On Fully-M-Cyclic Modules

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Abstract

The aim of this work was to generalize generator, M-generated modules in order to apply them to a wider class of rings and modules. We started by establishing a new concept which is called a fully-M-cyclic module. We defined this notation by using $Hom_R(M, *)$ operators which are helpful to contract the new construction and describe their properties. Finally, we could see the structure of fully-M-cyclic module and quasi-fully-cyclic module by the structure of M.

Keywords: Fully-M-cyclic modules, Quasi-fully-cyclic modules, Generator modules, Self-generator modules

1. Introduction

Throughout this paper, R is an associative ring with identity and M_R is the category of unitary right R-modules. Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. A right R-module N is called *M-generated* if there exists an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite, then N is called *finitely M-generated.* In particular, N is called *M-cyclic* if it is isomorphic to M/L for some submodule L of M. Following Wisbauer [1991], $\sigma[M]$ denotes the full subcategory of Mod-R, whose objects are the submodules of M-generated modules. A module M is called a *self-generator* if it generates all of its submodules. M is called a *subgenerator* if it is a generator of $\sigma[M]$.

2. On Fully-M-cyclic module

In this part, a module M be given as a right R-module.

Definition 2.1. Let $N \in M_R$. N is called a fully-M-cyclic module if every submodule A of N is of the form $s(M)$ for some s in $Hom_R(M, N)$.

Remark 2.2. Dealing directly from definition, the following statements are routine:
(1) Submodule of a fully-M-cyclic module is a fully-M-cyclic module.
(2) If M is simple module and N is fully-M-cyclic module, then any nonzero submodule of N is simple submodule.

Definition 2.3. The module $M \in M_R$ is called a quasi-fully-cyclic module if it is a fully-M-cyclic module.

Obviously, every semi-simple module is a quasi-fully-cyclic module.

Lemma 2.4. *Let N be a fully-M-cyclic module. If M is a noetherian module then $Soc(M) \cong Soc(N)$.*

Proof. Since N is a fully-M-cyclic module, a simple submodule B of N is of the form $s(M)$ for some $s \in Hom_R(M, N)$. By the simply property of B, there is $b \in B$ such that $B = bR$. Suppose that $s(a) = b$ for some $a \in M$. In noetherian module aR, there exists a simple submodule A containing a. It is easily to see that $A \cong B$. Conversely, if A is a simple submodule of M then $s(A) = B$ is a simple submodule of N and then $A \cong B$ for all $s \in Hom_R(M, N)$. This shows that $Soc(M) \cong Soc(N)$. □

Lemma 2.5. *If N is a fully-M-cyclic module then N has no nonzero small submodule.*

Proof. In a contrary, we suppose that there is a nonzero submodule A which is small in N. Let B be a submodule of N

such that $A + B = N$. Since N is a fully-M-cyclic module, there are $s, t \in Hom_R(M, N)$ such that $s(M) = A, t(M) = B$. Put $f = s + t$, then f is an epimorphism from M to N. Since A is a small submodule of N, t is an epimorphism and hence s is an epimorphism. It follows that $A = N$, a contradiction, showing that N has no nonzero small submodule. □

Corollary 2.6. *If N is a fully-M-cyclic module then $Rad(N) = 0$.*

Definition 2.7. Let N be a fully-M-cyclic module. For a submodule A of N there exists a homomorphism $s \in Hom_R(M, N)$ such that $s(M) = A$. s is called a *presented homomorphism* of A.

Lemma 2.8. *Let N be a fully-M-cyclic module. If s is a presented homomorphism of a submodule A of N then A is maximal if and only if every $t \in S = Hom_R(M, N)$ with $Im(t)$ containing the image of presented homomorphism of A is an epimorphism.*

Proof. Let $A = s(M) \subset_> Im(t)$ in N. Since A is a maximal submodule of N then $Im(t)$ must be N, and hence t is an epimorphism. Conversely, let $A = s(M)$ and $A \subset_> B$. Since N is a fully-M-cyclic module, there is an element $t \in Hom_R(M, N)$ such that $B = t(M)$. By assumption, the non equality $s(M) \subset_> t(M)$ follows that t is an epimorphism, and hence $B = N$. □

Leading directly from definition, the following properties in Lemma 2.9 are routine,

Lemma 2.9. *Let N be a fully-M-cyclic module and A be a submodule of N and s its a presented homomorphism.*

(1) *If M is an epimorphism image of M' then N is also a fully-M'-cyclic module.*
(2) *If M is a fully-M'-cyclic module then N is also a fully-M'-cyclic module.*
(3) *A is an essential in N if and only if for any nonzero element t of $Hom_R(M, N)$, $Im(t) \cap Im(s) \neq 0$.*
(4) *A is uniform if and only if every $t \in Hom_R(M, N)$ with $0 \neq Im(t) \subset_> Im(s)$ then $Im(t)$ is an essential in $Im(s)$.*
(5) *A is a direct summand of N if and only if there exists $t \in Hom_R(M, N)$ such that $Im(s) \cap Im(t) = 0$ and $s + t$ is an epimorphism.*

3. Quasi-fully-cyclic module

In this part, we put $S = End_R(M)$. We have known that for any right R-module M, the direct summand A of M is image of a presented homomorphism which is an idempotent of S but not all. Which is case of the form submodules such that every its presented homomorphisms are idempotents?. The following lemma is a clear answer:

Lemma 3.1. *Let M be a quasi-fully-cyclic module. If A is a simple submodule of M with s its a presented homomorphism then s is an idempotent of $S = End_R(M)$.*

Proof. Let s be a presented homomorphism of A. Because A is a simple submodule of M then $s^2(A) \neq 0$. Therefore, we have $0 \neq s^2(M) \subset_> s(M) = A$ and $s^2(M)$ must be equal to $A = s(M)$, showing that s is an idempotent of S. □

Right now, we suppose that M be a quasi-fully-cyclic module. If $e^2 = e$, the one gets a direct sum decomposition $M = e(M) \oplus (1 - e)(M)$. Conversely, if $M = A \oplus B$ then we can write $1 = \pi_A + \pi_B$ with π_A (resp. π_B) being a natural projection map from M to A (resp. B). π_A (resp. π_B) is an idempotent element of S which is a presented homomorphism of A (resp. B) so that we can get the following corollary.

Corollary 3.2. *In a quasi-fully-cyclic module, every simple submodule is a direct summand.*

Theorem 3.3. *Let M be a quasi-fully-cyclic module. M is a Noetherian (resp. Artinian) if and only if S is a right self Noetherian (resp. Artinian) ring.*

Proof. Suppose that M is Noetherian. We may easily analogize our self the proof of the case Artinian. Take any ascending chain of the right ideals $s_1S \subset_> s_2S \subset_> s_3S \subset_> ...s_nS \subset_> ...$ of the ring S. Since $s_iS \subset_> s_{i+1}S$, $s_i = s_{i+1}t$ for some $t \in S$. We have $s_i(M) \subset_> s_{i+1}(M)$ for all $i \in N$. The ascending chain of the submodules $s_1(M) \subset_> s_2(M) \subset_> s_3(M) \subset_> ... \subset_> s_n(M) \subset_> ...$ must be stationary in the noetherian module M so that $s_{n_0}(M) = s_{n_0+j}(M)$ for some n_0 and all $j \geq 0$. This implies that for $i \geq n_0$ there is a permutation function $t \in S$ such that $s_{i+1} = s_it$ for some $t \in S$. It follows that $s_{i+1}S \subset_> s_iS$, and hence $s_{i+1}S \subset_> s_iS$ for all $i \geq n_0$. It says that the ascending chain of the right ideals $s_1S \subset_> s_2S \subset_> s_3S \subset_> ...s_nS \subset_> ...$ must be exact stationary at n_0. Conversely, if S is a right self noetherian ring. Take any ascending chain of the submodules $A_1 \subset_> A_2 \subset_> A_3 \subset_> ... \subset_> A_n \subset_> ...$ of M. Since M is a quasi-fully-cyclic module, for every i index, there is $s_i \in S$ such that $s_i(M) = A_i$. Following that $s_1S \subset_> s_2S \subset_> s_3S \subset_> ...s_nS \subset_> ...$ is a ascending chain of the right ideals of S. By assumption, this ascending chain must be stationary at some n_0 index.

Therefore, $s_iS = S_{i+1}S$ for all $i \geq n_o$. This shows that $s_i(M) = s_{i+1}(M)$ for all $i \geq n_0$. And hence the given ascending chain of the submodules $A_1 \subset_> A_2 \subset_> A_3 \subset_> ... \subset_> A_n \subset_> ...$ is stationary at n_0. The proof now is completed. □

Lemma 3.4. *For each quasi-fully-cyclic-module, the following statements are equivalent:*

(1) *S is artinian;*

(2) *M is finitely co-generated;*

(3) *M is semisimple and finitely generated;*

(4) *M is semisimple and noetherian;*

(5) *M is the direct sum of a finite set of simple submodules.*

Proof. We refer to the ([Anderson, 1974], Proposition 10.15) for the proving of $3 \Longleftrightarrow 4 \Longleftrightarrow 5$. By the Theorem 3.3, we know that S is artinian if and only if M is artinian. By the Corollary 2.6, we have $Rad(M) = 0$. The proof is now completed by turning back to apply the ([Anderson, 1974], Proposition 10.15). □

Definition 3.5. Let M be a right R-module. M is called *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism of M is an automorphism.

Definition 3.6. Let M be a right R-module. M is called a *Fitting* module if every endomorphism f of M satisfies Fitting's lemma (i.e. there exists an integer $n \geq 1$ such that $M = Ker(f^n) \oplus Im(f^n)$).

Lemma 3.7. *Let M be a quasi-fully-cyclic-module. If M is finitely cogenerated and Hopfian then for any $s \in S$ there exists an integer number n such that $M = Ker(s^n) \oplus Im(s^n)$.*

Proof. Since M is both a quasi-fully-cyclic and finitely cogenerated and by the Lemma 3.4, we have M is artinian. Applying the ([Anderson, 1974], Lemma 11.6) to the Hopfian module M, we have M is a Fitting module. This shows that for any $s \in S$ there exists an integer number n such that $M = Ker(s^n) \oplus Im(s^n)$. □

Theorem 3.8. *Let M be a quasi-fully-cyclic module.*
(1) *For any $s, u \in S$, $l_S(Im(u)) + Ss \subset_> l_S(Im(u) \cap Ker(s))$.*
(2) If N is a maximal submodule of M then $l_S(N)$ is a minimal left ideal of S.

Proof. (1) According to the relationship $Im(u) \cap Ker(s) \subset_> Im(u)$ follows that $l_S(Im(u)) \subset_> l_S(Im(u) \cap Ker(s))$. Take any $ts \in Ss$ and $m \in Im(u) \cap Ker(s)$. We have $ts(m) = 0$. It implies that $ts \in l_S(Im(u) \cap Ker(s))$, and hence $Ss \subset_> l_S(Im(u) \cap Ker(s))$. Therefore, $l_S(Im(u)) + Ss \subset_> l_S(Im(u) \cap Ker(s))$.

(2) Since M is quasi-fully-cyclic module, there exists $s_0 \in S$ such that $s_0(M) = N$. Therefore, $l_S(N) = \{t \in S | ts_0 = 0\}$. It is easy to see that $l_S(N)$ is one of the form of left ideals of S. Take any $0 \neq t \in l_S(N)$ then $t(N) = 0$ saying that $N \subset_> Ker(t)$. By maximality of N, $Ker(t)$ is N. Right now, if we take any $k \in l_S(N)$, $k(N) = 0$ shows that $Ker(t) \subset Ker(k)$. It follows that there is $s \in S$ such that $k = st$, and hence $k \in St$. Thus it is $l_S(N) \subset St$, and hence $l_S(N) = St$, showing minimality of $l_S(N)$. □

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