field structure that can be shown to be consistent.

Keywords: Representational consistency, Idempotent conjectures, Automorphism involution

1. Introduction

The study of consistency has been viewed historically as very important to mathematics. For example Hilbert's second problem asks whether the axioms of arithmetic are consistent. There is more than one definition of consistency, the most straightforward being that a statement cannot be both true and false at the same time. Another definition is ω-consistency which is the opposite of ω-inconsistency. ω-inconsistency means a statement does not hold for some number, but is still provable for every number individually. Tarski has proved that the real numbers as a field are consistent see (Blum and Smale, 2000, p1477-1495).

Representational consistency is defined in a general sense using the notion of structure or interpretation from model theory (Crossley et al., 1972, p13).

Definition 1. Let S be a nonempty set, and U a collection of relations on S, $\langle S, U \rangle$ is called the structure of S. Let $R_1 : S \to V$, $R_2 : S \to W$ be two monomorphisms of a structure $\langle S, U \rangle$ and

$$\mathcal{U}(R_1, R_2) = \{(s, \{R_1(s)\} \cup \{R_2(s)\}) \mid s \in S\}. \tag{1}$$

This defines an operator $\mathcal{U}$ on mappings whose domain is S, but the range of $\mathcal{U}$ is

$$\{\{v, w\} \mid v \in V, w \in W\}.$$

Definition 2. The structure $\langle S, U \rangle$ is called *pair wise consistent* if for all pairs of monomorphisms R_1, R_2 the map $\mathcal{U}(R_1, R_2)$ is one to one. If for any pair of monomorphisms R_1, R_2 the map $\mathcal{U}(R_1, R_2)$ is not one to one, the structure is said to be *pair wise inconsistent*.

Let $S = kG$, where G is a group, and k is a field, and U are the relations given by the ring axioms on S. It will be shown that the structure $\langle S, U \rangle$ is pair wise consistent if G satisfies the idempotent conjecture and some other conditions. For example if G belongs to the class of groups $\mathcal{S}$ (Emmanouil, 2001, p151-160), or the class of groups C (Emmanouil, 1998, p307-330) then it satisfies the idempotent conjecture.

Irving Kaplansky made the idempotent conjecture in his classic book on fields and rings (Kaplansky, 1972, p122).

If G is a torsion free group, and k is a field then the group ring kG has no non-trivial zero-divisors.

The idempotent conjecture is currently open, although there are many cases in which it is known to hold. The idempotent conjecture is true for

- Torsion free abelian groups

- Torsion free polycyclic by finite groups

- Torsion free Noetherian groups

- Solvable groups with finite Hirsch number

In Emmanouil's paper (Emmanouil, 2001, p151-160) he introduces a class S of groups satisfying the idempotent conjectures, with nice closure properties. Consider for any group G the complex group algebra $\mathbb{C}G$ and the augmentation homomorphism

$$\varepsilon : \mathbb{C}G \to \mathbb{C}$$

and the augmentation ideal $I_G = \ker \varepsilon$. For any left $\mathbb{C}G$-module M let $M_G = M/I_G M$. The class S consists of groups that satisfy the following condition: For any non-zero projective $\mathbb{C}G$-module P, we have $P_G \neq 0$. The class S contains all torsion-free abelian groups, and is closed under subgroups, extensions, direct products and free products.

In addition to these examples, a larger class of groups with good closure properties can be found in (Emmanouil, 1998, p307-330). The class C consists of those groups that satisfy the following condition: For any element $g \in G$ of infinite order the image $\left(\alpha_g\right)_{\mathbb{Q}}$ of the class α_g in the cohomology ring $\oplus_i H^i\left(N_g, \mathbb{Q}\right)$ is nilpotent. A group G is said to be residually contained in C if for any element $g \in G$ with $g \neq 1$ there exists a normal subgroup $K \trianglelefteq G$, such that $g \notin K$ and $G/K \in C$. Emmanouil proves the following theorem.

Theorem 3. Let G be a group. If $G \in C$ is torsion-free, then G satisfies the idempotent conjecture.

Proof. See Proposition 4.4 in (Emmanouil, 1998, p307-330). □

The class C includes all torsion and all abelian groups, and is closed under subgroups, finite direct products and free products (Emmanouil, 1998, p307-330) Proposition 3.5. However there are examples of groups that do not satisfy the cohomological condition defining class C (Emmanouil, 2001, p159-171).

Further examples of pair wise consistent rings such as non-commutative ordered rings and rings other than group rings can be found in the book in progress by the author (Bhavnagri, 2010, p1-133).

2. Showing consistency

To show that structures with representational consistency exist we use the theorem from (Bhavnagri, 2009, p4).

Theorem 4. A set S with structure has (pairwise) representational consistency if

$$
\begin{aligned}
H : \operatorname{Aut}(S) \times S &\to S &&(g, x) \mapsto y \\
H(\operatorname{id}, x) &= x &&\forall x \in S \\
H(g, x) &\neq x &&\forall g \in \operatorname{Aut}(S) \setminus \{\operatorname{id}\}, \ \forall x \in S \\
g^2 &\neq \operatorname{id} &&\forall g \in \operatorname{Aut}(S) \setminus \{\operatorname{id}\}
\end{aligned}
$$

The identity automorphism is denoted by id.

Proof. If the representations of S are pair wise inconsistent then for some ϕ_1, ϕ_2 the mapping $\mathcal{U}(\phi_1, \phi_2)$ is not one to one. Hence there are $s, t \in S$ with $s \neq t$ such that

$$\{\phi_1(s), \phi_2(s)\} = \{\phi_1(t), \phi_2(t)\} \tag{2}$$

Since ϕ_1, ϕ_2 are monomorphisms they are one to one so

$$\phi_1(s) \neq \phi_1(t) \text{ and } \phi_2(s) \neq \phi_2(t). \tag{3}$$

Since $\operatorname{Aut}(S)$ is a group we have from equation 2

$$\phi_2^{-1}(\phi_1(s)) = t \tag{4}$$

and

$$\phi_1^{-1}(\phi_2(s)) = t. \tag{5}$$

From equations 4 and 5 we have $\phi_1^{-1}\phi_2\phi_1^{-1}\phi_2(s) = s$.

Since $\operatorname{Aut}(S)$ acts freely on S it must be that $\phi_1^{-1}\phi_2\phi_1^{-1}\phi_2 = \operatorname{id}$ where $\operatorname{id} \in \operatorname{Aut}(S)$ is the identity.

From equation 3 the sets in equation 2 cannot have only one element, as $\mathcal{U}(\phi_1, \phi_2)$ is not one to one, so $\phi_1 \neq \phi_2$.

$$\left(\phi_1^{-1}\phi_2\right)^2 = \operatorname{id} \tag{6}$$

The group $\operatorname{Aut}(S)$ has an involution. □

What is striking about the theorem above, is that we did not need to know anything about the structure in order to prove the theorem. It could perhaps apply to a large class of structures, but it leads to a formidable problem which has been unsolved since 1997. Identify structures which satisfy the following equations.

$$
\begin{aligned}
H : \operatorname{Aut}(S) \times S &\rightarrow S & (g,\, x) &\mapsto y \\
H(\mathrm{id},\, x) &= x & \forall x &\in S \\
H(g,\, x) &\neq x & \forall g \in \operatorname{Aut}(S) \setminus \{\mathrm{id}\},\ &\forall x \in S \\
g^2 &\neq g & \forall g \in \operatorname{Aut}(S) \setminus \{&\mathrm{id}\}
\end{aligned}
$$

In this paper we now assume S is a ring, not necessarily commutative, because it turns out that ring theory provides some solutions to the problem.

3. Involutions and idempotents

An element e of a structure is called an idempotent if $e^2 = e$. In a ring involutions and idempotents are equivalent provided the characteristic of the base field is not 2. If $e^2 = e$ then let $g = 2e - 1$ then $g^2 = 1$ and g is an involution. Conversely if g is an involution, $g^2 = 1$ and let $e = \frac{1}{2}(g + 1)$ so that $e^2 = e$, and e is an idempotent.

Caution is necessary when applying the equivalence of idempotent and involutory elements in a ring. This is because an involution ϕ with respect to composition in $\operatorname{Aut}(S)$ is $\phi(\phi(s)) = s$ which is not the same as an involutory element $g^2 = gg = 1$. Observe that the product

$$(2e - 1)(2e - 1)$$

is not the same as the composition

$$2(2e - 1) - 1.$$

Lemma 5. *Let S be a ring (with unit) and define $g_b : S \rightarrow S$ with $g_b(s) = bs$ and $b = 2a - 1$.*

(i) *If $a \in S$ is an idempotent then g_b is an involution.*

(ii) *If an idempotent is invertible it is trivial.*

(iii) *An involution is its own inverse.*

Proof. (i) By definition

$$g_b(g_b(s)) = b(bs) = b^2 s = s$$

so that g_b is an involution under composition.

(ii) $a^2 a^{-1} = a a^{-1} \Rightarrow a = 1$

(iii) For any involution b by definition $b^2 = 1$ so $b = b^{-1}$. Also

$$b^2 b^{-1} = 1 b^{-1} \Rightarrow b = b^{-1}.$$

$\square$

However g_b is not an automorphism as that would require $g_b(s) g_b(t) = g_b(st)$ for all $s,\, t \in S$ That would require the equality of $bsbt$ and bst.

On the other hand if we conjugate by two involutory elements $b^2 = 1$ and $d^2 = 1$ with $\phi(s) = bsd$ then ϕ is an involution with respect to composition in $\operatorname{Aut}(S)$. It is necessary that $db = 1$ if $\phi \in \operatorname{Aut}(S)$, such as an inner automorphism which has $d = b^{-1}$.

Lemma 6. *Let S be a ring with unit.*

(i) *If $a,\, c \in S$ are idempotents where $b = 2a - 1$ and $d = 2c - 1$ and $db = 1$ then $b = d$.*

(ii) *If $a \in S$ is a non-trivial idempotent then $\phi(s) = (2a - 1)s(2a - 1)$ is a non-trivial involution of $\operatorname{Aut}(S)$, furthermore ϕ is an inner automorphism of S.*

(iii) *If $\phi \in \operatorname{Aut}(S)$ is an inner automorphism involution $\phi = xsx^{-1}$ then $x^2 \in Z(S)$.*

(iv) *Suppose $Z(S) = k$ and $\operatorname{Aut}(S) = \{xsx^{-1} \mid x \in S\}$. If there is a non-trivial involution in $\operatorname{Aut}(S)$ and $x^2 = \lambda 1$ with $\lambda > 0$ then there is a non-trivial involution in S.*

Proof. (i)

Expanding $db = 1$ gives $(2c - 1)(2a - 1) = 4ca - 2a - 2c + 1$.

Then $db = 1$ if and only if

$$2ca = a + c \tag{7}$$

Multiplying equation 7 by a gives $2ca^2 = a^2 + ca$ and from idempotence of a

$$2ca = a + ca \tag{8}$$

Multiplying equation 7 by c gives $2c^2a = ca + c^2$ and from idempotence of c

$$2ca = ca + c \tag{9}$$

Equating 8 and 9 gives $a = c$ and also $b = d$.

(ii) We verify that $\phi(st) = \phi(s)\phi(t)$ and $\phi(\phi(s)) = s$ for all $s, t \in S$.

$$\phi(st) = (2a - 1)\,st\,(2a - 1)$$

From Lemma 5 (iii) $(2a - 1)^2 = 1$ and we have

$$(2a - 1)\,st\,(2a - 1) = (2a - 1)\,s\,(2a - 1)\,(2a - 1)\,t\,(2a - 1)$$

It follows that $\phi(st) = \phi(s)\phi(t)$. Also

$$(2a - 1)\,(2a - 1)\,s\,(2a - 1)\,(2a - 1) = 1s1 = s$$

so that $\phi(\phi(s)) = s$.

(iii) For all s, $x\left(xsx^{-1}\right)x^{-1} = s$ since ϕ is an inner automorphism involution. Right multiply by x^2 yielding $x^2 s = sx^2$. Since $x^2 \in Z(S)$ it follows that $\phi(s) = xsx^{-1} = sxx^{-1} = s$.

(iv) Suppose $Z(S) = k$ and $\operatorname{Aut}(S) \subseteq \left\{xsx^{-1} \mid x \in S\right\}$. If ϕ is an inner automorphism involution from (iii) $x^2 = \lambda 1$ for some $\lambda \in k$. If $\lambda > 0$ then $\lambda^{-1/2}x$ is a non-trivial involution in S.   □

Now what we are interested in is the extent to which the converse of Lemma 6 (ii) holds. That will give us conditions for the last equation in Theorem 4. That is under what conditions does knowing a ring has no non-trivial idempotents tell us it has no non-trivial automorphism involutions? The answer is provided by (iv) of the same Lemma 6.

There can be non-trivial inner automorphism involutions $s \mapsto xsx^{-1}$ with x not an involution. For example in the ring of 2×2 real matrices if $x^2 = c1$ and $x^{-2} = \frac{1}{c}1$ where c is a scalar. An example of a non-trivial involution in this ring is $x = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. However x^2 is in the centre of the ring S. The standard notation for the centre of a ring is $Z(S)$. It is well known the centre of $M_n(\mathbb{C})$ is

$$Z(M_n(\mathbb{C})) = \{\lambda I_n \mid \lambda \in \mathbb{C}\}$$

where I_n denotes the $n \times n$ identity matrix. So an inner automorphism involution is just a scaled involution in $M_n(\mathbb{C})$.

In particular $n = 1$ gives $\mathbb{C}$ itself, which has $i \mapsto -i$ as automorphism. So if j is such that $jij^{-1} = -i$ then $ji = -ij$ so $j \in \mathbb{H}$, not in $\mathbb{C}$ which is commutative. Even in $\mathbb{H}$ the map $x \mapsto ixi^{-1}$ is an involution automorphism, but $\mathbb{H}$ has no non-trivial idempotents, its automorphisms are all inner and it is central.

Also observe that for an inner automorphism the third equation in Theorem 4 is $\phi(s) = xsx^{-1} \neq s$, unless $\phi = \operatorname{id}$ (the second equation). That is $x \notin Z(S)$, unless $\phi = \operatorname{id}$. If $Z(S) = \{\lambda I_n \mid \lambda \in \mathbb{C}\}$ then that is true as $\lambda I_n s\left(\lambda^{-1}I_n\right) = \operatorname{id}$.

There are also examples of group rings $\mathbb{C}G$ whose centre is a direct sum of $\mathbb{C} \cdot I_{n_i}$, for example if G is a finite group in the book (Emmanouil, 2006, p195).

Automorphism groups can have outer automorphisms, that is automorphisms that are not inner. However, there are conditions under which all automorphisms are inner automorphisms. We will find an example of such rings after we deal with a simpler case first.

4. Ordered rings

An ordered ring is a ring with an order $<$ such that for any a, b, c

$$a < b \implies a + c < b + c$$

$$0 < a \text{ and } 0 < b \implies 0 < ab$$

An ordered ring has no idempotents other than 0 and 1. There are three cases. If $0 < x < 1$ then multiplying by x we have $x^2 < x$ so $x^2 \neq x$. If $x > 1$ then $x^2 > x$ so $x^2 \neq x$. Last if $x < 0$ multiply $x < 1$ then $x^2 > x$ so $x^2 \neq x$.

The real numbers are an ordered field, so they have no non-trivial idempotents. Likewise the rational numbers are another example of an ordered field. Both fields have no non-trivial automorphisms.

Any ordered field has positive squares $x^2 > 0$. The complex numbers are not an ordered field, otherwise $\sqrt{-1}$ would have positive square.

5. Inner automorphisms

If k is a field then any automorphism of the ring of matrices over k (denoted $M_n(k)$) can be extended to an inner automorphism. This is a corollary of the theorem of extension of isomorphisms we will now introduce.

A ring R is called simple if it has no non-trivial ideals and $R^2 \neq (0)$. The trivial ideals are R and (0). A division ring is a ring in which every non-zero element has a multiplicative inverse. Any division ring is simple.

Lemma 7. *The ring $M_n(D)$ where D is a division ring is simple.*

Proof. See the book (Kaplansky, 1972, p91-92).   □

The Jacobson density theorem follows, which can be used to prove the Wedderburn-Artin theorem.

Theorem 8. *Let M be a simple right R-module and let $D = \mathrm{End}_R(M)$. Let X be a finite subset of M and $I = \mathrm{ann}_R(X)$ the annihilator of X. If $u \in M$ is such that $uI = 0$ then $u \in XD$.*

Proof. See the book (Kaplansky, 1972, p95).   □

A ring is said to satisfy the descending chain condition on right ideals, if every properly descending chain of right ideals is finite. The next theorem is the Wedderburn-Artin theorem.

Theorem 9. *A simple ring with descending chain condition on right ideals is the full ring of linear transformations on a finite-dimensional vector space over a division ring.*

Proof. See the book (Kaplansky, 1972, p96).   □

A central simple k-algebra is a simple k-algebra whose center is the field k. For example $\mathbb{C}$ is not central simple and $(1 \otimes 1 + i \otimes i)$ is a non-trivial ideal because $(i \otimes i + 1 \otimes 1)(i \otimes i - 1 \otimes 1) = 0$, hence $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not simple. On the other hand the quaternion algebra $\mathbb{H}$ is central simple, the center being $\mathbb{R}$.

Lemma 10. *Let C be an algebra over F. Let A, B be subalgebras with A central simple and B simple. Suppose further that A and B commute elementwise. Then either $AB = 0$ or AB is isomorphic to $A \otimes B$.*

Proof. See the book (Kaplansky, 1972, p153).   □

Lemma 11. *Let D be a division algebra over F and A a finite dimensional algebra with unit element over F. Then $A \otimes D$ satisfies the descending chain condition on right ideals.*

Proof. See the book (Kaplansky, 1972, p155).   □

Theorem 12. *Let E be a vector space over a division ring D with center Z. Let R be the ring of all linear transformations on E and let B and C be simple subalgebras of R, finite dimensional over Z and containing the unit element of R. Then any isomorphism between B and C can be extended to an inner automorphism of R.*

Proof. See the book (Kaplansky, 1972, p161).   □

As a corollary to this theorem we have that if k is a field then any automorphism of $M_n(k)$ can be extended to an inner automorphism.

For further details about rings see for example the book by Irving Kaplansky (Kaplansky, 1972, p79-198). A similar result about all automorphisms being inner is also a corollary of the Skolem-Noether theorem (Drupieski and Hamblet, 2005, p1).

6. Showing inconsistency

We have seen thus far how to prove consistency, now we show a technique for proving inconsistency. We will show that the cyclic group is representationaly inconsistent.

Lemma 13. *The representations of $\mathbb{Z}_n$ are inconsistent if $n > 2$.*

Proof. Let $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ with $\phi : s_m \mapsto s_{m(n-1) \bmod n}$. Suppose $n - 1$ and n have a common factor $p > 1$. Then $n - 1 = pr$ and $n = ps$ for some integers r, s. Since $n = n - 1 + 1$, $ps = pr + 1$, so $p(r - s) = -1$. This would imply p and $r - s$ equal $+1$ or -1, contradicting $p > 1$. Thus $n - 1$ and n are relatively prime, so that the multiples of $\phi(s_1) = s_{n-1}$ generate $\mathbb{Z}_n$. Thus ϕ is a structure preserving mapping, hence a monomorphism of $\mathbb{Z}_n$. Since $\phi(s_{n-1}) = s_1$,

$$\mathcal{U}(\mathrm{id}, \phi)(s_1) = \{s_1, s_{n-1}\} = \mathcal{U}(\mathrm{id}, \phi)(s_{n-1})$$

so the representations of $\mathbb{Z}_n$ are inconsistent if $n > 2$.   □

The above lemma can be generalized. If any automorphism ϕ is an involution the representations will be inconsistent. Let $\phi(s_1) = s_2$ and $\phi(s_2) = s_1$. Then

$$\mathcal{U}(\mathrm{id}, \phi)(s_1) = \{s_1, s_2\} = \mathcal{U}(\mathrm{id}, \phi)(s_2)$$

and the representations are inconsistent.

The complex numbers $\mathbb{C}$ have complex conjugation as an automorphism. The composition of two conjugations is the identity, and hence it is an involution. Complex multiplication is commutative so there is no distinction between automorphism and anti-automorphism. Consequently the field of complex numbers $\mathbb{C}$ is representationaly inconsistent.

7. Consistency of circle

The field of real numbers $\mathbb{R}$ is consistent in both the classical sense proved by Tarski and in the sense of representational consistency. This is because it is both ordered and has trivial automorphism group. On the other hand the field of complex numbers $\mathbb{C}$ is inconsistent in the sense of representational consistency. Let us now consider the problem of giving the circle S^1 a ring structure, so that we can determine whether it is consistent or not.

A periodic function $f : \mathbb{R} \to S^1$ such as $f(\theta) = e^{i\theta}$ maps the real numbers to the circle. Consider ways in which $\mathbb{R}$ can be endowed with a product that is ordered. The lexicographic order $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or if $x_1 = x_2$ and $y_1 < y_2$.

The tensor product is not orderable. The tensor product $\mathbb{R} \otimes \mathbb{R}$ in a basis is $(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, y_1 y_2)$. If the product is ordered then $(x_1, y_1) > 0$ and $(x_2, y_2) > 0$ implies that $(x_1 x_2, y_1 y_2) > 0$. A counterexample is provided by any $x_1 > 0$, $y_1 < 0$, $x_2 = 0$, $y_2 > 0$, in which case $x_1 x_2 = 0$ but $y_1 y_2 < 0$. Both $(x_1, y_1) > 0$ and $(x_2, y_2) > 0$ but $(x_1, y_1) \otimes (x_2, y_2)$ is not positive.

Consider other products. Another product like $(x_1 x_2, x_1 y_2 + x_2 y_1)$ is also not ordered, $(0, 1)(0, 1) = (0, 0)$. Also $(x_1 x_2 + y_1 y_2, x_1 y_2 + x_2 y_1)$ is not ordered, for instance $(1, 1)(1, -1) = (0, 0)$.

A more interesting product $S^1 \times S^1 \to S^1$ can be defined using the concept of spin. $\mathrm{Spin}(\mathbb{R}^2)$ is the group consisting of products

$$(\cos(\vartheta_1) i + \sin(\vartheta_1) j)(\cos(\vartheta_2) i + \sin(\vartheta_2) j),$$

where $\vartheta \in S^1$ i.e. the unit circle in the above subspace $\mathbb{C} \subset \mathbb{H}$. Multiplying out the product it is

$$-\cos(\vartheta_1)\cos(\vartheta_2) - \sin(\vartheta_1)\sin(\vartheta_2) +$$
$$(\cos(\vartheta_1)\sin(\vartheta_2) - \cos(\vartheta_2)\sin(\vartheta_1)) k$$

So $\mathrm{Spin}(\mathbb{R}^2)$ is generated by 1, k. i and j act on $\mathbb{R}^2$ by reflection while k acts on $\mathbb{R}^2$ as a rotation. Thus $\mathrm{Spin}(\mathbb{R}^2)$ is isomorphic to $\mathrm{U}(1) \cong S^1$. Is this product ordered?

More generally an open problem would be whether all ordered structures are consistent. Also it is hoped some of the machinery developed for the idempotent conjecture can be carried over.

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