On the Existence Structure of One-dimensional Discrete Chaotic Systems

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Abstract

This paper investigates the existence and structure of one-dimensional discrete chaotic systems. It is expected to change the research perspective of discrete chaotic system, from being passively discovered to actively constructing. Firstly, some sufficient conditions guaranteeing a one-dimensional discrete system being chaotic are proposed and rigorously proved. Secondly, it presents a simple method for constructing the one-dimensional discrete chaotic system which can provide more models and inspecting tools for the study of the latter. Finally, by applying the method, some new discrete chaotic systems are constructed, and the numerical results with the bifurcation diagram and the Lyapunov exponent illustrate the chaos, which thus verifies the feasibility and effectiveness of the proposed method.

Keywords: Discrete Systems, Chaos, Lyapunov exponent

1. Introduction

"Chaos" is the paraphrase of the English word—chaos, and chaos itself means "a state of total confusion and lack of order". Since T.Y. Li and J. A. Yorke published the paper named "period three implies chaos" (Li.1975.p82) in 1975, many researches on the chaos theory and its applications have been established. Due to its inherent characteristics, such as the sensitive dependence on initial conditions, stochastic of the chaotic signals, the continuous broadband power spectrum, the difficulty in prediction and separation through the frequency domain or time domain processing, and so on, the chaos theory is of great applied value (Sheng, 2004,p53) in the following fields, for instance, the secure communication, the information encryption, etc. Especially since the 1990s, the pioneering works by Ott, Grebogi, and Yorke on chaos control (Yorke, 1990, p64) and Pecora, Carroll on chaos synchronization (Pecora, 1990, p64) has made the chaotic communication become one of the studying focus(Chua, 1997, p44)(Mou, 2003, p22)(Palacios, 2002, p303)(Chen, 1999, p9)(Kwok, 2003, p67)(Stojanoski, 2001, p48).

The traditional research method of chaotic systems is that, for a given dynamical system, one can judge it is chaotic or not by using various methods such as the various definitions or characteristic quantities of chaos, which is a process of being passively discovered. However, due to their great value of being widely applied, more chaos systems need be found or created actively to meet the needs of engineering applications. Therefore, this paper attempts to find a simple but convenient method to construct more discrete chaotic systems.

Lyapunov exponent is an important characteristic quantity to judge whether a dynamic system is chaotic or not, but it cannot provide methods to actively construct the chaotic systems. "Period three implies chaos" gives us an important illumination, that is, if a discrete system $x_{n+1} = f(x_n)$ has the period three point, then it is chaotic. However, what kind of map f has the period three point? In theory, the solution of $f^n(x) = x$ includes all period-n points, if the solutions of $f^3(x) = x$ are more than that of $f(x) = x$, then the map f has the period-three point. But in fact the equation $f^3(x) = x$ is not easy to solve. For example, by using the existing methods, we cannot derive formula solutions from $f^3(x) = x$ with $f(x) = e^x + sin(x)$. Through the numerical methods, we can find one or more solutions, but it is not possible to get the whole solutions, so it is difficult to directly identify its period-3 points and then to judge whether it is chaotic or not. In this paper, some sufficient conditions of the existence of chaos are presented, then we provide a simple method for actively constructing the one-dimensional discrete chaotic system, and further establish a series of chaotic discrete systems conveniently, thus to provide more models for the research of discrete chaotic system and its application.

2. Theoretical Background

2.1 One-dimensional Discrete System

The general form of one-dimensional discrete system is

$$x_{n+1} = f(x_n) \tag{1}$$

where f: interval $I \to$ interval I is a continuous map.

Here, $f^n(x)$ $(n \in N^+)$ means the n iterates of $f(x)$, and $O^+(x) = \{x, f(x), f^2(x), ...\}$ presents the forward trajectory of x ; if f is homeomorphic, then $O^-(x) = \{x, f^{-1}(x), f^{-2}(x), ...\}$ is the backward one of x . Hence, $O(x) = \{..., f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), ...\}$ is the whole trajectory of x.

Definition 1 If there exists a $x_0 \in I$ such that $f^n(x_0) = x_0$, but $f^k(x_0) \neq x_0$ for any natural number $k < n$, then x_0 is called a period-n point of map f and $\{x_0, f(x_0), f^2(x_0), ..., f^{n-1}(x_0)\}$ is called a n-periodic trajectory.

In order to derive the main results, the following lemma is needed.

Lemma 1 (Li, 1975, p82) Let f be a continuous map on interval I if f has the period-3 point, then f is chaotic in the sense of Li-York, that is, system (1) is chaotic.

2.2 Lyapunov Exponent

In order to investigate whether a orbit of a dynamic system is sensitivitive dependence on initial conditions, we introduce the Lyapunov exponent, a quantity which describes the average expansion of iterate along the whole forward orbit. It is an important characteristic quantity to judge that whether a system is chaotic or not.

Definition 2 (Robinson, 2004, p82) Let f be a map from interval I to itself that has a derivative. The Lyapunov exponent of an initial condition x_0 for the map f is defined to be

$$\sigma = \lim_{n\to\infty} \frac{1}{n} \ln|\,(f^n)'(x_0)\,|$$

For the one-dimensional discrete system (1), the above Lyapunov exponent can be calculated by

$$\sigma = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left|\frac{df(x)}{dx}\right|_{x=x_i} \tag{2}$$

Remark 1 When the Lyapunov exponent $\sigma < 0$, it indicates that the orbit is going to an attracting fixed point or periodic orbit. When $\sigma > 0$, it indicates that the system has sensitive dependence on initial conditions. Hence, $\sigma > 0$ can be the indicator for the chaotic behaviors. The change of σ from negative to positive presents that the motion is directing to the chaotic mechanism, while the point of $\sigma = 0$ being the bifurcation one.

In the following sections, based on Lemma 1, we are to prove a sufficient condition of one-dimensional discrete system being chaotic, and then to deduce a method for constructing a series of one-dimensional discrete chaotic system, and to verify the chaotic property of the constructed discrete system by the Lyapunov exponent.

3. The Main Result

Theorem 1 Let $f(x) \in C^2[a, b]$. Then the dynamical system $x_{n+1} = f(x_n)$ is chaotic on interval (a, b) if $f(x)$ satisfies the following conditions:

(1) $f(a) = f(b) = a$;

(2) $\max_{x\in[a,b]} f(x) = b$;

(3) $f''(x) \leq 0$ for all $x \in [a, b]$, and there exists finite number of solutions of equation $f''(x) = 0$.

Proof

(i) To prove that $f'(x)$ is strictly monotone decreasing on $[a, b]$.

Since $f''(x)|_{x\in[a,b]} \leq 0$, it follows that $f'(x)$ is monotone decreasing on $[a, b]$. Let $m_1, m_2, \cdots, m_n$ denote the solutions of $f''(x) = 0$,$m_1 < m_2 < \ldots < m_n$. For any $\varepsilon > 0$, assume that $\varepsilon < \min_{i=2,3,\cdots,n} (m_i - m_{i-1})$, then $f'(x)$ on $(m_{i-1} + \frac{\varepsilon}{2}, m_i - \frac{\varepsilon}{2})$ is strictly monotone decreasing. Thus there have $f'(m_{i-1} + \varepsilon) < f'(m_{i-1} + \frac{\varepsilon}{2}) \leq f'(m_{i-1})$ and $f'(m_i - \varepsilon) > f'(m_i - \frac{\varepsilon}{2}) \geq f'(m_i)$. By the randomicity of ε, $f'(x)$ is strictly monotone decreasing on $[m_{i-1}, m_i]$, also on $[m_1, m_n]$. If $f''(a) = 0$, then $m_1 = a$;

If $f''(a) < 0$, then $f'(x)$ is strictly decreasing on $[a, m_1)$. If $f''(b) = 0$, then $m_n = b$. If $f''(b) < 0$, then $f'(x)$ is strictly monotone decreasing on $(m_n, b]$. Above all, $f'(x)$ on $[a, b]$ is strictly monotone decreasing.

(ii) To prove that $f(x)$ on (a, b) has unique stationary point, x_0 which is also the maximum point, i.e. $f(x_0) = \max_{x\in[a,b]} f(x) = b$, and $f'(x) |_{x\in(x_0,b)} < 0$.

Obviously, $f(x) |_{x\in(a,b)} > a$. For any $\xi \in (a, b)$, it follows the Lagrange's mean value theorem, there exists $\xi_1 \in (a, \xi)$ and $\xi_2 \in (\xi, b)$ which respectively satisfy $f'(\xi_1) = \dfrac{f(\xi) - f(a)}{\xi - a} > 0$ and $f'(\xi_2) = \dfrac{f(\xi) - f(b)}{\xi - b} < 0$. It follows by (i) that there exists unique point $x_0 \in (\xi_1, \xi_2)$ such that $f'(x_0) = 0$, and $f'(x) |_{x\in(a,x_0)} > f'(x_0) = 0, f'(x) |_{x\in(x_0,b)} < f'(x_0) = 0$. Therefore x_0 is the unique stationary point of $f(x)$ on (a, b), thus $f(x_0) = \max_{x\in[a,b]} f(x) = b$.

(iii) To prove the equation $f(x) = x$ has only two roots on $[a, b]$.

Firstly, make a auxiliary function $F(x) = f(x)-x$, then there has $F'(x) = f'(x)-1$ and $F''(x) = f''(x)$. Since $F(a) = f(a)-a = a-a = 0$, $F(x_0) = f(x_0)-x_0 = b-x_0 > 0$, $F(b) = f(b)-b = a-b < 0$, by $f''(x) |_{x\in[a,b]} \leq 0$, we get $F''(x) |_{x\in[a,b]} \leq 0$, then $F(x)$ is the concave function. For all $\lambda \in (0, 1)$, there has $F((1 - \lambda)a + \lambda x_0) \geq (1 - \lambda)F(a) + \lambda F(x_0) = F(x_0) > 0$, by the randomicity of λ,one can obtain that $F(x) |_{x\in(a,x_0)} > 0$. By (ii), there has $F'(x) |_{x\in(x_0,b)} < 0$, $F(x)$ on (x_0, b) has only one zero point, thus $f(x) = x$ has only two roots on $[a, b]$.

(iv) To prove $f^3(x) = x$ at least has four roots on $[a, b]$.

Firstly, make a auxiliary function $U(x) = f(x) - x_0$, then there has $U(a) = f(a) - x_0 = a - x_0 < 0$, $U(x_0) = f(x_0) - x_0 = b-x_0 > 0$. It follows from the existence theorem of roots, there is at least one root $x_{11} \in (a, x_0)$ such that $U(x_{11}) = 0$, then $f(x_{11}) = x_0$. Also there is at least one root $x_{12} \in (x_0, b)$ such that $f(x_{12}) = x_0$.

Secondly, make a auxiliary function $V(x) = f^2(x) - x_{11}$, then $V(a) = f^2(a) - x_{11} = a - x_{11} < 0$, $V(x_{11}) = f^2(x_{11}) - x_{11} = b - x_{11} > 0$, $V(x_0) = f^2(x_0) - x_{11} = a - x_{11} < 0$, and there exists at least one root x_{21}, x_{22} respectively on (a, x_{11}) and (x_{11}, x_0) such that $f^2(x_{21}) = f^2(x_{22}) = x_{11}$. Similarly, there is at least one root x_{23}, x_{24} respectively on (x_0, x_{12}) and (x_{12}, b), such that $f^2(x_{23}) = f^2(x_{24}) = x_{12}$.

Thirdly, make a auxiliary function $H(x) = f^3(x) - x$. We calculate the function values at some points which are shown in table 1.

By the table 1, it shows that there is at least one root on every sub-interval of $[a, b]$, so $f^3(x) = x$ at least has four roots on $[a, b]$.

By (iii) and (iv), $f^3(x) = x$ has more roots than $f(x) = x$ on the interval $[a, b]$, so $f(x)$ has period three points on $[a, b]$. Obviously, $f(x)$ is a continuous map from $[a, b]$ to itself. By lemma 1, $x_{n+1} = f(x_n)$ is chaotic on (a, b), thus completes the proof.

Similarly, we can obtain the following conclusion.

Theorem 2 Let $f(x) \in C^2[a, b]$. Then the dynamical system $x_{n+1} = f(x_n)$ is chaotic on interval (a, b) if $f(x)$ satisfies the following conditions:

(1) $f(a) = f(b) = b$;

(2) $\min_{x\in[a,b]} f(x) = a$;

(3) $f''(x) \geq 0$ for all $x \in [a, b]$, and there exists finite number of solutions of equation $f''(x) = 0$.

4. Constructing Method

Based on theorem 1, we deduce a method for constructing one-dimensional discrete chaotic systems, the concrete steps are as follows:

step 1 Find a C^2 function $g(x)$ defined on an interval I where $g(x)$ is concave unimodal mapping on a sub interval $[a', b'] \subseteq I$.

step 2 Solve the equation $g(x) = max\{g(a'), g(b')\}$on $[a', b']$ and obtain the two solutions $a, b(a < b)$.

step 3 construct the function $G(x) = \dfrac{\lambda(b - a)}{\max_{x\in[a,b]} g(x) - g(a)}(g(x) - g(a)) + a$ while $\lambda \in [0, 1]$.

It is easy to prove that, when $\lambda = 1$, $G(x)$ on $[a, b]$ satisfies the conditions of the Theorem 1, therefore $x_{n+1} = G(x_n)$ is chaotic on (a, b).

5. Examples

In this section, several one-dimensional discrete chaotic systems will be constructed based on the above constructing method. We will prove the existence of chaos through the bifurcation diagram and the Lyapunov exponent, which verifies the validity of the constructing method.

Example 1 Based on the function $g(x) = e^x + sin(x)$, mentioned in the above section, we construct a one-dimensional discrete chaotic system and find its chaotic interval and the critical values of its parameters.

Obviously, the function $g(x)$ has continuously twice differentiable, and the Fig.1 shows that $g(x)$ is a convex function on interval $[-6, -4]$.

< Figure 1 >

Here, $g(-4) = 0.775118134196662$, $g(-6) = 0.281894250375592$. By solving the equation $g(x) = 0.775118134196662$, we can obtain that $x_1 = -5.403376918067952, x_2 = -4$; So we can find the interval $I_1 = [-5.403376918067952, -4]$. Construct function

$$\begin{aligned}G(x) &= \frac{-4-(-5.403376918067952))}{1.009024007938428-0.775118134196662}\lambda(g(x)-0.775118134196662)-5.403376918067952\\&= 5.999750650200821\lambda(g(x)-0.775118134196662)-5.403376918067952\\&= 5.999750650200821\lambda(e^x+sin(x)-0.775118134196662)-5.403376918067952\end{aligned}$$

then

$$x_{n+1} = 5.999750650200821\lambda(e^{x_n}+\sin(x_n)-0.775118134196662)-5.403376918067952 \tag{3}$$

From the above conclusion, the system (3) is chaotic on interval $(-5.403376918067952, -4)$ when $\lambda = 1$.

The Fig.2 shows that the system (3) is chaotic by (a) the bifurcation diagram which displays the orbital evolution with the parameter λ from 0 to 1 and (b) the Lyapunov exponent.

< Figure 2 >

Remark 2 Due to the sensitivity of chaos to the initial value, for accurate reasons, the decimal of coefficient in system (3) will be calculated to the accuracy of 16 decimal places.

Remark 3 Theorem 1 only gives the sufficient conditions for the existence of chaos, in fact, many chaotic systems are not only chaotic just at the point of $\lambda = 1$, but in a wider range of parameters. We could see that from the Fig.2, the parameter λ has come into the chaotic area when it is in a left neighborhood of $\lambda = 1$. Through the numerical calculation, we can obtain the critical point of the system parameter, that is $\lambda_\infty = 0.887729451801666$. That is, the system (3) is chaotic on the parameter interval $(\lambda_\infty, 1]$.

Similarly, using the constructing method, we can construct more simple chaotic systems, such as following examples.

Example 2 $x_{n+1} = \lambda\sin(\pi x_n), x_0 \in (0, 1)$.

Example 3 $x_{n+1} = \frac{3\sqrt{3}}{2}\lambda x_n - {x_n}^3, x_0 \in (0, \sqrt[4]{\frac{27}{4}})$.

For Example 1 and Example 2, their bifurcation diagrams, Lyapunov exponent evolution graphs, the critical value of parameters are shown in Fig.3 and Fig.4, respectively. From these graphs one can see that each discrete system is chaotic on its definition domain and $\lambda \in (\lambda_\infty, 1]$.

< Figure 3-4 >

6. Conclusion

By observing the graph of the Logistic model, we find that, when $\lambda = 1$, its image is just contained in a square whose diagonal line is on the line of $y = x$. To apply that into a more general situation, we derive the two theorems. This paper not only strictly proves the conclusion of the theorems , but also gives a concrete ways to construct the discrete chaotic system, and then illustrates the feasibility and effectiveness of the method with several examples. The method also provides some kinds of new models and tools for the research and application of discrete chaotic system. Finally, it is necessary to state that for a specific discrete chaotic system, using numerical methods one can always get the critical parameters of the systems which are in the chaotic state. From the examples of this paper, we can see that the critical values of different iterative functions are different, but the issue that which characteristic of the function has relationship with its critical value still needs to be further studied.

References

A. Palacios and H. Juarez. (2002). Crytography with cycling chaos. *Phys. Lett. A*, 303. 345.

G. Chen and T. Ueta. (1999). Yet another chaotic attractor. *Int. J. Bifurc. Chaos*, 9(7). 1465

Kwok W W, Sun W H and Ching K Y. (2003). A chaotic cryptography scheme for generating short ciphertext. *Phys. Lett. A*, 67. 310.

Li T Y and Yorke J A. (1975). Period three impiles chaos. *Amer. Math. Monthly*, 82. 985.

Li S J, Mou X Q, Ji Z,Zhang J and Cai Y L. (2003). Performance analysis of JakimoskiCKocarev attack on a class of chaotic cryptosystems. *Phys. Lett. A*, 22. 307.

Ott E, Grebogi C and Yorke J A. (1990). Controlling chaos. *Phys. Rev. Lett.*, 64.1196.

Pecora L M and Carroll T L. (1990). Synchronization in chaotic system. *Phys. Rev. Lett.*, 64(8). 821.

Robinson R.C. (2004). An introduction to dynamicalsystems:continuous and discrete. *Prentice Hall*.

Sheng L Y , Sun K H and Li C B. (2004). Study of a discrete chaotic system based on tangent delay for elliptic reflecting cavity and its properties, 53, 2871.

Stojanoski T and Kovarev L. (2001). Chaos-based random number generators, Part I.Analysis. *IEEE Tran CAS-I*, 48.3.

Tao Yang and Chua L O. (1997). Impulsive stabilization for control and syn chronization of chaotic systems: Theory and application to secure communication. *IEEE Trans Circuits Syst I*, 44. 976.

Table 1. The value of $H(x) = f^3(x) - x$ at some points

x	a	x_{21}	x_{11}	x_{22}	x_0	x_{23}	x_{12}	x_{24}	b
H	0	>0	<0	>0	<0	<0	<0	<0	<0

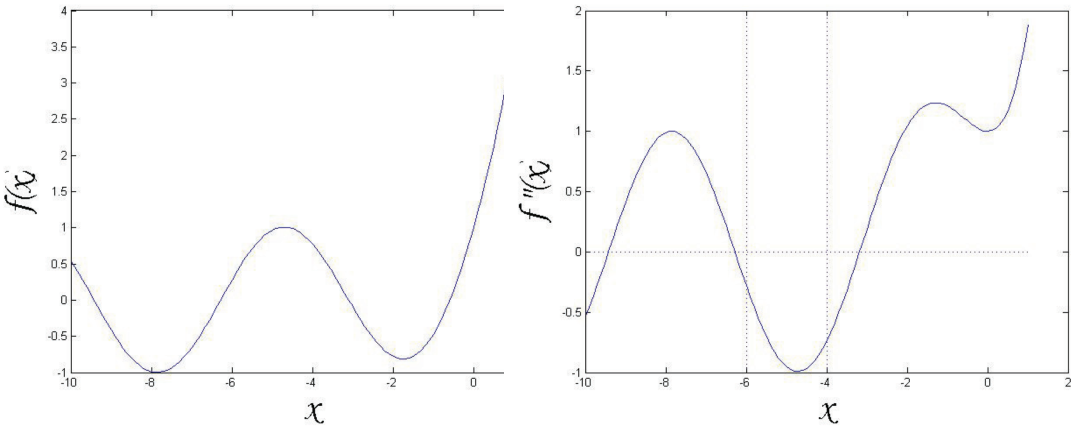

Figure 1. The graphs of $f(x)$ and $f''(x)$

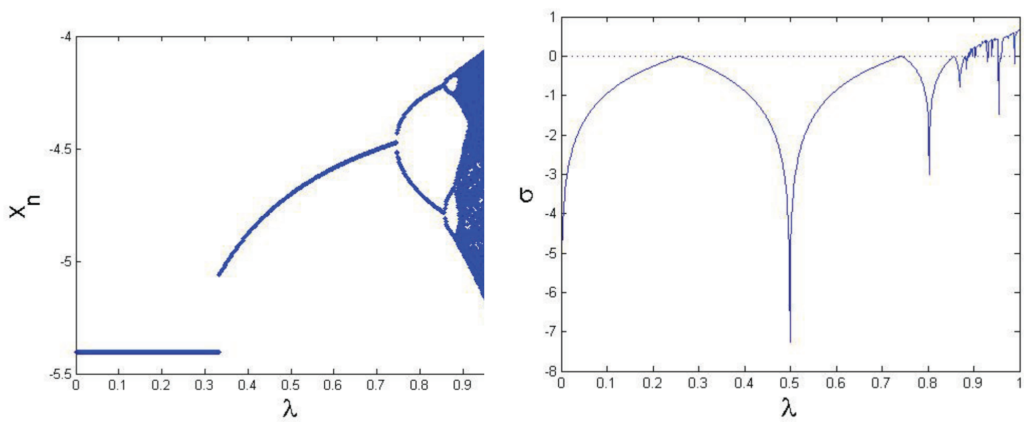

Figure 2. The bifurcation diagram and the plot of Lyapunov exponent evolution for system (3) with $x_0 \in (-5.403376918067952, -4)$, and $\lambda_\infty = 0.887729451801666$.

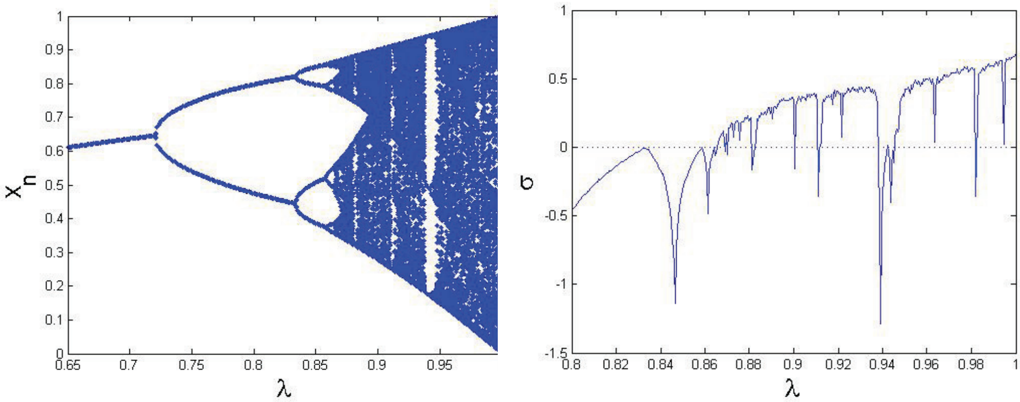

Figure 3. The bifurcation diagram and the Lyapunov exponent evolution of system $x_{n+1} = \lambda \sin(\pi x_n)$ with $x_0 \in (0, 1)$, and $\lambda_\infty = 0.865575993128005$.

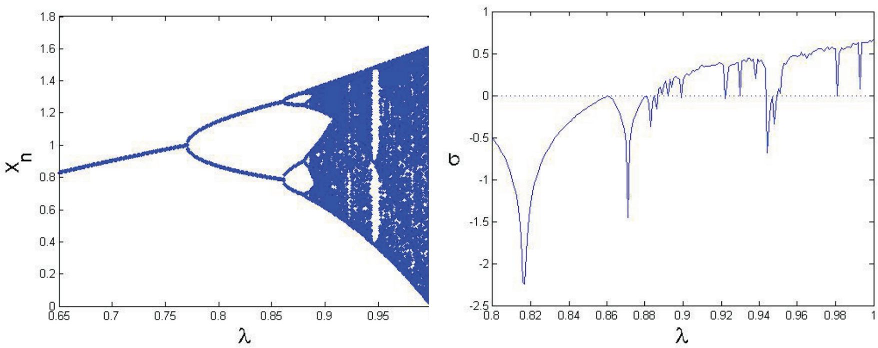

Figure 4. The bifurcation diagram and the Lyapunov exponent evolution of system $x_{n+1} = \frac{3\sqrt{3}}{2} x_n - {x_n}^3$ with $x_0 \in (0, \sqrt[4]{\frac{27}{4}})$, and $\lambda_\infty = 0.886075924343895$.