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Spatial Stochastic Framework for Sampling Time Parametric Max-stable Processes

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Abstract

Modelling the spatial extreme events uses the approach of max-stable processes which describe the stochastic behaviour of point-referenced data. Max-stable processes form the natural extension of multivariate extreme values distributions to infinite dimensions. In this paper we consider a max-stable stochastic process over space index. We extend the modelling to time-varying setting using new characterizations of the multivariate distribution underlying the process. A distortional measure is introduced to describe the marginal laws and joint dependence.

Keywords: spatial, max-stable, stochastic process, extreme values distributions

2000 Mathematics Subject Classification: 60G52, 60G70, 62G20, 62G32

1. Introduction

Stochastic processes play a fundamental role in modelling data behaviour on a set of spatial observations. Max-stable processes are stochastic distributions over some index set (space or time) such that all the finite dimensional distributions are max-stable. Space-time (ST) max-stable processes form a class of asymptotically justified models describing spatial dependence among extreme time valued. Max-stability is the foundation of multivariate extreme values (MEV) analysis. With a vast field of applications, statistics of extremes are concerned with modelling asymptotic behaviour of distributions mainly for component-wise maxima of laws when appropriatetly normalized. In one-dimensional extreme setting the well-known theorem of Fisher-Tippet-Gnedenko (See Beirlant) shows that three possible distributions can characterize these asymptotic behaviours such as:

$$\left.\begin{aligned} \Lambda(x) &= \exp\{-\exp(-x)\}; \infty < x < \infty, \text{Gumbel distribution} \\ \Phi_\theta(x) &= \exp\left\{-x^{-\theta}\right\}; x > 0, \theta > 0, \text{Fréchet distribution} \\ \Psi_\theta(x) &= \exp\left(-(-x)^{\theta}\right); x > 0, \theta > 0, \text{Weibul distribution} \end{aligned}\right\} \quad (1)$$

Multivariate extension of the univariate case leads instead to various non-trivial problems and no fully parametric model can summary the whole families.

In a spatial framework, modelling of extremes is an important adequat risk management in environment sciences. Indeed, many environment extremal problems are spatial or temporal in extent (sea height, annual maxima, daily rainfall, snow depth etc.). Specifically, the prospect of climatic change and its impact have brought spatial statistics of extreme events into sharper focus. Compared to their applications to classical statistics, the use of extreme values settings to spatial analysis is still in an evolutionary stage. In a pioneering work in this field (Smith, 1990; unpublished data) proposed a approach to model continuous max-stable processes using the canonical representations of de Haan (see Coles, 2001) of the underlying MEV distributions. This approach have been applied to ozone data in North Carolina by Naveau et al. (2006), to rainfall data by Smith and Stephenson (2009), Padoan et al. (2010) and Ribatet et al. (2010). More recently Blanchet and Davidson (2011) used max-stable processes to model maxima of annual snow height. All these approaches for modelling spatial aspects of rare events are based on generalization of the classical MEV distributions whose margins are usually standardized to unit-Fréchet models (Beirlant et al., 2005).

The main contribution of this paper is to use another approcah to extend continuous max-stable canonical representations to stochastic ST process. In particular, we characterize the marginal and the joint dependence of distribution functions underlying these processes but under some distortional contraints. The ST marginals of the process have been shown to be also max-stable and the joint distribution can be given via a spatial and parametric stable tail dependence function.

2. Materials and Methods

In this study, we consider a random process $\{Y(x), x \in \chi\}$ which describes point-referenced data on a spatial domain $\chi \subset \mathbb{R}^2$. The dependence of this process is captured by a parametric multivariate distribution H_θ, that is, $P(Y(x) \leq y) = H_\theta(y(x)); x \in \chi$. Assuming that the process is max-stable, then H_θ lies in the max-domain of attraction (MDA) of a parametric MEV distribution G_θ. The novelty here is that we are interested to characterize this spatial dependence using a time-varying extreme values distribution and under a distortional condition on the joint dependence. For this purpose, the spatial max-stable results (Schlather et al., 2002), the canonical representations of MEV structures and the conditional discordant measure (Barro, 2009) would be useful.

2.1 One-dimensional Model in Spatial Max-stability

Let $Y_1, ..., Y_n$ be independent copies of the process $\{Y(x)\}$ recorded on a domain χ. The stochastic spatial process is max-stable if there exist normalizing constants $\left\{\sigma_{n,\chi} > 0\right\}$ and $\left\{\mu_{n,\chi} \in \mathbb{R}\right\}$ such that Y is identical in law to the reduced process $Y^*_{n,\chi}$ such as

$$Y^*_{n,\chi} = \sigma^{-1}_{n,\chi}\left(\max_{1\leq i\leq n}\{Y_i\} - \mu_{n,\chi}\right).$$

In particular, if χ is reduced to a single site x, the law of Y^* is either the Fréchet distribution, the Gumbel or the Weibull distribution (Giné, 1990). These three models are referred to be the one-dimensional extreme values distributions and they are unified by a three real-valued parametric distribution, the so-called generalized extreme values (GEV) model, defined in spatial context here by

$$GEV_{(\mu_\chi,\sigma_\chi,\xi_\chi)}(y) = \exp\left\{-\left[1 + \xi_\chi(x)\left(\frac{y-\mu_\chi(x)}{\sigma_\chi(x)}\right)\right]^{\frac{-1}{\xi_\chi(x)}}_{+}\right\} \text{ if } \xi_\chi(x) \neq 0 \tag{2}$$

where $u_+ = \max(u, 0)$ and $\left\{\mu_\chi(x) \in \mathbb{R}\right\}$, $\left\{\sigma_\chi(x) > 0\right\}$ and $\left\{\xi_\chi(x) \in \mathbb{R}\right\}$ are respectively the parameters of location, scale and shape at the location site x of the domain χ. Even in a spatial context, it is usual to standardize the marginal to unit-Fréchet distribution. But, whatever may be the type of the marginal, there is no loss of generality in making this standardization since, for any continuous function f, the transformation $f(Y_i(x)) = \frac{-1}{\log(Y_i(x))}$ gives approximatively this distribution (See Brown, 1977).

In a spatial study Padoan and Ribatet have modeled the parameters of the GEV in (2) as smooth function of the explanatory variables (longitude, altitude, elevation etc.) such as:

$$Y(x) = \mu(x) + \frac{\sigma(x)}{\xi(x)}\left[Z(x)^{\xi(x)} - 1\right] \text{ where } Z(x) \sim \text{Unit-Fréchet}$$

for some partially correlation (Padoan, 2010). That needs to model both spatial behaviour of marginal parameters and spatial joint dependence.

2.2 Conditional Measure for Pareto Distributions

Dealing with the dependence of generalized Pareto distributions (Barro, 2009) has introduced a new measure to describe discordant probability of lower margins. Specifically he pointed out that, in extreme values context, every trivariate Pareto distribution describing a given conditional probability of order 1 can be expressed as a function of a convex parametric bivariate D_θ such as for all $i = 1, 2, 3$

$$H(x) = 1 + \left\{-\sum y_i(x_i) D_\theta\left(\frac{y_1(x_1)}{\sum y_i(x_i)}\right)\right\}; y_i(x_i) = \left[1 + \xi_i\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right]^{\frac{-1}{\xi_i}}_{+}$$

This paper makes the transition from classical extreme analysis to a spatial by making assumption that any realization of the spatial max-stable process is measured at a time-parametric station x_t.

3. Main Results

Let $\{Y(x)\}$ be a stochastic process observed at a finite number of locations $\chi_D = \{x_1, ..., x_D\} \in \mathbb{R}^D$. We describe the main properties of this process assuming that at all site $x_i, i = 1, ..., D$, the observation is made at the same date

$t \in T \subset]0, +\infty[$. This assumption allows us to consider the time varying vector of station $x_t = (x_1(t), ..., x_D(t))$. Then, the corresponding component-wise maxima vector is also ST dependent:

$$M_n(x_t) = \left(\left\{ \max_{1 \le k \le n} \{Y_k(x_1(t))\} \right\}; ...; \left\{ \max_{1 \le k \le n} \{Y_k(x_D(t)\} \right\} \right)^T$$

where n is the lenght of block e.g., n = 365, is large enough in the year.

3.1 Marginal Characterization of Space-time Max-stable Process

Like in the classical extreme values setting, several canonical representations of max-stable processes have been suggested. In this paper the following result allows us to characterize the general form of the one-dimensional marginal of the max-stable ST process $\{Y_t\}$ where $Y_t(x) = Y(x_t)$; $x_t \in \chi_D \times T \subset \mathbb{R}^3$.

Theorem 1 *Assume that the ST process is max-stable. Then the one-dimensional marginal law of the reduced process $Y^*_{t,\chi}$ is a ST parametric GEV. Equivalently, there exist ST-parametric normalizing sequences $\{\sigma_i(x_t) > 0\}$ and $\{\mu_i(x_t) \in \mathbb{R}\}$ such that, for all i,*

$$-\log P\left(Y^*_{i,\chi}(x_t) \le y_i\right) \underset{n \longrightarrow +\infty}{\longrightarrow} \begin{cases} \left[1 + \xi_i(x_t)\left(\frac{y_i(x_t) - \mu_i(x_t)}{\sigma_i(x_t)}\right)\right]_+^{\frac{-1}{\xi_i(x_t)}} \text{ if } \xi_i(x_t) \neq 0 \\ \exp\left\{-\left(\frac{y_i(x_t) - \mu_i(x_t)}{\sigma_i(x_t)}\right)\right\} \text{ if } \xi_i(x_t) = 0 \end{cases} \tag{3}$$

where $\{\mu_i(x_t) \in \mathbb{R}\}$, $\{\sigma_i(x_t) > 0\}$ and $\{\xi_i(x_t) \in \mathbb{R}\}$ are respectively the ST parameters of location, scale and shape of the observation at the parametric site x_t.

For proving Theorem 1, we require the following results on the well-known functional equations of Cauchy.

Lemma 2 *Let h a function defined on $\mathbb{R}_+$.*

i) If for all $v, w > 0$; $h(vw) = h(v)h(w)$then h has the form: $h(x) = x^\theta; \theta \in \mathbb{R}$.

ii) If for all $v, w > 0$; $h(v, w) = h(v) + h(w)$ then, h has the form: $h(x) = c \log(x)$ where $c > 0, x \in \mathbb{R}$.

Proof (of Theorem 1). A sufficient condition for proving the theorem is to establish that (3) holds for the standard form of the GEV, that is, for all i;

$$\lim_{n \longrightarrow +\infty} P\left(Y^*_{i,\chi}(x_t) \le y_i\right) = G^*_i(y_i(x_t)) = \exp\left\{-\left[1 + \xi_i(x_t) y_i(x_t)\right]^{\frac{-1}{\xi_i(x_t)}}\right\} \tag{4}$$

where $G^*_i(y_i(x_i)) = GEV_{(0,1,\xi_i(x_t))}(y_i(x_t))$ on $D_{\xi_i(x_t)} = \left\{x_t \in \chi_D \times T \subset \mathbb{R}^3; 1 + \xi_i(x_t) y_i(x_t) > 0\right\}$.

The process $\{Y_t\}$ being max-stable by assumption, there exist appropriate ST parametric normalizing sequences $(\mu_n(x_t), \sigma_n(x_t)) \in \mathbb{R} \times \mathbb{R}^+$ such that

$$Y^*_{n,\chi}(x_t) = \sigma_n^{-1}(x_t)\left(\max_{1 \le i \le n} \{Y_i(x_t)\} - \mu_n(x_t)\right) \overset{D}{=} Y(x_t). \tag{5}$$

Furthermore, since the observation date t is the same for all station x_i, we can follow Smith and Stephenson (2009) by assuming that the parametric process $\{Y_t\}$ is independent over t. Recall that the distribution of the time parametric maxima is given by $P(M_{n,t}(x) \le y(x_t)) = [H(y(x_t))]^n$.

Therefore, for any continuous transformation f, it follows that

$$I_n(x_t) = E\left[f\left(\frac{M_n(x_t) - \mu_n(x_t)}{\sigma_n(x_t)}\right)\right] \underset{n \longrightarrow +\infty}{\longrightarrow} E\left[f(H(y(x_t)))\right]$$

where

$$E\left[f\left(\frac{M_n(x_t) - \mu_n(x_t)}{\sigma_n(x_t)}\right)\right] = n \int_{\mathbb{R}} f\left(\frac{y(x_t) - \mu_n(x_t)}{\sigma_n(x_t)}\right) H(y(x_t))^{n-1} h(y(x_t))\, dy \tag{6}$$

Let Q denote the tail quantile function of H such as: $Q(u) = H^{-1}\left(1 - \frac{1}{u}\right); u > 0$ where $H^{-1}(u) = \inf\{x_t \in \chi_D \times T; H(y(x_t)) \ge u\}$ is the left-continuous inverse of H.

This result inserted in (3.4) gives

$$E\left[f\left(\frac{M_n(x_t) - \mu_n(x_t)}{\sigma_n(x_t)}\right)\right] = \int_{]0,n[} f\left(\frac{Q\left(\frac{n}{y(x_t)}\right) - \mu_n(x_t)}{\sigma_n(x_t)}\right)\left[1 - \frac{y(x_t)}{n}\right]^{n-1} dy$$

where it is known that $\lim_{n\longrightarrow+\infty}\left[1-\frac{y(x_t)}{n}\right]^{n-1}=\exp\left(-y\left(x_t\right)\right)$ as $y\left(x_t\right)$ is finite and fixed since t is fixed.

Moreover applying the theorem of Fisher-Tippet-Gnedenko (See Resnick) implies that every index i,

$$\lim_{n\longrightarrow+\infty}\frac{Q\left(w_i\left(x_t\right)n\right)-Q\left(n\right)}{\sigma_n\left(x_t\right)}=h_i\left(w_i\left(x_t\right)\right)\text{ on }\left\{x_t\in\chi_D\times T;w_i\left(x_t\right)>0\right\}$$

where the function h satisfies particularly the properties i) and ii) of Lemma 2.

Therefore, in a spatial framework, there exists ST-real parametric $\{\xi_i\left(x_t\right)\in\mathbb{R}\}$ such that $h\left(y_i\left(x_t\right)\right)=c\left(x_t\right)\left(w^{\xi_i(x_t)}-1\right)$ for a convenient constant $c\left(x_t\right)$.

Furthermore, the appropriate choice of $c\left(x_t\right)=\frac{1}{\xi_i(x_t)}$ and $Q\left(n\right)=\mu_n\left(x_t\right)$ gives

$$\lim_{n\longrightarrow+\infty}\frac{Q\left(\frac{n}{y(x_t)}\right)-\mu_n(x_t)}{\sigma_i(t)}=h\left(\frac{1}{y_i\left(x_t\right)}\right)=\begin{cases}\dfrac{y_i\left(x_t\right)^{-\xi_i(x_t)}-1}{\xi_i(x_t)}\text{ if }\xi_i\left(x_t\right)\neq0\\-\log\left(y_i\left(x_t\right)\right)\text{ if }\xi_i\left(x_t\right)=0\end{cases}$$

Thus, the well-known dominated convergence theorem of Lebesgue implies that

$$\lim_{n\longrightarrow+\infty}I_n\left(x_t\right)=\int_{D_{\xi_i(x_t)}}f\left(y_i\left(x_t\right)\right)\left[\exp\left\{-\left(1+\xi y\left(x_t\right)\right)\right\}^{-1/\xi}\right]dy.$$

Then, the law in (4) is obtained asymptotically by $H_i\left(y_i\left(x_t\right)\right)=\exp\left\{-\left[1+\xi_i\left(x_t\right)\left(y_i\left(x_t\right)\right)\right]_+^{\frac{-1}{\xi_i(x_t)}}\right\}$ provided $\xi_i\left(x_t\right)\neq0$ while $\lim_{\xi_i(x_t)\longrightarrow0}H_i\left(y_i\left(x_t\right)\right)=\exp\left(-y_i\left(x_t\right)\right)$.

Finally, the linear transform $\tilde{y}_i\left(x_t\right)=\frac{y_i(x_t)-\mu_i(x_t)}{\sigma_i(x_t)}$ gives relation (3) as disserted.

Remark 3 Notice that in Theorem 1, instead of the generalized type of marginal, it is not restrictive to consider any other of the three types given by (1). Indeed these models are linked each to others by the following functional transformations, even in spatial context:

$$Y\sim\Phi_\theta\Leftrightarrow\ln Y^\theta\sim\Lambda\Leftrightarrow\frac{-1}{Y}\sim\Psi_\theta\Leftrightarrow Z=\mu\left(x\right)+\frac{\sigma(x)}{\xi(x)}\left[Y^{\xi(x)}-1\right]\sim GEV_{(\sigma(x),\mu(x),\xi(x))}.$$

3.2 Space-time Parametric Multivariate Extreme Distributions

The following consequence gives a space-time version of a stable tail dependence function.

Corollary 4 *Let H be the joint distribution of a ST and max-stable process* $\{Y_t\}$*. Then, there are real constants* α *and* β *and a time parametric function* $l_{t,\xi}$*:* $[0,+\infty]^D\longrightarrow[0,+\infty]$ *such as*

$$\lim_{n\longrightarrow+\infty}H\left(\alpha y\left(x_t\right)+\beta\right)=\exp\left\{-l_{t,\xi}\left(x_1,...,x_D\right)\right\}\text{ for all }\mathrm{x}_t\in\chi_D\times T.$$

Moreover, the function $l_{t,\xi}$ *is convex, homogenuous of order one and satisfies:*

$$l_{t,\xi}\left(y\left(x\right)\right)=\lim_{s\longrightarrow0}\frac{1}{s}P\left[\bigcup_{i=1}^{n}\left\{H_i\left(y_i\left(x\right)>1-sy_i\left(x\right)\right)\right\}\right].$$

Definition 5 Two distributions F_1 and F_2 are of the same type if there are constants $a>0$ and b such that $F_1(ax+b)=F_2(x)$ for all x.

Proof. The Theorem 1 shows that the one-dimensional marginal of H lies in the MDA of a time-parametric GEV. Furthermore by assumption the margins are standardized to unit-Fréchet distribution. Finally proposition 5.15 of (Resnick,1987) allows us to conclud that their joint dependence function lies also in the MDA of a joint MEV distribution. Equivalently the distribution H is of the same type that a MEV distribution.

Furthermore, in classical extreme values setting, every MEV G model is canonicaly characterized via the stable tail dependence function (de Haan, 1984) as

$$G\left(x_1,...,x_n\right)=\exp\left\{-l\left(-\log G_1\left(x_1\right)\right),..,-\log G_n\left(x_n\right)\right\}\text{ for }\left(x_1,...,x_n\right)\in\mathbb{R}^n\tag{7}$$

In a ST framework, for all realization y_i of the process: $-\log\left[G_i\left(y_i x_t\right)\right] = 1 + \xi_i\left(x_t\right) y_i\left(x_t\right)^{\frac{-1}{\xi_i(x)}}$ for all $x_t \in \chi_D \times T$ provided that $\xi_i\left(x_t\right) \neq 0$.

Finally, we conclude by considering the transform $l_{t,\xi}$ of l defined by its univariate margins $l_{t,\xi_i}\left(x_i\right) = l\left(1 + \xi_i\left(x_t\right) y_i\left(x_t\right)^{\frac{-1}{\xi_i(x)}}\right)$

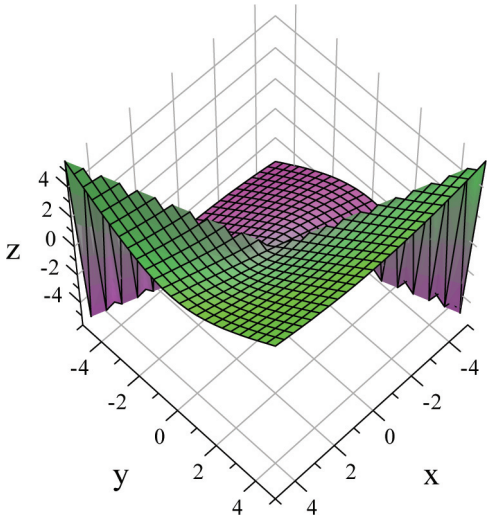

Figure 1. Graphic of $l_{t,\xi}$ for negative bilogistic trivariate $\theta_1 = \theta_2 = 2$

3.3 Space-time Distortional Dependence Measure

In spatial study, the class of areal data is one of the basic types of datasets. In areal data analysis the domain of interest χ is partitioned into a finite number of areal units $\chi_1, ..., \chi_n$ which can represent regions, zip codes, countries, etc. Suppose we are interested in modelling stochastic behaviour of annual maxima of climate for example but under following distortional contraint: at a given region $\chi_A \subset \chi$, the values of the process exceed a known value y_A given that somewhere else (on $\bar{\chi}_A = \complement_\chi^{\chi_A} = \chi \backslash \chi_A$) they do not reach an other given $y_{\bar{A}}$. That means in particular that if $\{Y_{t_A}\}$ denotes the restriction of the ST process $\{Y_t\}$ to the sub-domain χ_A we have to characterize the stochastic distortion defined in terms of probabilities as follows

Definition 6 Let χ_A be a given sub-region of the process domain. For all realizations y at a time parametric station x_t, we define the ST-upper coefficient of distortion at $y\left(x_t\right)$ as the conditional probability:

$$\delta_A^+(y\left(x_t\right)) = P(Y_A\left(x_{t_A}\right) > y_A / Y_{\bar{A}}\left(x_{t_{\bar{A}}}\right) \leq y_{\bar{A}}) \tag{8}$$

where $y_{\bar{A}}\left(x_t\right)$ is the real value of the realization $y\left(x_t\right)$ on the sub-domain.

Similarly, the ST-lower coefficient of distortion of the process is given by:

$$\delta_A^-(y\left(x_t\right)) = P\left(Y_A\left(x_{t_A}\right) \leq y_{D_A} / Y_{\bar{A}}\left(x_{t_{\bar{A}}}\right)\right)$$

In this case, χ_A is referred to be a ST-distortional sub-region of the process $\{Y\left(x\right)\}$.

It follows this result.

Theorem 7 *Let χ_A be a given ST-distortional region of the process $\{Y_t\}$. Then, under the condition of the max-stability of $\{Y_t\}$, the distributions underlying the marginal processes $\{Y_{t_A}\}$ and $\left\{Y_{t_{\bar{A}}}\right\}$ lie respectively in the MDA of two parametric MEV models G_A and $G_{\bar{A}}$. Moreover the distributions G_A and $G_{\bar{A}}$ are marginal distributions of G.*

Proof. The process $\{Y_t\}$ is max-stable by assumption, so Corollary 4 implies that the underlying distribution lies in the MDA of a parametric extreme values model G. Equivalently there exist the normalizing sequences as in (3.3) such as, for all $x_t \in \chi \times T$,

$$\lim_{n \longrightarrow +\infty} P\left\{\bigcap_{i=1}^{n}\left\{\frac{M_i\left(x_t\right) - \mu_i\left(x_t\right)}{\sigma_i\left(x_t\right)} \leq y_i\right\}\right\} = G\left(y_1\left(x_t\right), ..., y_n\left(x_t\right)\right). \tag{9}$$

Setting $x_t = (x_{t_A}, x_{t_{\bar{A}}}) \in \chi \times T$, the marginal distribution G_A of G defined on the sub-domain χ_A is obtained asymptotically by

$$\begin{aligned} G_A(y(x_{t_A})) &= \lim_{x_{t_{\bar{A}}} \longrightarrow x^*_{t_{\bar{A}}}} G(y(x_t)) \\ &= \lim_{x_{t_{\bar{A}}} \longrightarrow x^*_{t_{\bar{A}}}} \left[\lim_{n \longrightarrow +\infty} P\left(\bigcap_{i=1}^{n} \left\{ \frac{M_i(x_t) - \mu_i(x_t)}{\sigma_i(x_t)} \leq y_i \right\} \right) \right] \end{aligned}$$

where $x^*_{t_{\bar{A}}}$ is the right endpoint of the distribution $G_{\bar{A}}$. Then, it follows that

$$\begin{aligned} G_A(y(x_{t_A})) &= \lim_{n \longrightarrow +\infty} \left[\lim_{x_{t_{\bar{A}}} \longrightarrow x^*_{t_{\bar{A}}}} P\left(\bigcap_{i=1}^{n} \left\{ \frac{M_i(x_t) - \mu_i(x_t)}{\sigma_i(x_t)} \leq y_i \right\} \right) \right] \\ &= \lim_{n_A \longrightarrow +\infty} \left[P\left(\bigcap_{i_A=1}^{n_A} \left\{ \frac{M_{i_A}(x_{t_A}) - \mu_{i_A}(x_{t_A})}{\sigma_{i_A}(x_{t_A})} \leq y(x_{t_A}) \right\} \right) \right] \end{aligned}$$

where the index i_A is such as that $x_{t_A} \in \chi_A$. Therefore, there exist marginal ST normalizing sequences $(\sigma_{n_A}(x_{t_A}), \mu_{n_A}(x_{t_A})) \in \mathbb{R}^+ \times \mathbb{R}$ such that, the corresponding marginal component-wise maxima converge to G_A according equality (3.2). Finally, the underlying distribution of the ST marginal process Y_A lies in the MDA of the n_A-dimensional parametric MEV distribution G_A, $n_A = |\chi_A|$, the number of observations sites in the sub-domain χ_A.

Similarly, we establish that for the ST normalizing sequences $\sigma_{n_{\bar{A}}}(x_{t_A}) > 0$ and $\mu_{n_{\bar{A}}}\left(x_{t_{\bar{A}}}\right) \in \mathbb{R}$ the $(n - k_A)$-dimensional parametric MEV $G_{\bar{A}}$ is the limit of a suitably normalised maxima.

Corollary 8 *Let G be a time parametric spatial MEV distribution. Then, there exists a convex and ST measure D_t mapping $[\frac{-1}{2}, 1] \times T$ to $[0, 1]$ such as*

$$G_t(y(x_t)) = \exp\left\{ -\left[\sum_{i=1}^{n} y_i(x_t) D_t\left(\frac{y_i}{\sum_{i=1}^{n} y_i}, ..., \frac{y_{n-1}}{\sum_{i=1}^{n} y_i} \right) \right] \right\}; x_t \in \chi_{D_A} \times T.$$

Proof. Let's consider in (3.6) the simplest case where the sub-domain χ_A is reduced to a single observation station. Therefore, it is easy to check that:

$$\delta^+_A(y(x_t)) = P(Y_A(x_{t_A}) > y_A / Y_{\bar{A}}\left(x_{t_{\bar{A}}}\right) \leq y_{\bar{A}}) = 1 - \frac{P(Y_i(x_t) \leq y_i; 1 \leq i \leq n)}{P(Y_i(x_t) \leq y_i; 2 \leq i \leq n)}.$$

The process $\{Y_t\}$ being max-stable, the Theorem 7 allows us to suppose the existence of two non-degenerated distributions G and $G_{\bar{A}}$ respectively associated with the ST processes$\{Y_t\}$ and deal $\{Y_{t_A}\}$ such as

$$\lim_{n \longrightarrow +\infty} \delta^+_A(y_1(x_t), ..., y_n(x_t)) = 1 - \frac{G(y_1(x_t), ..., y_n(x_t))}{G_{\bar{A}}(y_2(x_t), ..., y_n(x_t))}.$$

Furthermore, writing equivalently their corresponding canonical representation of (4) in ST context, we get the following results

$$\frac{G(y(x_t))}{G_{\bar{A}}\left(y\left(x_{t_{\bar{A}}}\right)\right)} = \exp\{l_{\bar{A}}(y_2(x_t), ..., y_2(x_t)) - l_G(y_1(x_t), ..., y_1(x_t))\}$$

where $l_{\bar{A}}$ and l_G are the stable tail dependence function respectively of $G_{\bar{A}}$ and G.

Finally the restriction on unit simplex S_{m-1} of $\mathbb{R}^{n-1}$ gives the measure

$$D_t(t_1, ..., t_{m-1}) = l\left(1 - \sum t_i, t_2, \ldots, t_{m-1}\right) + \left(\sum t_i\right) l_{\tilde{N}_1}\left(\frac{t_2}{1 - \sum t_i}, \ldots, \frac{t_{m-1}}{1 - \sum t_i} \right)$$

where $S_{m-1} = \left\{ (t_1, ...t_{m-1}) \in [0, 1]^{m-1}, \sum_{i=1}^{m} t_i \leq 1 \right\}$

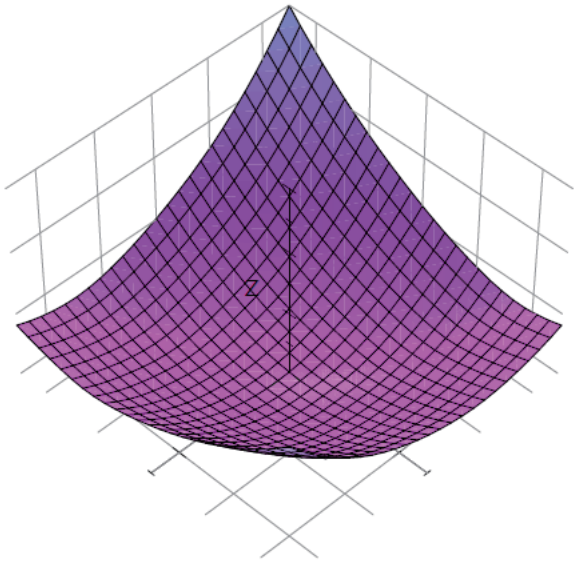

Figure 2. Graphic of D_θ for negative logistic trivariate $\theta_1 = \theta_2 = 2$

4. Discussion

The results of the study show that under the assumption of independence of copies of a max-stable and ST parametric process, the underlying joint distribution lies in the MDA of a ST distribution of MEV family. These results differ from the previous models because, at a fixed date t and for a given observation site x, they characterize both the joint dependence of the model and the possible distortional univariate margins

5. Conclusion

In this study, we have investigated about max-stability of continuous and stochastic model of ST process. We have built a new measure and function which describe this conditional dependence.The classical canonical representations have been extended to time-varying models. Specifically, we have introduced a new measure which describes a distortional constraint on the underlying marginal distributions.

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