The Distribution of Quadratic Expressions in Elliptically Contoured Vectors

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Abstract

A general representation of quadratic expressions in possibly singular elliptically contoured random vectors, as well as a procedure for the numerical evaluation of their distributions, are proposed in this paper. First, such quadratic expressions are represented as the difference of two positive definite elliptically contoured quadratic forms plus an independently distributed linear combination of spherically distributed random variables. Their distributions are then determined from a representation of elliptically contoured vectors in terms of scale mixtures of Gaussian vectors. Quadratic forms and quadratic expressions in various types of elliptically contoured vectors are considered. An accurate moment-based approximation to their density function is also provided. Several numerical examples illustrate the results.

Keywords: quadratic expressions, elliptically contoured vectors, spherically distributed vectors, quadratic forms, cumulant generating function, moments, density approximation

1. Introduction

Several fields of application involve elliptically contoured distributions, including, for instance, anomalous change detection in hyperspectral imagery: Theiler et al. (2010); option pricing: Hamada and Valdez (2008); filtering and stochastic control: Chu (1973); random input signal: McGraw and Wagner (1968); financial analysis: Zellner (1976) and the references therein; the analysis of stock market data: Mandelbrot (1963) and Fama (1965); and Bayesian Kalman filtering: Girón and Rojano (1994). Additionally, studies on the robustness of statistical procedures when the probability model departs from the multivariate normal distribution to the broader class of elliptically contoured distributions were carried out by King (1980) and Osiewalski and Steel (1993). Several multivariate applications are also discussed in Devlin et al. (1976). Results related to regression analysis can be found, for example, in Fraser and Ng (1980). Heavy-tailed time series models were discussed in Resnick (1997). A new family of life distributions, generated from an elliptically contoured distribution, is discussed by Díaz-García and Leiva-Sánchez (2005). Recently, Ip et al. (2007) derived some results applicable to Bayesian inference for a general multivariate linear regression model with matrix variate elliptically distributed errors. In fact, the class of elliptically contoured distributions, which contains the multivariate normal distribution, enjoys several of its properties while allowing for more flexibility in modeling various random processes.

A p-dimensional vector $\mathbf{X}$ has an *elliptically contoured* or *elliptical* distribution with mean vector $\boldsymbol{\mu}$ and scale parameter matrix Σ if its characteristic function $\phi(\mathbf{t})$ can be written as

$$\phi(\mathbf{t}) = e^{i \mathbf{t}' \boldsymbol{\mu}} \xi(\mathbf{t}' \Sigma \mathbf{t})$$

where $\boldsymbol{\mu}$ is a p-dimensional real vector, Σ is a $p \times p$ nonnegative definite matrix and $\xi(\cdot)$ is a nonnegative function, see, for instance, Cambanis et al. (1981); this will be denoted $\mathbf{X} \sim C_p(\boldsymbol{\mu}, \Sigma; \xi)$.

Moreover, the densities associated with p-dimensional elliptically contoured vectors $\mathbf{X}$ are of the form $h((\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}))$ where $h(\cdot)$ is a density defined on $(0, \infty)$ whose $(p/2 - 1)^{\text{th}}$ moment exists, see for example Fang et

al. (1990), Section 2.2.3. In particular, when $\boldsymbol{\mu}$ is the null vector and Σ is the identity matrix of order p, $\mathbf{X}$ is said to have a spherically symmetric or spherical distribution; this will be denoted $\mathbf{X} \sim \mathcal{S}_p(\xi)$.

In fact, whenever $\mathbf{Y} \sim C_p(\boldsymbol{\mu}, \Sigma; \xi)$ and Σ is a positive definite matrix, $\Sigma^{-\frac{1}{2}}(\mathbf{Y} - \boldsymbol{\mu}) \sim \mathcal{S}_p(\xi)$, where $\Sigma^{-1/2}$ denotes the inverse of the symmetric square root of Σ. Furthermore, spherical distributions are invariant under orthogonal transformations, that is, for any orthogonal matrix P, $\mathbf{X} \sim \mathcal{S}_p(\xi)$ and $P\mathbf{X}$ are identically distributed. Other characterizations and properties are available from Kelker (1970), Chmielewski (1981), Fang et al. (1990) and Mathai et al. (1995), among others.

A decomposition of quadratic expressions in possibly singular elliptically contoured vectors is introduced in Section 2 and representations of functions of elliptically contoured vectors such as the moments of a quadratic form, are obtained in Section 3. A density approximation methodology that combines these results is described and illustrated by several numerical examples in Section 4.

The distributional results derived in this paper for quadratic forms in elliptically contoured random vectors not only extend, but also make use of, their Gaussian counterparts. Given that elliptically contoured distributions are utilized as models in a host of applications, and quadratic forms are ubiquitous in statistics, the result presented herein should prove useful in a variety of contexts and lead to the development of improved statistical inference techniques.

2. A Decomposition of Quadratic Expressions in Elliptically Contoured Vectors

Consider the quadratic expression $Q^*(\mathbf{X}) = (\mathbf{X} - \boldsymbol{\alpha})'A(\mathbf{X} - \boldsymbol{\alpha}) + \mathbf{a}'(\mathbf{X} - \boldsymbol{\alpha}) + d$ where $\mathbf{X} \sim C_p(\boldsymbol{\mu}, \Sigma; \xi)$, $rank(\Sigma) = r \le p$, $\boldsymbol{\alpha}$ is a p-dimensional real vector and A is a real symmetric matrix. Letting $\mathbf{X} = \boldsymbol{\mu} + B\mathbf{S}$, where $B_{p \times r}$ is such that $BB' = \Sigma$ (cf. Example 4) and $\mathbf{S} \sim \mathcal{S}_r(\xi)$, one can write

$$
\begin{aligned}
Q^*(\mathbf{X}) \equiv Q^*(\mathbf{S}) &= (\boldsymbol{\mu} + B\mathbf{S} - \boldsymbol{\alpha})'A(\boldsymbol{\mu} + B\mathbf{S} - \boldsymbol{\alpha}) + \mathbf{a}'(\boldsymbol{\mu} + B\mathbf{S} - \boldsymbol{\alpha}) + d \\
&= [(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\mathbf{S}]'A[(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\mathbf{S}] + \mathbf{a}'[(\boldsymbol{\mu} - \boldsymbol{\alpha}) + B\mathbf{S}] + d \\
&= \boldsymbol{\mu}_1'A\boldsymbol{\mu}_1 + 2\boldsymbol{\mu}_1'A'B\mathbf{S} + \mathbf{S}'B'AB\mathbf{S} + \mathbf{a}'B\mathbf{S} + \mathbf{a}'\boldsymbol{\mu}_1 + d
\end{aligned}
$$

where $\boldsymbol{\mu}_1 = \boldsymbol{\mu} - \boldsymbol{\alpha}$. Let P be an orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, $\lambda_1, \ldots, \lambda_r$ denoting the eigenvalues of $B'AB$, with $\lambda_1, \ldots, \lambda_{r_1}$ positive, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$ and $\lambda_{r_1+\theta+1}, \ldots, \lambda_r$ negative, $\mathbf{m}' = (m_1, \ldots, m_r) = \mathbf{a}'BP$, $\mathbf{b}^* = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}_1'ABP$, $c_1 = \boldsymbol{\mu}_1'A\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d$. Then, letting $\mathbf{W} = (W_1, \ldots, W_{r_1}, \ldots, W_{r_1+\theta+1}, \ldots, W_r)' = P'\mathbf{S} \sim \mathcal{S}_r(\xi)$ and assuming that $B'AB \ne O$, one has

$$
\begin{aligned}
Q^*(\mathbf{X}) \equiv Q^*(\mathbf{W}) &= \mathbf{W}'P'B'ABP\mathbf{W} + 2\boldsymbol{\mu}_1'ABP\mathbf{W} + \mathbf{a}'BP\mathbf{W} + \boldsymbol{\mu}_1'A\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d \\
&= \mathbf{W}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{W} + (2\mathbf{b}^{*'} + \mathbf{m}')\mathbf{W} + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2\sum_{j=1}^{r_1} n_j W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + 2\sum_{j=r_1+\theta+1}^{r} n_j W_j \\
&\quad + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + c_1 \\
&= \sum_{j=1}^{r_1} \lambda_j \left(W_j + \frac{n_j}{\lambda_j}\right)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \left(W_j + \frac{n_j}{\lambda_j}\right)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j \\
&\quad + \left(c_1 - \sum_{j=1}^{r_1} \frac{n_j^2}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{n_j^2}{\lambda_j}\right) \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1 \\
&\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1, \qquad\qquad (1)
\end{aligned}
$$

where $Q_1(\mathbf{W}^+) = \mathbf{W}^{+'}\mathcal{D}iag(\lambda_1, \ldots, \lambda_{r_1})\mathbf{W}^+$ and $Q_2(\mathbf{W}^-) = \mathbf{W}^{-'}\mathcal{D}iag(\lambda_{r_1+\theta+1}, \ldots, \lambda_r)\mathbf{W}^-$ are positive definite quadratic forms with $\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim C_{r_1}(\boldsymbol{\nu}_1, I; \xi)$, $\boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\lambda_{r_1})'$, $\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, W_r + n_r/\lambda_r)' \sim C_{r-r_1-\theta}(\boldsymbol{\nu}_2, I; \xi)$, $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, n_r/\lambda_r)'$, θ being number of null eigenvalues of $B'AB$, $n_j = \frac{1}{2}m_j + b_j^*$, $c_1 = \boldsymbol{\mu}_1'A\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\mu}_1 + d$, $\kappa_1 = \left(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\right)$ and

$T_1 = (2 \sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim C_1(\kappa_1, 4 \sum_{j=r_1+1}^{r_1+\theta} n_j^2; \xi)$. If $rank(A\Sigma) = rank(\Sigma) = r$, $T_1 = \kappa_1$. Note that when $\alpha = 0$ and $\mu = 0$ (the central case), $\mu_1 = 0$ and $\mathbf{b}^* = 0$.

As a particular case, when $\alpha = 0$, $\mathbf{a} = 0'$ and $d = 0$, one has the following decomposition for the *quadratic form* $\mathbf{X}'A\mathbf{X}$ in the possibly singular elliptically contoured vector $\mathbf{X} \sim C_p(\mu, \Sigma; \xi)$, Σ being of rank $r \leq p$:

$$
\begin{aligned}
Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{r} \lambda_j W_j^2 + 2 \sum_{j=1}^{r} b_j^* W_j \\
&= \sum_{j=1}^{r_1} \lambda_j W_j^2 + 2 \sum_{j=1}^{r_1} b_j^* W_j - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| W_j^2 + 2 \sum_{j=r_1+\theta+1}^{r} b_j^* W_j \\
&\quad + 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + c \\
&= \sum_{j=1}^{r_1} \lambda_j \left(W_j + \frac{b_j^*}{\lambda_j}\right)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \left(W_j + \frac{b_j^*}{\lambda_j}\right)^2 + 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j \\
&\quad + \left(c - \sum_{j=1}^{r_1} \frac{b_j^{*2}}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r} \frac{b_j^{*2}}{\lambda_j}\right) \\
&\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + \kappa \\
&\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + T,
\end{aligned}
\tag{2}
$$

where $\mathbf{W}' = (W_1, \ldots, W_r) \sim S_r(\xi)$, $Q_1(\mathbf{W}_1) = \mathbf{W}_1' Diag(\lambda_1, \ldots, \lambda_{r_1}) \mathbf{W}_1$ and $Q_2(\mathbf{W}_2) = \mathbf{W}_2' Diag(\lambda_{r_1+\theta+1}, \ldots, \lambda_r) \mathbf{W}_2$ are positive definite quadratic forms with $\mathbf{W}_1 = (W_1 + b_1^*/\lambda_1, \ldots, W_{r_1} + b_{r_1}^*/\lambda_{r_1})' \sim C_{r_1}(\nu_1, I; \xi)$, $\nu_1 = (b_1^*/\lambda_1, \ldots, b_{r_1}^*/\lambda_{r_1})'$, $\mathbf{W}_2 = (W_{r_1+\theta+1} + b_{r_1+\theta+1}^*/\lambda_{r_1+\theta+1}, \ldots, W_r + b_r^*/\lambda_r)' \sim C_{r-r_1-\theta}(\nu_2, I; \xi)$, $\nu_2 = (b_{r_1+\theta+1}^*/\lambda_{r_1+\theta+1}, \ldots, b_r^*/\lambda_r)'$, θ is the number of null eigenvalues of $A\Sigma$, the λ_j's and b_j^*'s being as previously defined, $c = \mu'A\mu$, $\kappa = \left(c - \sum_{j=1}^{r_1} b_j^{*2}/\lambda_j - \sum_{j=r_1+\theta+1}^{r} b_j^{*2}/\lambda_j\right)$ and $T = 2 \sum_{j=r_1+1}^{r_1+\theta} b_j^* W_j + \kappa \sim C_1(\kappa, 4 \sum_{j=r_1+1}^{r_1+\theta} b_j^{*2})$, whenever $rank(A\Sigma) = r - \theta$, $\theta = 1, \ldots, r - 1$. When $rank(\Sigma) = rank(A\Sigma) = r$, $T = \kappa$.

3. Elliptically Contoured Distributions as Scale Mixtures of Gaussian Vectors

Normal scale mixtures have the stochastic representation $\mu + \Sigma^{1/2} L \mathbf{Z}$, where μ is the mean of the distribution, $\Sigma^{1/2}$ is such that $\Sigma^{1/2}(\Sigma^{1/2})' = \Sigma$, the positive semidefinite scale parameter matrix of the distribution, $\mathbf{Z}$ is a standard Gaussian random vector, and L is a positive random variable that is distributed independently of $\mathbf{Z}$. When Σ is positive definite, the density function of $\mathbf{Y} \sim C_p(\mu, \Sigma; \xi)$ can be expressed in terms of a scale mixture of normal densities as follows:

$$
g(\mathbf{y}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int_0^\infty r^{-p/2} \exp\left\{ -\frac{(\mathbf{y}-\mu)'\Sigma^{-1}(\mathbf{y}-\mu)}{2r} \right\} dU(r)
\tag{3}
$$

where $U(\cdot)$, the distribution function of L^2, is such that $U(0) = 0$. This representation can be found, for example, in Muirhead (1982). A result due to Chu (1973) is extended to non-central elliptically contoured distributions in the next theorem. This theorem enables one to express various distributional results involving elliptically contoured vectors in terms of their Gaussian counterparts.

Theorem 1 *Let* $\mathbf{Y} \sim C_p(\mu, \Sigma; \xi)$, *with* $\Sigma > 0$, $h(\mathbf{y})$ *denote the density of* $\mathbf{y}$ *and* $f(s)$ *be* $h(\mathbf{y})$ *wherein* $(\mathbf{y} - \mu)'\Sigma^{-1}(\mathbf{y} - \mu)/2$ *is replaced by* s. *Then, when the inverse Laplace transform of* $f(s)$ *exists, the density of* $\mathbf{Y}$ *denoted by* $h(\mathbf{y})$ *has the following integral representation:*

$$
h(\mathbf{y}) = \int_0^\infty w(t) \, \eta_{\mathbf{Y}}(\mu, t^{-1}\Sigma) \, dt
\tag{4}
$$

where $\eta_{\mathbf{Y}}(\mu, t^{-1}\Sigma)$ *denotes the density function of a p-dimensional Gaussian random vector with mean* μ *and covariance matrix* $t^{-1}\Sigma$, *and the weighting function* $w(t)$ *is obtained as*

$$
w(t) = (2\pi)^{p/2} |\Sigma|^{1/2} t^{-p/2} \mathcal{L}^{-1}(f(s)),
$$

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$\mathcal{L}^{-1}(f(s))$ *representing the inverse Laplace transform of* $f(s)$.

In fact, $\mathcal{L}^{-1}(f(s))$ exists whenever $f(s)$ is an analytic function and $f(s)$ is $O(s^{-k})$ as $s \to \infty$ for $k > 1$; for additional properties of the Laplace transform and its inverse, one may refer to Gradshteyn and Ryzhik (1980), Chapter 17. It follows from Theorem 1 that an elliptical distribution is completely specified by its mean $\boldsymbol{\mu}$, scale parameter matrix Σ and its weighting function $w(t)$, whenever the latter exists. On letting $t = 1/r$ and defining $w(t)$ to be the density function of $1/L^2$, it is seen that (3) and (4) are equivalent. On integrating $h(\mathbf{y})$ as defined in Theorem 1 over $\mathcal{R}^p$ and interchanging the order of integration, one can easily establish that $w(t)$ integrates to 1. Thus, $w(t)$ can be regarded as a weighting function. Explicit representations of $w(t)$ are given in Table 2 for several p-dimensional elliptically contoured distributions.

Theorem 1 enables one to determine the distribution of functions of elliptically contoured vectors in terms of their Gaussian counterparts. For instance, let $\mathbf{Y} \sim C_p(\boldsymbol{\mu}, \Sigma; \xi)$ and its associated weighting function be $w(t)$. Then, the *moment-generating function* of the non-central quadratic form $\mathbf{Y}'A\mathbf{Y}$ can be obtained as follows:

$$
\begin{aligned}
M_{\mathbf{Y}'A\mathbf{Y}}(\theta) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} e^{\theta \mathbf{y}'A\mathbf{y}} \, w(t) \, \eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma) \, dt \, d\mathbf{y} \\
&= \int_0^{\infty} w(t) \, M_{Q(\mathbf{W})}^*(\theta) \, dt
\end{aligned}
\tag{5}
$$

where

$$
M_{Q(\mathbf{W})}^*(\theta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\theta \mathbf{y}'A\mathbf{y}} \, \eta_{\mathbf{Y}}(\boldsymbol{\mu}, t^{-1}\Sigma) \, d\mathbf{y}
$$

is the moment-generating function of the quadratic form $Q(\mathbf{W}) = \mathbf{W}'A\mathbf{W}$ wherein $\mathbf{W} \sim \mathcal{N}_p(\boldsymbol{\mu}, t^{-1}\Sigma)$.

Similarly, the *moments* of $\mathbf{Y}'A\mathbf{Y}$ can be evaluated as follows:

$$
E(\mathbf{Y}'A\mathbf{Y})^h = \int_0^{\infty} w(t) \, E[(\mathbf{W}'A\mathbf{W})^h] \, dt,
\tag{6}
$$

where $\mathbf{W} \sim \mathcal{N}_p(\boldsymbol{\mu}, t^{-1}\Sigma)$ and $E[(\mathbf{W}'A\mathbf{W})^h]$ can be determined from (7).

In general, the moments of a random variable can be obtained from its cumulants by means of a recursive relationship derived in Smith (1995), which can also be deduced for instance from Theorem 3.2b.2 in Mathai and Provost (1992). For example, the hth moment of $Q(\mathbf{W}) = \mathbf{W}'A\mathbf{W}$ is given by

$$
E(\mathbf{W}'A\mathbf{W})^h = \mu_h = \sum_{i=0}^{h-1} \frac{(h-1)!}{(h-1-i)! \, i!} \, k(h-i) \, \mu_i
\tag{7}
$$

where $k(h)$, the hth cumulant of $Q(\mathbf{W})$, is

$$
k(h) = \begin{cases} 2^{h-1} h! \left(\frac{\operatorname{tr}(t^{-1}A\Sigma)^h}{h} + \boldsymbol{\mu}'(t^{-1}A\Sigma)^{h-1}A\boldsymbol{\mu} \right), & h \geq 2, \\[2ex] \operatorname{tr}(t^{-1}A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu}, & h = 1. \end{cases}
$$

4. Illustrative Examples

Four numerical examples involving quadratic forms and quadratic expressions in various types of elliptically contoured vectors are presented in this section. The steps of the proposed methodology for determining their distributions are described in the first example.

Example 1 Consider the quadratic form $Q^{\mathrm{I}}(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X}$ has a noncentral t-distribution with 10 degrees of freedom whose density function is as given in Table 1 with $s = (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})/2$, $\boldsymbol{\mu} = (0, 1, 3, 2)'$,

$$
\Sigma = \begin{pmatrix} 1 & 1/2 & 2/5 & 1/2 \\ 1/2 & 1 & 1/4 & 3/8 \\ 2/5 & 1/4 & 1 & 1/3 \\ 1/2 & 3/8 & 1/3 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -6 & 2 & 1 \\ -6 & 7 & 0 & 4 \\ 2 & 0 & -4 & 1 \\ 1 & 4 & 1 & 2 \end{pmatrix}.
$$

The proposed methodology comprises the following steps:

1) $Q^I(\mathbf{X})$ is expressed as $Q_1^I(\mathbf{W}_1) - Q_2^I(\mathbf{W}_2) + \kappa$ in accordance with Equation (2).

2) The moments of $Q_i^I(\mathbf{W}_i)$, $i = 1, 2$ are determined from Equations (6) and (7).

3) A generalized gamma density function,

$$\psi(z) = \frac{\gamma}{\beta^{\alpha\gamma}\Gamma(\alpha)} z^{\alpha\gamma-1} e^{-(z/\beta)^\gamma} \, \mathcal{I}_{(0,\infty)}(z), \quad \alpha > 0, \, \beta > 0, \, \gamma > 0, \tag{8}$$

is taken as base density for $Q_i^I(\mathbf{W}_i)$, $i = 1, 2$.

4) The parameters α, β and γ are determined by simultaneously solving the following nonlinear equations

$$\mu_j = m_j \quad \text{for } j = 1, 2, 3,$$

where

$$m_j = \frac{\beta^j \, \Gamma(\alpha + j/\gamma)}{\Gamma(\alpha)}, \quad j = 0, 1, \ldots$$

are the moments associated with a generalized gamma density function and μ_j can be determined from the recursive formula (7).

5) A polynomial adjustment of degree d can be made as explained in the Appendix, the resulting density approximation being

$$f_d(z) = \psi(z) \sum_{j=0}^{d} \xi_j \, z^j \,;$$

in this case, we set $d = 7$.

6) Given the density approximations determined for $Q_1^I(\mathbf{W}_1)$ and $Q_2^I(\mathbf{W}_2)$, the approximate density of the difference is obtained by applying the transformation of variables technique. Shifting this density by κ then yields the desired approximation.

Certain values of the resulting approximate distribution function of $Q^I(\mathbf{X})$ are included in Table 1. The percentiles were obtained by simulation on the basis of 1,000,000 replications. The plot shown in Figure 1 confirms that the proposed approach yields a very accurate approximation to the distribution of $Q^I(\mathbf{X})$.

Table 1. Approximate cdf of $Q^I(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul%)

CDF	Simul%	Approx.
0.01	−93.013	0.009429
0.05	−56.312	0.052101
0.10	−40.539	0.101342
0.50	7.9373	0.509682
0.90	77.615	0.893073
0.95	106.51	0.942475
0.99	179.76	0.988139
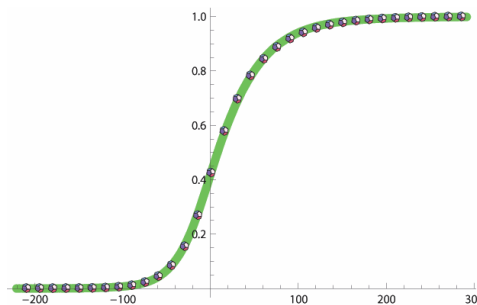
Figure 1. Simulated cdf of $Q^I(\mathbf{X})$ and cdf approximation (dots)

Example 2 Consider the quadratic form $Q^{\mathrm{II}}(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X}$ is a contaminated normal random vector as specified in Table 2, for which $\phi = 0.4$, $\boldsymbol{\mu} = (1, 2, 3)'$,

$$\Sigma = \begin{pmatrix} 1 & 0.2 & 0.7 \\ 0.2 & 1 & 0.2 \\ 0.7 & 0.2 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 5 & 3 & 2 \\ 3 & -5 & 5 \\ 2 & 5 & -2 \end{pmatrix}.$$

It this case, a gamma distribution (as defined by (8) with $\gamma = 1$) was utilized as base density to obtain an approximate distribution for each quadratic form in decomposition of $Q^{\mathrm{II}}(\mathbf{X})$. Letting the integer moments of a non-negative definite quadratic form be denoted by μ_j, $j = 1, 2, \ldots$, a gamma approximation can be specified by equating its first two moments to μ_1 and μ_2, respectively, and solving for α and β, that is, $\alpha\beta = \mu_1$ and $\alpha(\alpha + 1)\beta^2 = \mu_2$, which yields

$$\alpha = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \quad \text{and} \quad \beta = \frac{\mu_2}{\mu_1} - \mu_1. \tag{9}$$

The methodology described in Example 1 was applied in conjunction with polynomial adjustments of degree six to determine the approximate distribution of $Q^{\mathrm{II}}(\mathbf{X})$. The plot shown in Figure 2 indicates that the resulting approximation is very accurate.
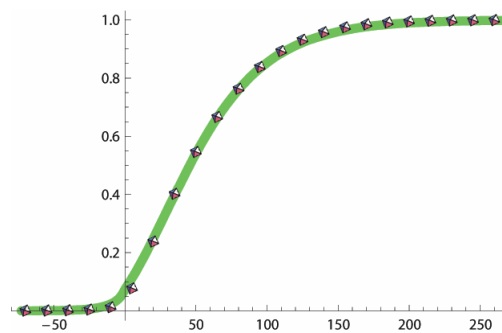
Figure 2. Simulated cdf of $Q^{\mathrm{II}}(\mathbf{X})$ and cdf approximation (dots)

Table 2. Some elliptically contoured distributions and their weighting functions

Distribution	Density function	Weighting function
Gaussian	$e^{-s}/((2\pi)^{p/2}\,\lvert\Sigma\rvert^{1/2})$ with $s = \mathbf{x}'\Sigma^{-1}\mathbf{x}/2$ throughout	$\delta(t-1)$ The Dirac delta function
Contaminated Normal	$\{\phi\lambda^{p/2}e^{-\lambda s} + (1-\phi)e^{-s}\}/$ $\{(2\pi)^{p/2}\,\lvert\Sigma\rvert^{1/2}\}$	$\phi\,\delta(t-\lambda) + (1-\phi)\,\delta(t-1)$
$t-$distribution with ν d.f.	$\{\nu^{\nu/2}\Gamma((\nu+p)/2)\lvert\Sigma\rvert^{-1/2}$ $\times(\nu+2s)^{-(\nu+p)/2}\}/\{\pi^{p/2}\Gamma(\nu/2)\}$	$\{\nu(\nu t/2)^{(\nu/2)-1}e^{-\nu t/2}\}/\{2\Gamma(\nu/2)\}$
Multivariate Analog of the Bilateral Exponential Density	$\{\Gamma(p/2)\,e^{-\sqrt{2s}}\}/$ $\{2\pi^{p/2}\Gamma(p)\lvert\Sigma\rvert^{1/2}\}$	$\{\Gamma(p/2)\,e^{-1/2t}\}/\{\Gamma(p)$ $\times 2^{(3-p)/2}\,\sqrt{\pi}\,t^{(p+3)/2}\}$
The Generalized Slash Distribution	$\nu s^{-p/2-\nu}\lvert\Sigma\rvert^{-1/2} \times \{\Gamma(p/2+\nu)$ $-\Gamma(p/2+\nu, s)\}/(2\pi)^{p/2}$	$\begin{cases} \nu\,t^{\nu-1}, & 0 < \nu < 1 \\ 0, & \nu \geq 1 \end{cases}$

Example 3 Consider the quadratic form $Q^{\mathrm{III}}(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X}$ follows a generalized slash distribution whose density function is as defined in Table 2 with $\boldsymbol{\mu} = (0, 1, 2)'$,

$$\Sigma = \begin{pmatrix} 1 & 1/2 & 2/5 \\ 1/2 & 1 & 1/4 \\ 2/5 & 1/4 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -6 & 2 \\ -6 & 7 & 0 \\ 2 & 0 & -4 \end{pmatrix}.$$

By making use of the weighting function associated with the generalized slash distribution in order to determine the moments (Equation (7)) of the quadratic forms occurring in its decomposition and implementing the steps described in Example 1 in conjunction with a gamma distribution or a generalized gamma distribution whose associated densities are taken as base densities, one can determine an approximate distribution for $Q^{\mathrm{III}}(\mathbf{X})$.

The left and right panels of Figure 3 respectively show the distribution functions resulting from gamma and generalized gamma approximations, which are superimposed on the simulated distribution function determined on the basis of 1,000,000 replications.
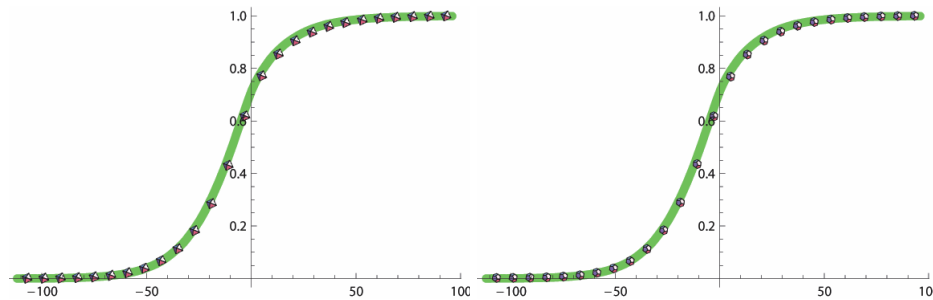
Figure 3. Simulated cdf of $Q^{\mathrm{III}}(\mathbf{X})$ and approximations based on polynomially adjusted gamma (left panel) and generalized gamma (right panel) distributions (dots)

Example 4 Let $Q_1^*(\mathbf{X}) = (\mathbf{X} - \boldsymbol{\alpha})' A (\mathbf{X} - \boldsymbol{\alpha}) + \mathbf{a}'(\mathbf{X} - \boldsymbol{\alpha}) + d$ be a quadratic expression in a singular $t-$vector with 10 degrees of freedom where $\mathbf{X} \sim C_5(\boldsymbol{\mu}, \Sigma; \xi)$, $\boldsymbol{\mu} = (4, 1, -1, 3, 2)'$,

$$
\Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

which is singular, $\boldsymbol{\alpha} = (1, 1, 0, 1, 1)'$, A is the following indefinite matrix

$$
A = \begin{pmatrix} 1 & 1 & 2 & 3 & -5 \\ 1 & 1 & 2 & 3 & -5 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ -5 & -5 & 0 & 0 & -26 \end{pmatrix},
$$

$\mathbf{a} = (1, 2, 3, 4, 5)'$ and $d = 6$.

When $\Sigma_{p \times p}$ is a singular matrix of rank $r < p$, we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements being equal to zero. Next, we let $B^*_{p \times p} = UW^{1/2}$ and remove the last $p - r$ columns of B^*, which are null vectors, to obtain the matrix $B_{p \times r}$. Then, it can be verified that $\Sigma = BB'$. In this case, the matrices B and P were found to be

$$
B = \begin{pmatrix} 1.66591 & 0.39015 & 0 & -0.26930 \\ 1.66591 & 0.39015 & 0 & -0.26930 \\ 2.03287 & -0.92672 & 0 & 0.09291 \\ 1.18171 & 0.49418 & 0 & 0.59945 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

and

$$
P = \begin{pmatrix} -0.97731 & 0.00042 & -0.14936 & -0.15022 \\ 0.05695 & -0.58347 & -0.72923 & 0.35290 \\ 0.13922 & 0.69384 & -0.66277 & -0.24484 \\ -0.14916 & 0.42208 & 0.08157 & 0.89048 \end{pmatrix},
$$

respectively. One can utilize the decomposition of $Q^*(\mathbf{X})$, which is provided in Equation (1), to determine an approximation to the distribution function of $Q_1^*(\mathbf{X})$. The approximate density functions of $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are obtained by making use of a gamma approximation, as explained in Example 2. We first apprximated density of $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-)$ and then, determined the density function of $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1$ by applying the transformation of variables technique.

Referring again to the decomposition (1), the eigenvalues of $B'AB$ were found to be $\lambda_1 = 65.8197$, $\lambda_2 = -29.5759$, $\lambda_3 = -2.24383$, $\lambda_4 = 0$, and it was determined that $n_1 = -43.6247$, $n_2 = 31.6913$, and $n_3 = 2.87613$, and $n_4 = -0.154303$, and that $\mu_1 = -0.662791$, $\mu_2 = (-1.07153, -1.2818)'$ and $c_1 = -4$. The resulting distribution function was evaluated at certain simulated percentiles obtained on the basis of 500,000 replications. The results are presented in Table 3 and the approximate cdf is plotted in Figure 4.

Table 3. Approximate cdf of $Q_1^*(\mathbf{X})$ evaluated at certain percentiles obtained by simulation (Simul%)

CDF	Simul%	Approx.
0.01	−364.29	0.011478
0.05	−188.33	0.058634
0.10	−123.21	0.110176
0.50	6.1069	0.506286
0.90	233.11	0.893207
0.95	360.83	0.943116
0.99	722.56	0.988926
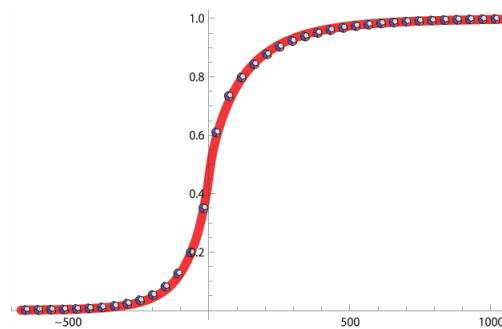
Figure 4. Simulated cdf of $Q_1^*(\mathbf{X})$ and approximation (dots)

Appendix

A.1 Polynomially Adjusted Gamma Density Approximations

As explained in Provost (2005), the density functions of numerous statistics distributed on the positive half-line can be approximated from their exact moments by means of gamma-type density functions that are adjusted with linear combinations of Laguerre polynomials. For conditions ensuring that a distribution be uniquely defined by its moments, the reader is referred to Rao (1965).

Consider a random variable Y defined on the interval $[0, \infty)$, whose j^{th} moment is denoted by μ_j, $j = 0, 1, 2, \ldots$, and let $c = (\mu_2 - \mu_1^2)/\mu_1$, $v = (\mu_1/c) - 1$ and $X = Y/c$. Denoting the jth moment of X by $\mu_j^* = E\left[(Y/c)^j\right]$, the density function of the random variable X, also defined on the interval $[0, \infty)$, can be expressed as

$$f(x) = x^v e^{-x} \sum_{j=0}^{\infty} \delta_j L_j(v, x), \tag{10}$$

where

$$L_j(v, x) = \sum_{k=0}^{j} (-1)^k \frac{\Gamma(v + j + 1) \, x^{j-k}}{k! \, (j-k)! \, \Gamma(v + j - k + 1)} \tag{11}$$

110

is a Laguerre polynomial of order j in x with parameter v and

$$\delta_j = \sum_{k=0}^{j} (-1)^k \frac{j!}{k!\,(j-k)!\,\Gamma(v+j-k+1)} \mu_{j-k}^*,\tag{12}$$

see for instance Szegö (1959) or Devroye (1989). Then, on truncating the series appearing in Equation (10) and making the change of variable $Y = cX$, one obtains the following density approximant for Y:

$$f_d(y) = \frac{y^v e^{-y/c}}{c^{v+1}} \sum_{j=0}^{d} \delta_j L_j(v, y/c)\,.\tag{13}$$

Remark A.1 Note that $f_0(y)$ is a gamma density function with parameters $\alpha \equiv v + 1 = \mu_1^2/\left(\mu_2 - \mu_1^2\right)$ and $\beta \equiv c = (\mu_2 - \mu_1^2)/\mu_1$ whose mean, $\alpha\beta = \mu_1$, and variance, $\alpha\beta^2 = \mu_2 - \mu_1^2$, match the mean and variance of Y and that, in light of Equation (13), one can express $f_d(y)$ as the product of an initial gamma density approximation specified by $f_0(y)$ times a polynomial adjustment:

$$f_d(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \,\Gamma(\alpha)} \sum_{j=0}^{d} \omega_j L_j\left(\alpha-1, \frac{y}{\beta}\right),\tag{14}$$

where $\omega_j = \Gamma(\alpha)\,\delta_j$.

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