International Journal of Statistics and Probability; Vol. 2, No. 2; 2013
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

A Note on Closure Properties of Classes of Discrete Lifetime Distributions

Bander Al-Zahrani[1], Javid Gani Dar[2], & Abdulkader Daghistani[1]

[1] Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia

[2] Department of Statistics, GDC (Boys), Anantnag, Kashmir, India

Correspondence: Bander Al-Zahrani, Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia. Tel: 966-2-965-2295. E-mail: bmalzahrani@kau.edu.sa

Received: December 29, 2012 Accepted: February 16, 2013 Online Published: February 28, 2013

doi:10.5539/ijsp.v2n2p42 URL: http://dx.doi.org/10.5539/ijsp.v2n2p42

Abstract

The main purpose of this note is to provide further properties of discrete lifetime distributions based on variance residual lifetimes (VRL). New discrete aging classes are introduced in terms of discrete version of VRL. We demonstrate closure of discrete variance residual lifetime under convolution and mixing.

Keywords: life distribution, discrete aging classes, variance residual lifetime, closure property

1. Introduction

Lifetime analysis has been of great interest to many researchers. It is important to obtain a lifetime distribution that can appropriately describe the aging behavior of a system or a component of that system. In nature, most lifetime distributions are continuous and hence many continuous distributions have been studied and presented in the literature. However, in realistic situations it is possible to find discrete failure data, for example, a series of reports collected weekly or monthly displaying the number of failures of a device, with no specification for failure times. Another example which can be seen is when a device operating on demand and the worker staff observe the number of successful demands executed before failure.

A wide range of aging classes of continuous life distributions have been studied and successfully used in reliability and survival analysis. The properties of continuous classes like increasing failure rate (IFR) and decreasing mean residual lifetime (DMRL) and their dual classes have been broadly studied in the literature. However, for the discrete case, studying the behavior of such classes needs more care. The reason is that at each value of the lifetime there is now a strictly positive mass, while in the continuous case the probability of any fixed value is equal to zero.

Interest in discrete life distributions came later compared to the continuous analogue. (Barlow & Proschan, 1965) defined the discrete version of the failure rate function and gave properties of discrete IFR and DRF distributions. Ebrahimi (1986) characterized discrete DMR lifetime based on discrete failure rate function. Many authors have studied characterizations for discrete life distributions, e.g. (Xekalaki, 1983; Ruiz & Navarro, 1995; Gupta et al., 1997; Kemp, 2004).

The paper is organized as follows: In Section 2, we introduce the definition of the discrete variance residual lifetime (D-VRL). New discrete aging classes are introduced in terms of D-VRL in Section 3. In Section 4, we demonstrate closure of D-VRL under convolution and mixing.

2. Definitions and Properties

Suppose that the lifetime of a component or a device is described by a discrete random variable $X \sim (P, \mathbb{N}_+)$, where $P = \{p_k = Pr(X = k), k \in \mathbb{N}_+\}$ is the probability mass function, and $\mathbb{N}_+$ is the set of all positive integer numbers. Let A_k denote the cumulative distribution function. The reliability function, denoted by B_k, is defined as the probability that the component is still alive at time k.

$$B_k = Pr(X > k) = \sum_{k+1}^{\infty} p_i = p_{k+1} + p_{k+2} + \dots$$

Clearly, $A_k + B_k = 1$ and hence $B_0 = 1$.

The time k can be any value in the set $\mathbb{N}_+$ which means that the component can fail only at times in $\mathbb{N}_+$. However, possibility of failure at time $k = 0$ can be assumed, for example, a component that is damaged at the time of purchase. In such a case a random variable $Y = X - 1$ with support in $\mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$, will be considered. It is important to note that some authors, e.g. (Kemp, 2004), have defined B_k as $B_k = Pr(X \geq k)$. Clearly, $\{p_k\}$, $\{A_k\}$ and $\{B_k\}$ are equivalent as describing the functioning of the component.

The discrete failure rate function, denoted by $r(k)$, was defined by Barlow and Proschan (1965). It is the probability that a component failed or stopped functioning at time k given that it is still properly functioning at time k.

$$r(k) = Pr(X = k | X \geq k) = \frac{p_k}{B_k}, \quad \text{for all } k \in \mathbb{N}_+. \tag{1}$$

The probabilities $\{p_k\}$, the survival function $\{B_k\}$ and the failure rate function $\{r(k)\}$ are equivalent in the sense that given one of them, the other two can be uniquely determined. This fact can be seen by noting that the failure rate function can be expressed in terms of the survival as follows:

$$r(k) = 1 - \frac{B_{k+1}}{B_k}, \quad \text{for all } k \in \mathbb{N}_+. \tag{2}$$

There are two definitions for the discrete mean residual lifetime (D-MRL). We study some properties of D-MRL that are related to reliability. Following Kalbfleisch and Prentice (2002) and Kemp (2004), D-MRL for a random variable X can be defined as follows:

$$\mu(k) = \mathbf{E}[X - k | X \geq k] = \frac{1}{B_k} \sum_{j=k}^{\infty} (j - k) Pr(X = j), = \frac{1}{B_k} \sum_{j=k+1}^{\infty} B_j, \quad k \in \mathbb{N}_+. \tag{3}$$

Obviously, if $k = 0$, then the first order moment of X, or simply the mean of the life distribution, μ, is obtained, i.e.

$$\mu(0) = \mu = \mathbf{E}[X | X \geq 0] = \sum_{j=1}^{\infty} B_j.$$

There is an alternative definition for the D-MRL function which is slightly different from that given in (3). The definition was considered by (Roy & Gupta, 1999):

$$L(k) = \mathbf{E}[X - k | X > k] = \frac{1}{B_{k+1}} \sum_{j=k+1}^{\infty} B_j, \quad k \in \mathbb{N}_+. \tag{4}$$

An interpretation of the function $\mu(k)$ was given by (Ruiz & Navarro, 1994). They defined the D-MRL function over the whole nonnegative real line by taking $L(k)$ as the right-hand limit of $\mu(t)$, where $t \in (k, k + 1)$ and $t \to k$.

Remark 1 It should be noted that $L(0) = 1 + \mu(1)$ while $\mu(0) = \mu$, hence the use of $\mu(k)$ is preferable. The D-MRL function, $\mu(k)$, and its alternative $L(k)$ are related to each other by the following simple relationship:

$$L(k) = 1 + \mu(k + 1), \quad k \in \mathbb{N}_+.$$

Using equations (3) and (4) and on simple algebraic simplification gives the following relation:

$$\frac{\mu(k)}{L(k)} = \frac{B_{k+1}}{B_k}, \quad k \in \mathbb{N}_+. \tag{5}$$

Based on relation (5) we can formulate the following result:

Proposition 1 *The reliability function B_k is uniquely determined by the ratio of $\mu(k)$ and $L(k)$ and is given based on the following inversion formula:*

$$B_k = \prod_{i=0}^{k-1} \frac{\mu(i)}{L(i)}, \quad k \in \mathbb{N}_+. \tag{6}$$

Thus the distribution $A_k = 1 - B_k$ is uniquely determined by the discrete conditional mean function $\mu(k)$, $k \in \mathbb{N}_+$.

Now, we give definitions for the second moment of the residual lifetime, and the discrete version of variance residual lifetime (VRL). We use the abbreviation D-VRL to denote such a class.

Definition 1 Suppose that $X \sim \{P, \mathbb{N}_+\}$ and X has a finite second moment, i.e. $\mathbf{E}[X^2] < \infty$. Then the second moment of the residual lifetime at time k is denoted by $\mu_2(k) = \mathbf{E}[(X-k)^2 | X \geq k]$, $k \in \mathbb{N}_+$ and it is defined as follows:

$$\mu_2(k) = \frac{1}{B_k} \sum_{j=k}^{\infty} (j-k)^2 Pr(X = j) = 2 \sum_{i=k}^{\infty} \sum_{j=i+1}^{\infty} \frac{B_j}{B_k} - \sum_{j=k+1}^{\infty} \frac{B_j}{B_k}. \tag{7}$$

Clearly, $\mu_2(0) = \mu_2$ is the second moment.

We notice from relation (7) that the function $\mu_2(k)$, $k \in \mathbb{N}_+$ is not exactly the same as its continuous counterpart.

Definition 2 Suppose that X is a discrete lifetime and $X \sim \{P, \mathbb{N}_+\}$. Furthermore, suppose that X has a finite second moment, so the mean and the variance are well defined. Then the discrete variance residual lifetime is denoted by $\sigma^2(k) = \mathbf{Var}[X - k | X \geq k]$, $k \in \mathbb{N}_+$ and is defined as follows:

$$\sigma^2(k) = \mu_2(k) - \mu^2(k) = 2 \sum_{i=k}^{\infty} \sum_{j=i+1}^{\infty} \frac{B_j}{B_k} - \sum_{j=k+1}^{\infty} \frac{B_j}{B_k} - \Big(\sum_{j=k+1}^{\infty} \frac{B_j}{B_k} \Big)^2. \tag{8}$$

Looking at relation (8), we immediately notice that the D-VRL function $\sigma^2(k)$ is different from its continuous counterpart.

3. Discrete Aging Classes

As noted before, the discrete aging notions are defined analogously to their continuous counterparts. However, in some cases there are differences. In this section we recall the definitions of the discrete aging classes. New discrete aging classes are introduced in terms of discrete variance residual lifetime.

Definition 3 We are given a discrete random variable $X \sim \{P, \mathbb{N}_+\}$. Suppose that the mean μ is finite. Then X is said to have:

(a) Discrete increasing failure rate, denoted by $X \in$ D-IFR, if $r(k)$, $k \in \mathbb{N}_+$ is increasing.

(b) Discrete decreasing mean residual life, denoted by $X \in$ D-DMRL, if $\mu(k)$, $k \in \mathbb{N}_+$ is decreasing.

Definition 4 A discrete random variable X, or its corresponding discrete lifetime distribution, with a finite second moment, is said to have discrete decreasing variance residual life, denoted by $X \in$ D-DVRL, if the function $\sigma^2(k)$, $k \in \mathbb{N}_+$ is decreasing.

The dual classes of the classes given in Definition 3 and Definition 4 are obtained by changing the word 'increasing' for the word 'decreasing' or vice versa.

The well known monotonicity of failure rate of a life distribution plays a very important role in modeling failure time data. However, determination of the D-IFR and D-DFR property is not easy for some distributions. For this reason, Gupta et al. (1997) define η function for a discrete random variable analogously to the Glaser's function. Thus, their results are parallel to Glaser's result (Glaser, 1980). Define η function as follows:

$$\eta(k) = \frac{p_k - p_{k+1}}{p_k}, \text{ and take } \Delta\eta(k) = \eta(k+1) - \eta(k),\ k \in \mathbb{N}_+.$$

If $\Delta\eta(k) > 0$, then $r(k)$ is increasing and if $\Delta\eta(k) < 0$, then $r(k)$ is decreasing for all $k \in \mathbb{N}_+$. However, if $\Delta\eta(k) = 0$, then $r(k)$ is constant and hence leads to the geometric distribution.

Theorem 1 *A discrete random variable X, or its distribution, is D-DMRL if and only if the following inequality holds*

$$r(k)\, L(k) \leq 1, \quad \text{for all} \quad k \in \mathbb{N}_+. \tag{9}$$

Proof. Assuming that $\mu(k)$ is decreasing for all $k \in \mathbb{N}_+$ and noting that $1 - r(k) = B_{k+1}/B_k$, we obtain the following chain of relations:

$$\begin{aligned} \mu(k) \text{ is decreasing} \iff \quad & \mu(k+1) \leq \mu(k) \\ & L(k) - 1 \leq L(k)\, \frac{B_{k+1}}{B_k} \\ & L(k) - 1 \leq L(k)[1 - r(k)]. \end{aligned}$$

Clearly this implies (9), as required. □

Theorem 2 *Suppose that* $X \sim \{P, \mathbb{N}_+\}$*. Denote by* $\pi(k)$ *the ratio of the D-MRL functions, i.e.* $\pi(k) = \mu(k)/L(k)$*. Furthermore, let* $\Delta\pi(k) = \pi(k) - \pi(k+1)$*. Then:*

(a) If $\Delta\pi(k) > 0$*, then* $X \in$ *D-DMRL.*

(b) If $\Delta\pi(k) = 0$*, then* $\mu(k)$ *is constant.*

(c) If $\Delta\pi(k) < 0$*, then* $X \in$ *D-IMRL.*

Proof. We prove (a) and proofs for (b) and (c) can be obtained similarly. Noting relation (5), we can write $\Delta\pi(k)$ as follows:

$$\Delta\pi(k) = \frac{B_{k+1}}{B_k} - \frac{B_{k+2}}{B_{k+1}}, \quad k \in \mathbb{N}_+.$$

Suppose that $\Delta\pi(k) > 0$, then $B_{k+1}^2 > B_k B_{k+2}$. This means that $X \in$ IFR (log-concavity property) and hence $X \in$ DMRL. □

Theorem 3 *Suppose that* $X \sim \{P, \mathbb{N}_+\}$*. Then X is a D-DVRL if and only if the following inequality holds*

$$\sigma^2(k) \le \mu(k)\, L(k), \quad \text{for all} \quad k \in \mathbb{N}_+. \tag{10}$$

We construct a relationship between two consecutive values of the D-VRL function and the mean residual life function.

Lemma 1 *Two consecutive D-VRL function values,* $\sigma^2(k)$*,* $\sigma^2(k+1)$*,* $k \in \mathbb{N}_+$*, and the D-MRL functions* $\mu(k)$*,* $L(k)$*,* $k \in \mathbb{N}_+$*, are linked by the following relationship:*

$$\sigma^2(k) = \frac{\mu(k)}{L(k)}\, \sigma^2(k+1) + \mu(k)[L(k) - \mu(k)]. \tag{11}$$

The proof of Theorem 3 is now obvious.

The dual class D-IVRL can be defined by reversing the inequality sign in relation (10), i.e. $X \in$ D-IVRL if and only if $\sigma^2(k) \ge \mu(k)L(k)$, for all $k \in \mathbb{N}_+$.

3. Preservation under Convolution and Mixing

It is known that the classes D-IFR and D-DMRL are not closed under convolution. It is concerned to investigate whether the D-DVRL class is closed or not under convolution.

Theorem 4 *Let the discrete life distributions* F_1 *and* F_2 *be D-DVRL. Then their convolution* $F = F_1 * F_2$ *is not necessarily D-DVRL.*

Proof. As usual, we have to find at least two discrete life distributions with decreasing variance residual life such that their convolution does not have this property. Consider two independent components with life times X_1 and X_2, where X_1 takes values $0, 4$ with probabilities $1/4, 3/4$, while X_2 takes values $0, 4$ with probabilities $1/2, 1/2$, respectively. Their survival functions are:

$$\bar{F}_{X_1}(k) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \frac{3}{4}, & \text{if} \quad k = 1, 2, 3, \\ 0, & \text{if} \quad k = 4, 5, \ldots \end{cases} \quad \text{and} \quad \bar{F}_{X_2}(k) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \frac{1}{2}, & \text{if} \quad k = 1, 2, 3, \\ 0, & \text{if} \quad k = 3, 4, \ldots \end{cases}$$

Then we can write the D-MRL functions explicitly:

$$\mu_{X_1}(k) = \begin{cases} \frac{3}{2}, & \text{if} \quad k = 0, \\ 1, & \text{if} \quad k = 1, \\ 0, & \text{if} \quad k = 2, 3, \ldots. \end{cases} \quad \text{and} \quad \mu_{X_2}(k) = \begin{cases} 1, & \text{if} \quad k = 0, 1, \\ 0, & \text{if} \quad k = 2, 3, \ldots \end{cases}$$

We can also find the D-VRL functions as follows:

$$\sigma^2_{X_1}(k) = \begin{cases} 5\frac{1}{4}, & \text{if} \quad k = 0, \\ 4, & \text{if} \quad k = 1, \\ 2, & \text{if} \quad k = 2, \\ 0, & \text{if} \quad k = 3, 4, \ldots. \end{cases} \quad \text{and} \quad \sigma^2_{X_2}(k) = \begin{cases} 4, & \text{if} \quad k = 0, 1, \\ 2, & \text{if} \quad k = 2, \\ 0, & \text{if} \quad k = 3, 4, \ldots \end{cases}$$

It is clear that both functions $\mu(k)$ and $\sigma^2(k)$ are decreasing in $k \in \mathbb{N}_+$ which implies that F_i, $i = 1, 2$ are D-DMRL and hence D-DVRL. However, their convolution $\bar{F}$ is not D-DMRL nor D-DVRL, because the survival function of F is

$$\bar{F}_k = \sum_{i=0}^{\infty} \bar{F}_1(k-i)p_2(i) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \frac{7}{8}, & \text{if} \quad k = 1, 2, 3, \\ \frac{3}{8}, & \text{if} \quad k = 4, 5, 6, \\ 0, & \text{if} \quad k = 7, 8, \ldots \end{cases}$$

It is easy to calculate $\mu(k)$ to show that $F \notin$ D-DMRL since $\mu(4) = 1 > \mu(3) = 0.86$. It is also easy to find $\sigma^2(k)$ to show that $F \notin$ D-DVRL since $\sigma^2(4) = 4 > \sigma^2(3) = 3.5$. □

It is known that the classes D-IFR and D-DMRL are not closed under mixing operation. The proof of closure results for discrete classes can be found in (Pavlova et al., 2006).

As illustrated below, Theorem 5 gives similar result for the class D-DVRL, for some $p \in [0, 1]$.

Theorem 5 *If the discrete life distributions F_1 and F_2 are D-DVRL, then the p-mixture distribution $F = pF_1 + (1 - p)F_2$ is not necessarily D-DVRL.*

Proof. Take F_1 and F_2 such that their survival functions are:

$$\bar{F}_1(k) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \frac{3}{4}, & \text{if} \quad k = 1, 2, 3, 4, \\ 0, & \text{if} \quad k = 5, 6, \ldots \end{cases} \quad \text{and} \quad \bar{F}_2(k) = \begin{cases} 1, & \text{if} \quad k = 0, \\ \frac{3}{4}, & \text{if} \quad k = 1, 2, \\ 0, & \text{if} \quad k = 3, 4, \ldots \end{cases}$$

Choose $p = 0.1$. Then we can find the p-mixture distribution G. Its survival function is found explicitly:

$$\bar{G}(k) = \begin{cases} 1 & \text{if} \quad k = 0, \\ \frac{3}{4} & \text{if} \quad k = 1, 2, \\ \frac{3}{10} & \text{if} \quad k = 3, 4, \\ 0 & \text{if} \quad k = 5, 6, \ldots \end{cases}$$

It can be verified that $\sigma^2(2) = 1/2 < \sigma^2(3) = 2$. Hence, G is not D-DVRL. □

Acknowledgements

The authors are grateful to the editor and anonymous referees for their valuable comments which improved the presentation of the paper.

References

Barlow, R. E., & Proschan, F. (1965). *Mathematical Theory of Reliability*. New York: Wiley.

Ebrahimi, N. (1986). Classes of discrete decreasing and increasing mean residual life distributions. *Reliability, IEEE Transactions on, 35*(4), 403-405.

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association, 75*(371), 667-672.

Gupta, P. L., Gupta, R. C., & Tripathi, R. C. (1997). On the monotonic properties of discrete failure rates. *Journal of Statistical Planning and Inference, 65*(2), 255-268.

Kalbfleisch, J. D., & Prentice, R. L. (2002). *The Statistical of Failure Time Data* (2nd ed.). New York: Wiley.

Kemp, A. W. (2004). Classes of discrete lifetime distributions. *Communications in Statistics-Theory and Methods, 33*(12), 3069-3093.

Pavlova, K. P, Cai, J., & Willmot, G. E. (2006). The preservation of classes of discrete distributions under convolution and mixing. *Insurance: Mathematics and Economics, 38*(2), 391-405.

Roy, D., & Gupta, R. P. (1999). Characterizations and model selections through reliability measures in the discrete case. *Statistics & Probability Letters, 43*(2), 197-206.

Ruiz, J. M., & Navarro, J. (1994). Characterizations by relationships between failure rate and mean residual life. *Reliability, IEEE Transactions on, 43*(4), 640-644.

Xekalaki, E. (1983). Hazard functions and life distributions in discrete time. *Communications in Statistics-Theory and Methods, 12*(21), 2503-2509.