A Decomposition for the Low Dimensional Cohomology of Semidirect Product of Topological Groups

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Abstract

Let G be a topological group and A a trivial G-module. Suppose G is the semidirect product of a closed normal subgroup N and a subgroup T. In this paper we find, $H^i(G, A)$ $i = 1, 2$, the first and the second cohomology of G in terms of its factors.

Keywords: Semidirect product of topological groups, Complement, Topological extension

1. Introduction

The concept of semidirect product is one of the basic notions in group theory.In recent years it has found its way into Banach algebra theory [Palmer, T.W. 1978; Thomas, M.P. 1991; Berndt O. 1994] and categorical group [Garzon, A.R., 2001]. In this paper we use it in the category of topological groups. In section 2, we recall the semidirect product and cohomology of topological groups [Sahleh, H. 2007]. In section 3, we show that if G is the semidirect product of a normal subgroup N and a subgroup T then $H^i(T, A)$ is a direct summand of $H^i(G, A)$, $i = 1, 2$.

Spaces are assumed to be completely regular and Hausdorff. A *topological extension* of Q by K is a short exact sequence $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$, where i is a topological embedding onto a closed subgroup and π an open continuous onto homomorphism. The extension is *central* if K is in the center of G. We consider extensions with a continuous section i.e. $u : Q \to G$ such that $\pi u = Id$. For example,if Q is a connected locally compact group, then any topological extension of Q by a connected simply connected Lie group has a continuous section [Shtern, A. 2001, theorem 2]. Notation and definitions as in [Berndt, O. 1998].

2. Semidirect product and cohomology of topological groups

2.1 Semidirect product

In this part we define the semidirect product in the category of topological groups.

Definition 2.1.1 Let K and Q be topological groups. The *semidirect product* of Q and K is an exact sequence $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ with a continuous homomorphism $u : Q \to G$ such that $\pi u = Id_Q$. Sometimes G itself is called a semidirect product of Q and A.

Examples:

(1) A direct product $K \times Q$ is a semidirect product of K by Q (also Q by K)

(2) An abelian group is a semidirect product iff it is a direct (usually called a direct sum) since every subgroup of an abelian group is normal.

(3) cyclic groups of prime power order are not semidirect product since they can not be direct sum of two proper subgroups.

For the extensions there is a standard notion of equivalence.

Definition 2.1.2 Let K and Q be topological groups and

$$(e) : 0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$$

$$(e^{'}) : 0 \to K \xrightarrow{i} G^{'} \xrightarrow{\pi} Q \to 0$$

be two semidirect product of Q and K with the homomorphisms $u : Q \to G$, $u^{'} : Q \to G^{'}$, respectively. Then (e) and $(e^{'})$ are *equivalent*, denoted by $e \sim e^{'}$, if there is an open continuous isomorphism $\alpha : G \to G^{'}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & G \arrow[r, "\pi"] \arrow[d, "\alpha"] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & G^{'} \arrow[r, "\pi^{'}"] & Q \arrow[r] & 0
\end{tikzcd}$$

$\pi\alpha = \pi^{'}, \pi u = Id_Q, \pi^{'} u^{'} = Id_Q$

Remark. In definition 2.1.2 it is sufficient to demand that α be continuous isomorphism. It can be shown that α is open as follows: Let U_1 be a neighborhood of identity in G. Since π is open and $\pi^{'}$ is continuous we can choose a neighborhood of identity $V_1 \subset U_1$ in G and a neighborhood of identity U in $G^{'}$ such that $\pi^{'}(U) \subseteq \pi(V_1)$ and $(U\alpha(V_1)^{-1}) \cap K \subseteq U_1$. Then for every $u \in U$, there is a $v \in V_1$ such that $\pi^{'}(u) = \pi(v) = \pi^{'}\alpha(v)$. Hence $u = q\alpha(v)$ where $q \in (U\alpha(V_1)^{-1}) \cap K \subseteq U_1$. Thus $qv \in U_1U_1$ and $\alpha(qv) = u$. We have shown that $\alpha(U_1U_1) \supseteq U$ and so is a neighborhood of identity in G. This is sufficient to say that α is open.

We consider the case where K is abelian.

Proposition 2.1.3 *Let* $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ *be an extension with a continuous section* $u : Q \to G$.

(1) for every $x \in G$, *conjugation* $\theta_x : K \to K$ *defined by* $x.a = u(x)au(x)^{-1}, a \in K$ *is independent of the choice of u.*

(2) The map $\theta : Q \to Aut(K)$, $x \mapsto \theta_x$ *is a homomorphism.*

Proof. (1). Let $u : Q \to G$, and $u^{'} : Q \to G$,$\pi u^{'}(x) = x$, $\pi u(x) = x$. Then $u(x)u^{'}(x) \in ker\pi = K$ Therefore, $u^{'}(x) = u(x)b$ for some $b \in K$.Now $u^{'}(x)au^{'}(x)^{-1} = u(x)a(u(x)b)^{-1} = u(x)bb^{-1}u(x)^{-1} = u(x)au(x)^{-1}$ since K is abelian.

(2). Since K is normal in G, then $\theta_x(a) = u(x)au(x)^{-1} \in K$. So θ_x is a map from K to K. Also θ_X is an automorphism because conjugations are. If $x, y \in Q$, then

$$\theta_x(\theta_y(a)) = \theta_x(u(x)au(x)^{-1}) = u(x)u(y)au(y)^{-1}u(x)^{-1}$$

while

$$\theta_{xy}(a) = u(xy)au(xy)^{-1}$$

But $u(xy)$ and $u(x)u(y)$ both are lifting of xy,$\pi u(xy) = \pi(u(x)u(y))$. So by part (1), $\theta_x\theta_y = \theta_{xy}$.

Remark. The homomorphism θ indicates how K is normal in G. For example let K be a cyclic group of order 3 and $Q =< x >$ be the cyclic group of order 2. If G is the semidirect product, then G is abelian and K lies in the center of G. In this case $u(x)au(x)^{-1} = a$ for all $a \in K$ and $\theta_x = 1_K$.

proposition 2.1.4 *Let K and Q be topological groups with K abelian. Then* $\theta : Q \to Aut(K)$ *makes K into a ZQ-module* $xa = \theta_x(a)$ *for all* $a \in Q$. *Conversely if K is a left ZQ-module then* $x \mapsto \theta_x$ *defines a homomorphism* $\theta \to Aut(K)$.

Proof. Let $b \in ZQ$.Then b has a unique expression of the form $w = \sum_{x \in Q} m_x x$ where $m_x \in Z$ and almost all $m_x = 0$. Define

$$(\sum m_x x)a = \sum m_x \theta_x(a) = \sum m_x(xa)$$

Since θ is a homomorphism, $\theta(1) = 1 + K$, and so $1a = \theta_1(a)$. Since $\theta_x \in Aut(K)$, $x(a + b) = xa + xb$ It follows that $w(a + b) = wa + wb$ for all $w \in ZQ$. Similarly, $(w + v)a = wa + va$ $w, v \in ZQ$. Finally $(wv)a = w(va)$ since $(xy)a = x(ya)$, $x, y \in Q$. But $(xy)A = \theta_{xy}(a) = \theta_x(\theta_y(a)) = \theta_x(ya) = x(ya)$.

Corollary 2.1.4 *If* $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ *is an extension with a continuous section* $u : Q \to G$. *Then K is a left ZQ-module by* $xa = u(x)au(x)^{-1}$ $x \in Q$, $a \in K$. *The multiplication is independent of the choice of u.*

Proof. By propositions 2.1.3 and 2.1.4.

Now we express the semidirect product as a product of groups.

Proposition 2.1.5 *Let K be a normal subgroup of G*

(1) If $0 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ *is a splitting with* $j : Q \to G$, $\pi j = 1_Q$, *then* $i(K) \cap j(Q) = 0$ *and* $i(k)j(Q) = G$

(2) Every $g \in G$ *has a unique form* $g = aj(x), a \in K, x \in Q$

(3) If K and Q are subgroups of G with K normal in G then G is a semidirect product of K by Q iff $K \cap Q = \{1\}$, $KQ = G$ *and each* $g \in G$ *has a unique form* $g = ax$, $a \in K$, $x \in Q$

Proof. (1) : If $g \in i(K) \cap j(Q)$, then $g = i(a) = j(x)$ for some $a \in K, x \in Q$. Now $g = j(x)$ implies that $\pi(g) = \pi j(x) = x, \pi(g) = \pi i(a) = 0$. Therefore, $x = 0$ and $g = j(x) = 0$. If $g \in G$ then $\pi(g) = \pi j\pi(g)$ and so $gj(\pi(g)^{-1}) \in ker\pi$; hence there is $a \in K$ with $a(j\pi(g))^{-1} = i(a)$ and so $g = i(a)j(\pi(g)) \in (i(K)(j(Q))$.

(2) : We identify $i(a)$ as a. If $g = aj(x) = a^{'}j(x^{'})$, then $i(a)i(a^{'})^{-1} = j(x)j(x^{'})^{-1}$. Hence $0 = \pi i(a)i(a^{'})^{-1} = \pi j(x)j(x^{'})^{-1} = xx^{'}$.So $x = x^{'}$. Similarly $a = a^{'}$.

(3) : Since i and j are inclusions, necessity is the spacial case of (2). Conversely if each $g \in G$ has a unique expression $g = ax,\ a \in K,\ x \in Q$. Define $\pi : G \to Q$ by $\pi(ax) = x$. It is easy to check that π is a continuous homomorphism.

Definition 2.1.6 Let G be a topological group and N a G-module. Then a subgroup K of the semidirect product $N \propto G$ is called a complement (or complement to N) if

(1) $N \cap K = 0$

(2) $NK = N \propto G$

Example. The symmetric group, S_3, is the semidirect product of cyclic groups of order 2 and 3 ; $S_3 = Z_3 \propto Z_2$. Let $N = Z_3$. This has complements $<(12)>, <(13)>$ and $<(23)>$, which are all conjugate.

2.2. Cohomology of topological groups

In this part we recall the cohomology of topological groups [Sahleh, H. 2007]. When G is a topological group the theory of cohomology gets more interesting since we have both algebraic and topological notions of cohomology and there are different ways to combine them.

Let G be a topological group and A an abelian topological group on which G acts continuously.

Let $C^n(G, A)$ be the continuous maps $\phi : G^n \to A$ with the coboundary map

$$C^n(G, A) \overset{\delta_n}{\to} C^{n+1}(G, A))$$

given by

$$\delta\phi(g_1, ..., g_n) = g_1.\phi(g_2, ..., g_n)$$

$$+ \sum_{i=1}^{n-1} (-1)^i \phi(g_1, ..., g_i g_{i+1}, ..., g_n) + (-1)^n \phi(g_1, ..., g_{n-1})$$

Note that this is analogous to the inhomogeneous resolution for the discrete case [Fulp R.O., 1976].

Definition 2.2.1. The continuous group cohomology of G with coefficient in A is

$$H^n(G, A) = ker\delta_n / Im\delta_{n-1}$$

Let $Ext_s(G, A)$ be the set of extensions of GA by A with a continuous section. It is known, by the Baer sum, that $Ext_s(G, A)$ is an abelian topological group. By [Hu, S.T. 1952], if G is a topological group and A a trivial G-module then there is an isomorphism between the second cohomology of G and the group of extensions of A by G with continuous sections, namely

$$H^2(G, A) \simeq Ext_s(G, A)$$

Note that if the extension $0 \to A \overset{i}{\to} M \overset{\pi}{\to} G \to 0$ has a continuous section then $M \simeq A \times G$, as topological spaces [Berndt, O. 1998].

3. A decomposition

Let G be the semidirect product of a normal subgroup N and a subgroup T and A a G-module. Suppose N acts trivially on A. In this section we express $H^1(G, A)$ and $H^2(G, A)$ in terms of the first and the second cohomology of N and T.

Remark. Let S be a subgroup of G. By Z_S we mean the additive group of continuous homomorphisms $z : S \to A$ such that

$$\sigma_1(z(\sigma_2)) - z(\sigma_1\sigma_2) + z(\sigma_1) = 0 \ , \quad \sigma_1, \sigma_2 \in S$$

Theorem 3.1 *Let G be the semidirect product of a normal subgroup N and a subgroup T and A a G-module on which N acts trivially. Let I be the subgroup of* $Hom(N, A)$ *such that*

$$\tau h(\nu) = h(\tau_\nu), \forall \tau \in T, \nu \in N$$

Then

$$H^1(G, A) \cong H^1(T, A) \oplus I$$

Proof. Let $f \in Z_G$. Then the restrictions f_T and f_N are in Z_T and Z_N, respectively. Now define a map $i : Z_G \to Z_T \oplus Z_N$ by $i(f) = (f_T, f_N)$. It is clear that i is continuous. We show that $f_N \in I$ and $im(i) = Z_N \oplus I$. The map i is an injective homomorphism since $G = TN$. We also have

$$f(n_1, n_2) = n_1(f(n_2)) + f(n_1) = f(n_1) + f(n_2), \;\; n_1, n_2 \in N$$

since N acts on A trivially. Also for each $n \in N, t \in T$

$$tf(n) = ftn) - f(t) = f(tnt^{-1}) - f(t) = tnt^{-1}f(t) + f(tnt^{-1}) - f(t) = f(tnt^{-1})$$

Thus $f_N \in I$.

Let $g \in Z_T, h \in I$. Since $T \cap N = 0$, define $F : G \to A$

$$F(nt) = g(t) + h(n), \;\; `n \in N, t \in T$$

For all $n_1, n_2 \in N, \; t_1, t_2 \in T$
$n_1t_1(F(n_2t_2) - f(n_1t_1n_2t_2) + F(n_1t_1)$
$= n_1t_1(g(t_2) + h(n_2)) - h(n_1t_1n_2t_1^{-1}) + g(t_1) + h(n_1)$
$= t_1(g(t_2)) + t_1(h(n_2)) - g(t_1t_2) - n_1(h(t_1n_2t_1^{-1})) - h(n_1) + g(t_1) + h(n_1)$
$= t_1(h(n_2)) - h(t_1n_2t_1^{-1})$
$= 0.$

Since $f_T = g$, $F_N = h$, it follows that

$$Im(i) = Z_T \oplus I$$

Let B_S be the addetive subgroup of all continuous maps $b : S \to A$ such that

$$b(s) = sa - a, \;\; , s \in S$$

with some $a \in A$. It is clear that $B_S \subseteq Z_S$, $H^1(S, A) = Z_S/B_S$ and

$$i(B_S) = B_T \oplus B_N, \quad , B_N = \{0\}$$

Hence, i induices an isomorphism from $H^1(G, A)$ to $H^1(T, A) \oplus I$.

Note. For each subgroup S of G we denot

$$S^* = Hom(S, A)$$

By the tensor product of two groups G, H we mean [Fulp R.O., 1976]

$$G \otimes H = Hom(G, \widehat{\hat{H})}$$

If G is finitely generated then $G \otimes H$ is locally compact. In this case the definition of the tensor product coincides with the definition of Moskowitz[1967]. If G and H are descrete then $G \otimes H$ is the usual tensor product of discrete abelian groups.

In the following theorem $N \wedge N$ denotes the usual exterior product of N.

Theorem 3.2 *Let A be a locally compact abelian divisiable group. Suppose N is the direct product of its r (discrete) finite cyclic subgroups,* $N_1, N_2, ..., N_r$ $r \geq 1$ *and for each* $t \in T$ *there exists an integer k such that* $tnt^{-1} = n^k$ *for every* $n \in N$. *Then*

$$H^2(G, A) \cong H^2(T, A) \oplus (N \wedge N) \oplus (\oplus_{i=1}^{r} H^1(T, N_i^*))$$

Proof. Induction on r. Note that the action of T on N_i^* is given by

$$(tf)(n) = f(tnt^{-1}) \;\; , t \in T \; , \; f \in N_i^* \; , \; n \in N_i$$

Let s be a positive integer. Suppose the theorm holds for $r < s$.

Let $N' = N_1N_2...N_{s-1}$, so that G is the semiduirect product of N_s by TN'. Let $R : H^2(G, A) \to H^2(TN', A)$ be the restriction map. Then by [Tahara K, 1972,theorem 2]

$$H^2(G, A) \cong H^2(TN', A) \oplus KerR$$

and there is an exact sequence

$$0 \to H^1(TN^{'}, N_i^*) \to KerR \to H^2(N_S, A)$$

where the action of $TN^{'}$ is given by

$$(\sigma f)(\mu) = f(\sigma\mu\sigma^{-1})\ ,\ \sigma \in TN^{'}\ ,\ f \in N_i^*\ ,\ \mu \in N_S$$

By [Fulp R.O., 1976], $H^2(N_s, A) \cong Ext(N_s, A) \cong A/sA = 0$. Thus

$$(1) \qquad H^2(G, A) \cong H^2(TN^{'}, A) \oplus H^1(TN^{'}, N_i^*)$$

Now by inductiuon

$$H^2(TN^{'}, A) \cong H^2(T, A) \oplus H^2(N^{'}, A) \oplus (\oplus_{i=0}^{s-1} H^1(T, N_i^*)$$

Let $h \in Hom(N^{'}, N_i^*)\ ,\ t \in T\ ,\ t \in T\ ,\ \nu \in N^{'}\ .\ \mu \in N_i$. By assumption

$$t\mu t^{-1} = \nu^k \quad for\ some\ k \in Z$$

Hence

$$(t(h(\mu))(\nu) = (h(\nu))(t\mu t^{-1}) = (h(\nu))(\mu^t)) = t(h(\nu)(\mu)) = (th(\nu))(\mu) = (h(t\nu t^{-1})(\mu)$$

Therefore, $t(h(\nu)) = h(t\nu t^{-1})$. So by theorem 3.1,

$$H^1(TN^{'}, N_i^*) = H^1(T, N_i^*) \oplus Hom(N^{'}, N_i^*)$$

Since $Hom(N^{'}, N_i^*) = \bigoplus_{i=1}^{s-1} Hom(N_i, N_i^*)$ it follows that

$$\begin{aligned} H^2(G, A) &\cong H^2(T, A) \oplus (N^{'} \wedge N^{'}) \oplus H^1(T, N_i^*) \oplus \bigoplus_{i=1}^{s-1} N_i \otimes N_i \\ &\cong H^2(T, A) \oplus (N \wedge N) \oplus \bigoplus_{i=1}^{s} H^1(T, N_i^*). \end{aligned}$$

References

Berndt, O. (1998). A caregorical definition of semidirect products , *Applied categorical structures*, 6, 37-62.

Berndt O. (1994). On semidirect products of commutative Banach algebras, *Quaest. Math*, 17, 67-81.

Fulp R.O., & Griffith, P. (1976). Homological study of purity in locally compact groups, *Proc.London Math.Soc.*, 21, 501-512.

Garzon, A.R., & Inassaridze, H. (2001). Semidirect products of categorical groups. Obstrection theory, *HHA*, 3(6), 111-138.

Hu, S.T. (1952). Cohomology theory in topological groups, *Michigan Math.J.*, 1(1), 11-59.

Mader, A.. (1967). A note on direct and semi-direct products of groups, *Math.Zeischr.*, 95, 272-275.

Moskowiz, M. (1967). Homological algebra in locally compact abelian groups I, *Trans.Amer.Math.Soc.*, 127, 361-401.

Palmer, T.W. (1978). *Banach Algebras and the General theory of *-algebras, Volum 1:Algebras and Banach Algebras*, Cambridge ,Encyclopedia of mathematics and its applications 49,Cambridge University Press.

Sahleh, H. (2007). The semidirect product and first cohomology of topological groups, *Int.Math.Forum.*, 2, 1363-1373.

Scott, W.R. (1964). *Group theory*. Englewood Cliffs: Prentic Hall.

Shtern, A. (2001). Remarks on Pseudochrscters and the Real Continuous Bounded Cohomology of Connected Locally Compact Groups, *Annals of Global analysis and Geometry*, 20, 199-221.

Tahara K. (1972). On the second cohomology groups of semidirect products, *Math.Z.*, 129, 365-379.

Thomas, M.P. (1991). Principal ideals and semi-direct products in commutative Banach algebras, *J.Funct.Anal.*, 101, 312-328.