Estimation of the Parameters of Bivariate Normal Distribution with Equal Coefficient of Variation Using Concomitants of Order Statistics

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Abstract

To provide an optimum estimator for the parameters, the use of priori information has a crucial role in univariate as well as bivariate distributions. One such prior information is to utilize the knowledge on coefficient of variation in the inference problems. In the past plenty of work was carried out regarding the estimation of the mean μ of the normal distribution with known coefficient of variation. Also inference about the parameters of bivariate normal distribution in which X and Y have the equal (known) coefficient of variation c, are extensively discussed in the available literature. Such studies arise in clinical chemistry and pharmaceutical sciences. It is interesting to note that concomitants of order statistics are applied successfully to deal with statistical inference problems associated with several real life situations. A problem of interest considered here is the estimation of parameters of bivariate normal distribution in which X and Y have the same coefficient of variation c using concomitants of order statistics. For that consider a sample of n pairs of observations from a bivariate normal distribution in which X and Y have the same coefficient of variation c, we derive the best linear unbiased estimator (BLUE) of θ_2 and derive some estimators of ϑ. Efficiency comparisons are also made on the proposed estimators with some of the usual estimators, finally we conclude that efficiency of our best linear unbiased estimator (BLUE) $\tilde{\theta}_2$ is much better than that of the estimators $\hat{\theta}_2$ and θ_2^*.

Keywords: coefficient of variation, concomitants of order statistics, best linear unbiased estimator, bivariate normal distribution

1. Introduction

The use of prior information in inference is well established in the Bayesian arena of statistical methodology. Such information is usually incorporated into a model by choosing an appropriate prior distribution. In some instances prior information can be incorporated in classical model as well. Searls and Intarapanich (1990) considered an estimator of variance when the kurtosis of the sampled population is known. But in some situations of biological and physical sciences, where the scale parameter is proportional to the location parameter, knowing the proportionality constant is equivalent to knowing the population coefficient of variation (Gleser & Healy 1976), however, inferential testing procedures become more complex, and the property of completeness is no longer held true, and, thus the standard theory of uniformly minimum variance unbiased estimation (UMVUE) is not applicable. Several authors, have studied these problems; Searls (1964) used the coefficient of variation to improve the precision of sample mean as an estimator of the population mean. If the parent distribution is normal with mean θ and standard deviation $c\theta$ thus, distributed as $N(\theta, c^2\theta^2)$, the problem of estimating θ has been studied (Kunte, 2000; Guo & Pal, 2003); the best linear unbiased estimator (BLUE) of θ for $N(\theta, c^2\theta^2)$ distribution for different values of c using order statistics are discussed in Thomas and Sajeevkumar (2003); and estimating the location parameter θ of the exponential distribution with known coefficient of variation by ranked set sampling are discussed by Irshad and Sajeevkumar (2011); and Sajeevkumar and Irshad (2011) have also dicussed estimation of the location parameter θ of the exponential distribution using censored samples.

Estimation of means of two normal populations is frequently undertaken, assuming that the variances of the populations are known. Sen and Gerig (1975) developed an estimator under the equality of the two coefficients of

variation assumption; Azen and Reed (1973) consider the problem of estimation of parameters of bivariate normal distribution with equal coefficient of variation, however, they fail to give explicit expression for the estimates of the parameters. Sajeevkumar and Irshad (2012) have recently studied the estimation parameters of bivariate normal distribution with equal coefficient of variation using concomitants of record values. This paper discuss the estimation of parameters of bivariate normal distribution with equal coefficient of variation say $BVND(\theta_1, \theta_2, c\theta_1, c\theta_2, \vartheta)$ with means $\theta_1 > 0$ and $\theta_2 > 0$, standard deviations $c\theta_1$ and $c\theta_2$, and correlation coefficient ϑ using concomitants of order statistics.

Let $(X_i, Y_i), i = 1, 2, \cdots, n$ be a random sample from an arbitrary bivariate distribution with probability density function (pdf) $f(x, y)$. If the sample values on the variate X are ordered as $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$, then the accompanying Y-observation in an ordered pair with X-observation equal to $X_{r:n}$ is called the concomitant of the rth order statistic $X_{r:n}$ and is denoted by $Y_{[r:n]}$. There is extensive literature available on the application of concomitants of order statistics such as in: biological selection problems (Yoe & David, 1984), ocean engineering (Castillo, 1988), development of structural designs (Coles & Tawn, 1994). Harrel and Sen (1979) used concomitants of order statistics to estimate the correlation coefficient ϑ of a bivariate normal distribution. An excellent review of work on concomitants of order statistics is available in David and Nagaraja (2003).

David (1973) considered the bivariate normal model say $BVND(\theta_1, \theta_2, \phi_1, \phi_2, \vartheta)$ in which the variable Y is linked with the variable X through the regression model:

$$Y = \theta_2 + \vartheta\phi_2\left(\frac{X - \theta_1}{\phi_1}\right) + Z,$$

where $Z \sim N(0, \phi_2^2(1 - \vartheta^2))$ and error term Z is independent of X. Thus for $r = 1, 2, \cdots, n$,

$$Y_{[r:n]} = \theta_2 + \vartheta\phi_2\left(\frac{X_{r:n} - \theta_1}{\phi_1}\right) + Z_{[r]},$$

where $Z_{[r]}$ denotes the particular error Z_r associated with $X_{r:n}$. Due to the independents of X_r and Z_r, we have, set of $X_{r:n}$ is independent of $Z_{[r]}$. Yang (1977) showed that:

$$E[Y_{[r:n]}] = E[m(X_{r:n})], \tag{1}$$

$$Var[Y_{[r:n]}] = Var[m(X_{r:n})] + E[\sigma^2(X_{r:n})] \tag{2}$$

and,

$$Cov[Y_{[r:n]}, Y_{[s:n]}] = Cov[m(X_{r:n}), m(X_{s:n})], \quad 1 \le r \le s \le n, \tag{3}$$

where $m(X_{r:n}) = E(Y/X_{r:n} = x)$ and $\sigma^2(X_{r:n}) = Var(Y/X_{r:n} = x)$.

2. Inference on θ_2 When c and ϑ are Known Using Concomitants of Order Statistics

Let $(X_i, Y_i); i = 1, 2, \cdots, n$ be a random sample of size n drawn from $BVND(\theta_1, \theta_2, c\theta_1, c\theta_2, \vartheta)$. Let $Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]}$ be the concomitants of order statistics arising out of ordering $X_1, X_2, \cdots, X_n$, also let $\mathbf{Y}_{[n]} = (Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]})'$ be the vector of concomitants of order statistics. Using (1), (2) and (3), we obtain means and vaiances of $Y_{[r:n]}$ and covariances of $Y_{[r:n]}, Y_{[s:n]}$ from $BVND(\theta_1, \theta_2, c\theta_1, c\theta_2, \vartheta)$, and are given by:

$$E[Y_{[r:n]}] = \theta_2 + \vartheta c\theta_2 \, \delta_{r:n} = \theta_2(1 + \vartheta c\delta_{r:n}), \tag{4}$$

$$Var[Y_{[r:n]}] = c^2\theta_2^2[\vartheta^2(V_{r,r:n} - 1) + 1], \tag{5}$$

and,

$$Cov[Y_{[r:n]}, Y_{[s:n]}] = c^2\theta_2^2\vartheta^2 V_{r,s:n}, \quad r \ne s, \tag{6}$$

where,

$$\delta_{r:n} = E\left(\frac{X_{r:n} - \theta_1}{c\theta_1}\right),$$

$$V_{r,r:n} = Var\left(\frac{X_{r:n} - \theta_1}{c\theta_1}\right),$$

and,

$$V_{r,s:n} = Cov\left(\frac{X_{r:n} - \theta_1}{c\theta_1}, \frac{X_{s:n} - \mu_1}{c\theta_1}\right), \quad r, s = 1, 2, \cdots, n.$$

Let $\mathbf{Y}_{[n]}=(Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]})'$ be the vector of concomitants of order statistics, $\delta = \vartheta(\delta_{1:n}, \delta_{2:n}, \cdots, \delta_{n:n})'$ and $\mathbf{H}=(1 - \vartheta^2)\mathbf{U} + \vartheta^2\mathbf{V}$, $\mathbf{U}$ is an identity matrix of dimension n and $\mathbf{V} = ((V_{i,j}))$. Then by consider θ_2 as the location parameter of Y variable, a linear unbiased estimator of θ_2 based on concomitants of order statistics is given by (same argument that of Balakrishnan & Rao, 1998, p. 13):

$$\hat{\theta}_2 = -\frac{\delta'\mathbf{H}^{-1}(\mathbf{1}\ \delta' - \delta\ \mathbf{1}')\mathbf{H}^{-1}}{(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2}\mathbf{Y}_{[n]}, \tag{7}$$

and,

$$Var(\hat{\theta}_2) = \frac{(\delta'\mathbf{H}^{-1}\delta)c^2\theta_2^2}{(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2}. \tag{8}$$

Since the X variable is symmetric about θ_1, then by the same argument that of (David & Nagaraja, 2003, p. 188),

$$\delta'\mathbf{H}^{-1}\mathbf{1} = 0. \tag{9}$$

Using (7), (8) and (9) reduces to:

$$\hat{\theta}_2 = \frac{\mathbf{1}'\mathbf{H}^{-1}}{\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}}\mathbf{Y}_{[n]}, \tag{10}$$

and,

$$Var(\hat{\theta}_2) = \frac{c^2\theta_2^2}{\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}}. \tag{11}$$

Introducing $c\theta_2$ as the scale parameter of Y variable, a linear unbiased estimator of $c\theta_2$ based on concomitants of order statistics is given by (same argument that of Balakrishnan & Rao, 1998, p. 13):

$$T = \frac{\mathbf{1}'\mathbf{H}^{-1}(\mathbf{1}\ \delta' - \delta\ \mathbf{1}')\mathbf{H}^{-1}}{(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2}\mathbf{Y}_{[n]}, \tag{12}$$

and,

$$Var(T) = \frac{(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1})c^2\theta_2^2}{(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2}. \tag{13}$$

Using (9), (12) and (13) reduces to:

$$T = \frac{\delta'\mathbf{H}^{-1}}{\delta'\mathbf{H}^{-1}\delta}\mathbf{Y}_{[n]}, \tag{14}$$

and,

$$Var(T) = \frac{c^2\theta_2^2}{\delta'\mathbf{H}^{-1}\delta}. \tag{15}$$

From (12) we can obtain another linear unbiased estimator θ_2^* of θ_2 based on concomitants of order statistics and is given by:

$$\theta_2^* = \frac{1}{c}\frac{\mathbf{1}'\mathbf{H}^{-1}(\mathbf{1}\ \delta' - \delta\ \mathbf{1}')\mathbf{H}^{-1}}{\left[(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2\right]}\mathbf{Y}_{[n]}, \tag{16}$$

and,

$$Var(\theta_2^*) = \frac{(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1})\theta_2^2}{(\delta'\mathbf{H}^{-1}\delta)(\mathbf{1}'\mathbf{H}^{-1}\mathbf{1}) - (\delta'\mathbf{H}^{-1}\mathbf{1})^2}. \tag{17}$$

using (9), (16) and (17) reduces to:

$$\theta_2^* = \frac{\delta'\mathbf{H}^{-1}}{c\delta'\mathbf{H}^{-1}\delta}\mathbf{Y}_{[n]}, \tag{18}$$

and,

$$Var(\theta_2^*) = \frac{\theta_2^2}{\delta'\mathbf{H}^{-1}\delta}. \tag{19}$$

where $\mathbf{1}$ is an $n \times 1$ matrix. The BLUE $\tilde{\theta}_2$ of θ_2 is given in the following theorem.

Theorem 1 *Let $(X_i, Y_i); i = 1, 2, \cdots, n$ be a random sample of size n drawn from BVND $(\theta_1, \theta_2, c\theta_1, c\theta_2, \vartheta)$. Let $\mathbf{Y}_{[n]}=(Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]})'$ be the vector of concomitants of order statistics, $\delta = \vartheta(\delta_{1:n}, \delta_{2:n}, \cdots, \delta_{n:n})'$ and $\mathbf{H}=(1-\vartheta^2)\mathbf{U} + \vartheta^2\mathbf{V}$, $\mathbf{U}$ is an identity matrix of dimension n and $\mathbf{V} = ((V_{i,j}))$ then the BLUE of θ_2 is given by:*

$$\tilde{\theta}_2 = \frac{(c\delta + \mathbf{1})'\mathbf{H}^{-1}}{(c\delta + \mathbf{1})'\mathbf{H}^{-1}(c\delta + \mathbf{1})}\mathbf{Y}_{[n]}, \tag{20}$$

and,

$$Var(\tilde{\theta}_2) = \frac{c^2\theta_2^2}{(c\delta + \mathbf{1})'\mathbf{H}^{-1}(c\delta + \mathbf{1})}. \tag{21}$$

Proof. If $\mathbf{Y}_{[n]}=(Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]})'$ be the vector of concomitants of order statistics. From (4) we have:

$$E[\mathbf{Y}_{[n]}] = \theta_2\mathbf{1} + c\theta_2\delta$$

$$= (c\delta + \mathbf{1})\theta_2, \tag{22}$$

where $\mathbf{1}$ is an $n \times 1$ matrix and $\delta = \vartheta(\delta_{1:n}, \delta_{2:n}, \cdots, \delta_{n:n})'$. From (5) and (6) the dispersion matrix of $\mathbf{Y}_{[n]}$ is obtained as:

$$D(\mathbf{Y}_{[n]}) = \mathbf{H}c^2\theta_2^2, \tag{23}$$

where $\mathbf{H} = (1-\vartheta^2)\mathbf{U} + \vartheta^2\mathbf{V}$, $\mathbf{U}$ is an identity matrix of dimension n and $\mathbf{V} = ((V_{i,j}))$, where ϑ and c are known, then (22) and (23) defines a generalised Gauss-Markov setup, the BLUE $\tilde{\theta}_2$ of θ_2 is given by:

$$\tilde{\theta}_2 = \frac{(c\delta + \mathbf{1})'\mathbf{H}^{-1}}{(c\delta + \mathbf{1})'\mathbf{H}^{-1}(c\delta + \mathbf{1})}\mathbf{Y}_{[n]}$$

and,

$$Var(\tilde{\theta}_2) = \frac{c^2\theta_2^2}{(c\delta + \mathbf{1})'\mathbf{H}^{-1}(c\delta + \mathbf{1})}.$$

Using (9), (20) and (21) reduces to:

$$\tilde{\theta}_2 = \frac{\left(c\delta'\mathbf{H}^{-1} + \mathbf{1}'\mathbf{H}^{-1}\right)}{c^2\delta'\mathbf{H}^{-1}\delta + \mathbf{1}'\mathbf{H}^{-1}\mathbf{1}}\mathbf{Y}_{[n]} \tag{24}$$

and,

$$Var(\tilde{\theta}_2) = \frac{c^2\theta_2^2}{c^2\delta'\mathbf{H}^{-1}\delta + \mathbf{1}'\mathbf{H}^{-1}\mathbf{1}}. \tag{25}$$

This proves the theorem. □

Remark 1 The estimator $\tilde{\theta}_2$ defined in (24) is a convex combination of the estimators $\hat{\theta}_2$ and θ_2^* defined respectively in (10) and (18).

Proof.

$$\text{Let} \quad R = a\hat{\theta}_2 + (1-a)\theta_2^*$$

and,

$$Var(R) = a^2 Var(\hat{\theta}_2) + (1-a)^2 Var(\theta_2^*).$$

Now we should find a such that $Var(R)$ is minimum. That is we have to find a such that $\frac{\partial Var(R)}{\partial a} = 0$. which implies:

$$a = \frac{Var(\theta_2^*)}{Var(\hat{\theta}_2) + Var(\theta_2^*)}.$$

For this values of a, we have $R = \tilde{\theta}_2$. Hence the proof. □

We have calculated the coefficients of $Y_{[i]}$ in the BLUE $\tilde{\theta}_2$, for different values of n, c and ϑ. Also we have computed the numerical values of $Var(\hat{\theta}_2)$ given in (11), $Var(\theta_2^*)$ given in (19), $Var(\tilde{\theta}_2)$ given in (25), the efficiency $E_1 = \frac{Var(\hat{\theta}_2)}{Var(\tilde{\theta}_2)}$ of $\tilde{\theta}_2$ compared to $\hat{\theta}_2$, the efficiency $E_2 = \frac{Var(\theta_2^*)}{Var(\tilde{\theta}_2)}$ of $\tilde{\theta}_2$ compared to θ_2^*, for different values of n, c and ϑ, and are given in the following tables. It may be noted that in all the cases efficiency of our estimator $\tilde{\theta}_2$ is much better than that of the estimators $\hat{\theta}_2$ and θ_2^*.

Table 1. Coefficients of $Y_{[i]}$ in the BLUE $\tilde{\theta}_2$, $Q_1 = \frac{Var(\tilde{\theta}_2)}{\theta_2^2}$, $Q_2 = \frac{Var(\theta_2^*)}{\theta_2^2}$, $Q_3 = \frac{Var(\bar{\theta}_2)}{\theta_2^2}$ and the relative efficiencies E_1 and E_2 for $c = 0.15$

n	ϑ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	Q_1	Q_2	Q_3	E_1	E_2
							Coefficients									
2	0.50	0.47383	0.52405									0.01125	5.28318	0.01123	1.00178	470.45236
	0.75	0.44775	0.54601									0.01125	1.79252	0.01118	1.00626	160.33274
3	0.50	0.30743	0.33228	0.35713								0.00750	2.36988	0.00748	1.00267	316.82888
	0.75	0.28262	0.33031	0.37799								0.00750	0.81847	0.00743	1.00942	110.15747
4	0.50	0.22652	0.24259	0.25553	0.27160							0.00563	1.48694	0.00560	1.00536	265.52500
	0.75	0.20452	0.23519	0.25945	0.29012							0.00563	0.51892	0.00556	1.01259	93.33094
5	0.50	0.17887	0.19057	0.19916	0.20776	0.21946						0.00450	1.07237	0.00448	1.00446	239.36830
	0.75	0.15935	0.18164	0.19764	0.21364	0.23593						0.00450	0.37684	0.00445	1.01124	84.68315
6	0.50	0.14754	0.15666	0.16302	0.16883	0.17518	0.18430					0.00375	0.83456	0.00373	1.00536	223.74263
	0.75	0.13005	0.14741	0.15921	0.16994	0.18174	0.19909					0.00375	0.29473	0.00370	1.01351	79.65676
7	0.50	0.12541	0.13284	0.13784	0.14219	0.14653	0.15154	0.15897				0.00321	0.68132	0.00320	1.00313	212.91250
	0.75	0.10958	0.12370	0.13298	0.14098	0.14899	0.15826	0.17239				0.00321	0.24151	0.00317	1.01262	76.18612
8	0.50	0.10896	0.11520	0.11930	0.12275	0.12603	0.12948	0.13358	0.13983			0.00281	0.57473	0.00280	1.00357	205.26071
	0.75	0.09449	0.10635	0.11395	0.12029	0.12631	0.13266	0.14026	0.15211			0.00281	0.20432	0.00277	1.01444	73.76173
9	0.50	0.09626	0.10162	0.10508	0.10793	0.11055	0.11318	0.11602	0.11948	0.12485		0.00250	0.49650	0.00249	1.00402	199.39759
	0.75	0.08294	0.09312	0.09953	0.10476	0.10956	0.11437	0.11959	0.12600	0.13619		0.00250	0.17693	0.00247	1.01215	71.63158
10	0.50	0.08617	0.09086	0.09385	0.09626	0.09843	0.10054	0.10272	0.10513	0.10811	0.11280	0.00225	0.43672	0.00224	1.00446	194.96429
	0.75	0.07381	0.08272	0.08824	0.09267	0.09665	0.10050	0.10448	0.10891	0.11443	0.12334	0.00225	0.15593	0.00222	1.01351	70.23874

Table 2. Coefficients of $Y_{[i]}$ in the BLUE $\tilde{\theta}_2$, $Q_1 = \frac{Var(\hat{\theta}_2)}{\theta_2^2}$, $Q_2 = \frac{Var(\theta_2^*)}{\theta_2^2}$, $Q_3 = \frac{Var(\tilde{\theta}_2)}{\theta_2^2}$ and the relative efficiencies E_1 and E_2 for $c = 0.20$

n	ϑ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	Q_1	Q_2	Q_3	E_1	E_2
						Coefficients										
2	0.50	0.46469	0.53154									0.02000	5.28318	0.01992	1.00402	265.21988
	0.75	0.42929	0.55968									0.02000	1.79252	0.01978	1.01112	90.62285
3	0.50	0.29841	0.33147	0.36452								0.01333	2.36988	0.01326	1.00528	178.72398
	0.75	0.26485	0.32799	0.39113								0.01333	0.81847	0.01312	1.01601	62.38388
4	0.50	0.21836	0.23973	0.25693	0.27830							0.01000	1.48694	0.00993	1.00705	149.74219
	0.75	0.18868	0.22924	0.26131	0.30187							0.01000	0.51892	0.00981	1.01937	52.89704
5	0.50	0.17154	0.18710	0.19852	0.20994	0.22549						0.00800	1.07237	0.00794	1.00756	135.05919
	0.75	0.14526	0.17471	0.19584	0.21698	0.24643						0.00800	0.37684	0.00783	1.02171	48.12771
6	0.50	0.14092	0.15304	0.16149	0.16920	0.17765	0.18977					0.00667	0.83456	0.00661	1.00908	126.25719
	0.75	0.11740	0.14031	0.15590	0.17006	0.18565	0.20856					0.00667	0.29473	0.00652	1.02301	45.20399
7	0.50	0.11938	0.12925	0.13589	0.14167	0.14744	0.15409	0.16396				0.00571	0.68132	0.00567	1.00705	120.16226
	0.75	0.09811	0.11675	0.12899	0.13956	0.15012	0.16236	0.18100				0.00571	0.24151	0.00558	1.02329	43.28136
8	0.50	0.10342	0.11171	0.11716	0.12174	0.12610	0.13069	0.13613	0.14442			0.00500	0.57473	0.00496	1.00806	115.87298
	0.75	0.08400	0.09965	0.10967	0.11804	0.12599	0.13436	0.14438	0.16003			0.00500	0.20432	0.00488	1.02459	41.86885
9	0.50	0.09114	0.09826	0.10286	0.10664	0.11013	0.11361	0.11739	0.12199	0.12911		0.00444	0.49650	0.00441	1.00680	112.58503
	0.75	0.07327	0.08670	0.09516	0.10205	0.10839	0.11472	0.12162	0.13007	0.14351		0.00444	0.17693	0.00434	1.02304	40.76728
10	0.50	0.08141	0.08764	0.09160	0.09480	0.09769	0.10049	0.10338	0.10658	0.11055	0.11678	0.00400	0.43672	0.00396	1.01010	110.28283
	0.75	0.06484	0.07659	0.08387	0.08971	0.09496	0.10004	0.10529	0.11113	0.11841	0.13016	0.00400	0.15593	0.00390	1.02564	39.98205

Table 3. Coefficients of $Y_{[i]}$ in the BLUE $\tilde{\theta}_2$, $Q_1 = \frac{Var(\hat{\theta}_2)}{\theta_2^2}$, $Q_2 = \frac{Var(\theta_2^*)}{\theta_2^2}$, $Q_3 = \frac{Var(\tilde{\theta}_2)}{\theta_2^2}$ and the relative efficiencies E_1 and E_2 for $c = 0.25$

n	ϑ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	Q_1	Q_2	Q_3	E_1	E_2
						Coefficients										
2	0.50	0.45537	0.53875									0.03125	5.28318	0.03107	1.00579	170.04119
	0.75	0.41044	0.57242									0.03125	1.79252	0.03071	1.01758	58.36926
3	0.50	0.28924	0.33043	0.37162								0.02083	2.36988	0.02065	1.00872	114.76416
	0.75	0.24684	0.32506	0.40327								0.02083	0.81847	0.02032	1.02509	40.27904
4	0.50	0.21008	0.23669	0.25811	0.28472							0.01563	1.48694	0.01546	1.01099	96.17982
	0.75	0.17269	0.22286	0.26253	0.31269							0.01562	0.51892	0.01517	1.02966	34.20699
5	0.50	0.16412	0.18348	0.19770	0.21191	0.23127						0.01250	1.07237	0.01236	1.01133	86.76133
	0.75	0.13108	0.16746	0.19358	0.21969	0.25608						0.01250	0.37684	0.01210	1.03306	31.14380
6	0.50	0.13422	0.14930	0.15981	0.16941	0.17992	0.19501					0.01042	0.83456	0.01029	1.01263	81.10398
	0.75	0.10470	0.13299	0.15223	0.16972	0.18896	0.21725					0.01042	0.29473	0.01006	1.03579	29.29722
7	0.50	0.11328	0.12555	0.13382	0.14101	0.14820	0.15646	0.16874				0.00893	0.68132	0.00881	1.01362	77.33485
	0.75	0.08662	0.10962	0.12473	0.13776	0.15080	0.16591	0.18891				0.00893	0.24151	0.00861	1.03716	28.04994
8	0.50	0.09782	0.10813	0.11491	0.12061	0.12604	0.13174	0.13851	0.14883			0.00781	0.57473	0.00771	1.01297	74.54345
	0.75	0.07351	0.09281	0.10517	0.11550	0.12530	0.13562	0.14799	0.16728			0.00781	0.20432	0.00752	1.03856	27.17021
9	0.50	0.08597	0.09483	0.10054	0.10524	0.10958	0.11391	0.11861	0.12433	0.13319		0.00694	0.49650	0.00685	1.01314	72.48175
	0.75	0.06361	0.08018	0.09060	0.09910	0.10691	0.11473	0.12323	0.13365	0.15022		0.00694	0.17693	0.00668	1.03892	26.48653
10	0.50	0.07660	0.08434	0.08927	0.09326	0.09685	0.10033	0.10392	0.10791	0.11283	0.12058	0.00625	0.43672	0.00616	1.01461	70.89610
	0.75	0.05589	0.07037	0.07935	0.08655	0.09302	0.09928	0.10574	0.11294	0.12192	0.13640	0.00625	0.15593	0.00601	1.03993	25.94509

3. Inference on ϑ When c and θ_2 are Known Using Concomitants of Order Statistics

In this part, we derive certain estimators for ϑ of a $BVND(\theta_1, \theta_2, c\theta_1, c\theta_2, \vartheta)$ with known coefficient of variation by concomitants of order statistics. Take

$$Y^*_{[r:n]} = \frac{Y_{[r:n]} - \theta_2}{c\theta_2},$$

using (4), (5) and (6) we have:

$$E[Y^*_{[r:n]}] = \vartheta\delta_{r:n}, \tag{26}$$

$$Var[Y^*_{[r:n]}] = 1 + \vartheta^2(V_{r,r:n} - 1)$$

and,

$$Cov[Y^*_{[r:n]}, Y^*_{[s:n]}] = \vartheta^2 V_{r,s:n}, r \neq s.$$

Suppose $Y^*_{[n]} = (Y^*_{[1:n]}, Y^*_{[2:n]}, \cdots, Y^*_{[n:n]})'$. Therefore,

$$E[Y^*_{[n]}] = \vartheta\xi \tag{27}$$

and,

$$D[Y^*_{[n]}] = (1 - \vartheta^2)\mathbf{U} + \vartheta^2\mathbf{V}, \tag{28}$$

$\xi = (\delta_{1:n}, \delta_{2:n}, \cdots, \delta_{n:n})'$ and $\mathbf{V} = ((V_{i,j}))$. From (27) and (28) we noted that Gauss-markov theorem does not applicable to derive the BLUE of ϑ. Hence we derive two linear unbiased estimators of ϑ, and are given below.

Theorem 2 *Suppose W be an n dimensional coloumn vector and $W'Y^*_{[n]}$ be a linear function of $Y^*_{[1:n]}, Y^*_{[2:n]}, \cdots, Y^*_{[n:n]}$, with*

$$Var[W'Y^*_{[n]}] = (1 - \vartheta^2)W'W + \vartheta^2 W'VW. \tag{29}$$

Then an estimator $\hat{\vartheta}_1$ obtained by minimising $W'W$ subject to $W'Y^*_{[n]}$ is unbiased for ϑ is obtained as:

$$\hat{\vartheta}_1 = \frac{\xi'}{\xi'\xi}Y^*_{[n]} \tag{30}$$

and an estimator $\hat{\vartheta}_2$ obtained by minimising $W'VW$ subject to $W'Y^*_{[n]}$ is unbiased for ϑ is obtained as:

$$\hat{\vartheta}_2 = \frac{\xi'V^{-1}}{\xi'V^{-1}\xi}Y^*_{[n]}. \tag{31}$$

Also:

$$Var(\hat{\vartheta}_1) = (1 - \vartheta^2)\frac{1}{\xi'\xi} + \frac{\xi'V\xi}{(\xi'\xi)^2}\vartheta^2 \tag{32}$$

and,

$$Var(\hat{\vartheta}_2) = (1 - \vartheta^2)\frac{\xi'V^{-2}\xi}{(\xi'V^{-1}\xi)^2} + \frac{1}{(\xi'V^{-1}\xi)}\vartheta^2. \tag{33}$$

The proof of the theorem is omitted, since it is similar to the proof of Chacko and Thomas (2008).

The estimators $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ given in (30) and (31) can also written as $\hat{\vartheta}_1 = \Sigma^n_{i=1} d_i Y^*_{[i]}$ and $\hat{\vartheta}_2 = \Sigma^n_{i=1} e_i Y^*_{[i]}$.

Theorem 3 *If $\hat{\vartheta}_1 = \frac{\xi'}{\xi'\xi}Y^*_{[n]}$ and $\hat{\vartheta}_2 = \frac{\xi'V^{-1}}{\xi'V^{-1}\xi}Y^*_{[n]}$ are two unbiased estimators of ϑ with variances given by (32) and (33) then $\hat{\vartheta}_1$ is more better than $\hat{\vartheta}_2$ if $|\vartheta| < Z$, where $Z = \sqrt{\frac{Z_2}{Z_1 + Z_2}}$, $Z_1 = \frac{\xi'V\xi}{(\xi'\xi)^2} - \frac{1}{\xi'V^{-1}\xi}$ and $Z_2 = \frac{\xi'V^{-2}\xi}{(\xi'V^{-1}\xi)^2} - \frac{1}{\xi'\xi}$, provided $Z_1 \neq 0$ and $Z_2 \neq 0$.*

The proof of the theorem is omitted, since it is similar to the proof of Chacko and Thomas (2008).

We have calculated the values d_i and e_i contained in $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$, their variances and the interval $(-Z, Z)$ in which $\hat{\vartheta}_1$ is more better than $\hat{\vartheta}_2$ for different values of n and are given in Table 4.

Remark 2 If $Z_1 = Z_2 = 0$ then $\hat{\vartheta}_1 = \hat{\vartheta}_2$.

Table 4. Coefficients d_i/e_i of the estimators $\hat{\vartheta}_1 = \sum_{i=1}^n d_i Y^*_{[i]}$, $\hat{\vartheta}_2 = \sum_{i=1}^n e_i Y^*_{[i]}$, $R_1 = Var(\hat{\vartheta}_1)$, $R_2 = Var(\hat{\vartheta}_2)$) and the value of Z such that when $\vartheta \in (-Z, Z)$ $\hat{\vartheta}_1$ is more better than $\hat{\vartheta}_2$

n	d_i/e_i	1	2	3	4	5	6	7	8	9	10	R_1/R_2	Z_1	Z_2	Z
2	d_i	-0.88623	0.88623									$1.57079 - \rho^2$	0	0	...
	e_i	-0.88623	0.88623									$1.57079 - \rho^2$			
3	d_i	-0.59082	0	0.59082								$0.69813 - \rho^2 0.42265$	0	0	...
	e_i	-0.59082	0	0.59082								$0.69813 - \rho^2 0.42265$			
4	d_i	-0.44840	-0.12938	0.12938	0.44840							$0.43561 - \rho^2 0.25548$	0.00008	0.00080	0.95432
	e_i	-0.45394	-0.11018	0.11018	0.45394							$0.43640 - \rho^2 0.25635$			
5	d_i	-0.36399	-0.15493	0	0.15493	0.36399						$0.31298 - \rho^2 0.17956$	0.00010	0.00092	0.94975
	e_i	-0.37238	-0.13521	0	0.13521	0.37238						$0.31390 - \rho^2 0.18058$			
6	d_i	-0.30783	-0.15590	-0.04896	0.04896	0.15590	0.30783					$0.24292 - \rho^2 0.13712$	0.00009	0.00086	0.94650
	e_i	-0.31752	-0.13856	-0.04321	0.04321	0.13856	0.31752					$0.24378 - \rho^2 0.13807$			
7	d_i	-0.26761	-0.14989	-0.06980	0	0.06980	0.14989	0.26761				$0.19791 - \rho^2 0.11032$	0.00009	0.00075	0.94408
	e_i	-0.27781	-0.13510	-0.06246	0	0.06246	0.13510	0.27781				$0.19866 - \rho^2 0.11116$			
8	d_i	-0.23729	-0.14205	-0.07881	-0.02542	0.02542	0.07881	0.14205	0.23729			$0.16668 - \rho^2 0.09199$	0.00008	0.00065	0.94220
	e_i	-0.24759	-0.12945	-0.07131	-0.02296	0.02296	0.07131	0.12945	0.24759			$0.16734 - \rho^2 0.09272$			
9	d_i	-0.21355	-0.13407	-0.08225	-0.03948	0	0.03948	0.08225	0.13407	0.21355		$0.14380 - \rho^2 0.07872$	0.00007	0.00057	0.94071
	e_i	-0.22373	-0.12327	-0.07510	-0.03597	0	0.03597	0.07510	0.12327	0.22373		$0.14437 - \rho^2 0.07936$			
10	d_i	-0.19443	-0.12653	-0.08290	-0.04748	-0.01550	0.01550	0.04748	0.08290	0.12653	0.19443	$0.12635 - \rho^2 0.06869$	0.00007	0.00049	0.93951
	e_i	-0.20438	-0.11720	-0.07625	-0.04359	-0.01422	0.01422	0.04359	0.07625	0.11720	0.20438	$0.12685 - \rho^2 0.06925$			

Concluding Remarks

It may be noted that our estimator $\tilde{\theta}_2$ defined in (24) is much better than the estimators $\hat{\theta}_2$ and θ_2^* defined respectively in (10) and (18). Also the estimator $\tilde{\theta}_2$ is convex combination of the estimators $\hat{\theta}_2$ and θ_2^*.

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