Explicit Solution to Optimal Growth Models

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Abstract

This paper shows that the standard optimal growth model can be solved explicitly by using a utility function describing preferences for consumption and savings. Such a maximising criterion including the flow of savings can actually be strongly motivated by two arguments. First, the basic assumption of a representative agent who wishes to consume and save a part of his income each time, can be interpreted as an implicit assumption of some degree of preference for thriftiness. Second, this function formalizes also the concept of Max Weber's spirit of capitalism (with a direct preference for wealth), which makes the model similar to the one of Heng-Fu Zou (1994) except that his specification includes the capital stock. The resulting model offers an interesting application of the Pontryagin's Maximum Principle, as well as elegant closed form solutions.

Keywords: The Ramsey-Cass-Koopmans model, Saddle path, Optimal Control Theory, Saving rate

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1. Introduction

Considering the general definition of utility as an abstract variable indicating goal-attainment or want-satisfaction, the standard assumption of an agent who desires to consume and save a part of his income each time, can imply an agent gaining satisfaction from saving with some degree of preference for thriftiness. Reformulated generally, it might be obvious (or implicit) that an agent who desires something (no matter what it is), means that he is naturally satisfied to obtain it, or to realize it. In that case, he wishes to preserve some income each time because his goal of maximizing consumption on the overall period induces the implicit goal of saving each time for capital accumulation, and/or capital renovation. A Utility function implying the flow of saving is a way of formalizing this initial assumption, and controlling for the degree of thriftiness.

This agent might be also satisfied to enhance wealth as argued by H.F Zou (1994) who introduces the concept of Max Weber's spirit of capitalism in his growth model. According to the sociologist, capital is accumulated in a society not only for the future consumption that it can serve to bring, but also for its own sake. Many reasons support this idea that was not ignored by some major economists in their growth analysis (like JM Keynes), when facing empirical evidence. The realistic fact that property and wealth provides satisfaction to the owner even if not consumed (political power, social influence, prestige...), is one reason of capital accumulation in a country that can be strong enough to invalidate standard theories...

The originality of this paper is to present a simple way of deriving an explicit equation for the Ramsey saddle path, with the particularity of preserving the general assumptions of the model (apart from the usual CRRA utility function). Indeed, several papers in the literature have already raised the question of a determined saddle path, but they all work under restrictive cases, like Benhabib and Rustichini (1994), working on a class of stochastic equilibrium models admitting a constant saving rate, or like Halvor Mehlum (2005) working with a particular fixed coefficient production function, or even like McCallum (1989) with a particular assumption on depreciation of capital...

Following this introduction, Section II presents the model, and Section III analyzes the results before some concluding remarks.

2. The Model

The instantaneous utility function supposed is of the simple Cobb Douglas form:

$$U_t(c_t,s_t) = c_t^{\theta} s_t^{1-\theta} \tag{1}$$

where c_t is consumption, and s_t is savings at time t. The parameter θ measures the relative preference for consumption, whereas $(1-\theta)$ measures the relative preference for thriftiness. Actually, this parameter will also measure the sensitivity of the agent to the interest rate, as well as the degree of the 'capitalist spirit' of the agent.

Introducing the budget constraint $c_t + s_t = f(k_t) = y_t$ where $f(k_t)$ is the standard Cobb Douglas production

function, y_t is the output and k_t is the capital stock per worker, this utility function becomes:

$$U_{t}(c_{t},k_{t}) = c_{t}^{\theta} [f(k_{t}) - c_{t}]^{1-\theta}$$
(2)

A simplifying but non-necessary assumption is to let the rate of preference for consumption be the time preference rate also. There is no loss of logic when assuming that the agent appreciates consuming now at the same rate he discounts future utility. From the social planner's point of view, the optimization program is to maximize lifetime utility of the agent subject to the standard dynamic constraint:

$$\begin{cases} Max \int_{0}^{\infty} U_{t}(c_{t},k_{t}) = c_{t}^{\theta} [f(k_{t}) - c_{t}]^{1-\theta} e^{(n-\theta)t} dt \qquad (3) \\ \text{subject to } \dot{k}_{t} = f(k_{t}) - c_{t} - (n+\delta)k_{t} \qquad (4) \end{cases}$$

subject to
$$\dot{k}_t = f(k_t) - c_t - (n+\delta).k_t$$
 (4)

where *n* is the natural growth rate of population (with $\theta > n$), and where δ is the rate of depreciation of capital. The resolution procedure used is the Hamiltonian:

$$H_{t} = c_{t}^{\theta} \cdot (f(k_{t}) - c_{t})^{(1-\theta)} \cdot e^{(n-\theta)t} + \lambda_{t} e^{(n-\theta)t} \cdot [f(k_{t}) - c_{t} - (n+\delta)k_{t}]$$
(5)

Differentiating totally the first condition (which is the derivative of H_t with respect to c_t) gives:

$$\theta.(1-\theta)c_{t}^{\theta-1}s_{t}^{-\theta}\left[\left(-\frac{s_{t}}{c_{t}}-\frac{c_{t}}{s_{t}}-2\right)\dot{c}_{t}+f'.(1+\frac{c_{t}}{s_{t}})\dot{k}_{t}\right]=\dot{\lambda}_{t}$$
(6)

The second condition provides:

$$(1-\theta)c_t^{\theta}s_t^{-\theta}.f' + \lambda_t(f' - \delta) = -\dot{\lambda}_t + \theta.\lambda_t$$
(7)

Combining (6) and (7), and introducing the dynamic constraint of capital accumulation, leads to the differential system describing the optimal path. To define a feasible solution to this problem, recall the first condition of the Hamiltonian which implies the maximization of H_t with respect to the control variable c_t . Optimal control theory informs that one can choose a solution c_t^* among all admissible c_t 's (avoiding corner solutions) which maximizes H_t , for given values of k_t and λ_t . A feasible optimal path should satisfy the first two correspondences:

$$\begin{cases} c_t \in [0; \theta. f(k_t)] & \text{if} \quad \lambda_t > 0 \\ c_t = \theta. f(k_t) & \text{if} \quad \lambda_t = 0 \\ c_t \in [\theta. f(k_t); 0] & \text{if} \quad \lambda_t < 0 \end{cases}$$

Using the second one (Note 1) which is: $c_t^* = \theta . y_t$; $s_t^* = (1 - \theta) . y_t$; $\lambda_t = 0$

the resulting differential system of the program becomes:

$$\begin{cases} \dot{c}_t = f_k(k_t) [2.(1-\theta).c_t - \theta.(n+\delta).k_t] \\ \dot{k}_t = y_t - c_t - (n+\delta).k_t \end{cases}$$
(8)
(9)

Equation (8) is the major contribution of this paper to the literature of optimal growth. It provides a new shape for the consumption isoclines on the phase diagram, and enables to specify explicitly the transition path towards a unique stationary solution. It states that:

$$\begin{aligned} & \text{if } c_t > \frac{\theta}{2(1-\theta)}(n+\delta)k_t \,, \quad \text{then } \dot{c}_t > 0 \\ & \text{if } c_t = \frac{\theta}{2(1-\theta)}(n+\delta)k_t \,, \quad \text{then } \dot{c}_t = 0 \\ & \text{if } c_t < \frac{\theta}{2(1-\theta)}(n+\delta)k_t \,, \quad \text{then } \dot{c}_t < 0 \end{aligned}$$

3. Analysis of the results

3.1 The saddle path and steady state

A simple graphical resolution shows clearly that the saddle path is given by the $\dot{c}_t = 0$ locus (see the phase

diagram on Figure 1).

$$c_t = \frac{\theta}{2(1-\theta)}(n+\delta)k_t \tag{10}$$

Equation (10) states that, along the optimal path, the level of consumption is a particular constant proportion of the capital stock. Precisely, it is one half of the ratio of relative preferences for consumption and thriftiness multiplied by the total depreciation of capital. Combined with the dynamic constraint of capital accumulation, this equation determines the following stationary solution for the capital stock:

$$k_t^* = \left[\frac{2(1-\theta)}{(2-\theta)(n+\delta)}\right]^{\frac{1}{1-\alpha}} = \left[\frac{s(\theta)^*}{n+\delta}\right]^{\frac{1}{1-\alpha}}$$
(11)

Obviously, we obtain a great similarity with the basic exogenous model of R.Solow (1956). Here, the saving rate is endogenous and converges to its equilibrium value $s(\theta)^*$. The level of consumption at equilibrium is given by:

$$c_t^* = \frac{\theta}{2(1-\theta)}(n+\delta)k_t^* = \frac{\theta}{2-\theta}y_t^*$$
(12)

The author verifies that adding-up conditions are satisfied with: $\frac{2(1-\theta)}{2-\theta} + \frac{\theta}{2-\theta} = s(\theta)^* + c(\theta)^* = 1$

This means that equation (10) can be expressed in terms of equilibrium rates:

$$c_t = \frac{c(\theta)^*}{s(\theta)^*} (n+\delta)k_t \tag{13}$$

Before analyzing growth rates and variations, Proposition 1 formulates a general remark on this model which is demonstrated through the resolution of the dynamic program. Formally,

Proposition 1: Let $U_t(c_t, s_t) = c_t^{\gamma} s_t^{1-\gamma}$ where γ represents the relative preference for consumption, let $f(k_t) = k_t^{\alpha}$ and $(n+\delta) > 0$, then $\forall \theta$ (the discounting rate of future utility), there will be a feasible optimal path characterized

by:

$$\begin{cases} c_t = \frac{\gamma}{2(1-\gamma)}(n+\delta)k_t \\ \dot{k}_t = f(k_t) - c_t - (n+\delta).k_t \end{cases}$$

with a unique stable solution $(c_t^*; k_t^*)$

The discounting rate of future utility is neutralized through the various simplifications arising in this feasible resolution.

3.2 The analysis of growth rates and variations

To start analyzing growth rates of variables (denoted with prefix g), simple algebra applied to the previous differential system shows that along the optimal path:

$$gk_{t} = k_{t}^{\alpha - 1} - \frac{(n + \delta)}{s(\theta)^{*}}$$
(14)

where:

i)
$$gk_t = gc_t > 0$$
 and $gy_t = \alpha gk_t > 0$ if $k_t < k_t^*$
ii) $gk_t = gc_t = gy_t = 0$ if $k_t = k_t^*$

iii) $gk_t = gc_t < 0$ and $gy_t = \alpha gk_t < 0$ if $k_t > k_t^*$

The expression of the growth rate of k(t) is almost the same as the one of the exogenous growth model of R.Solow (1956). Letting $s(\theta)^* = s^{Solow}$, then $(gk_t)^{Solow} = s.gk_t$, which means that the presented optimal growth model

describes a faster economic growth than the Solow model towards the same BGP (everything equal).

The growth rate of the saving rate is also easily characterized using its closed form equation (we denote $s_{[t]}$ the saving rate path):

$$s_{[t]} = 1 - \frac{c_t}{y_t} = 1 - \frac{c(\theta)^*}{s(\theta)^*} (n + \delta) . k_t^{1 - \alpha}$$
(15)

Formally,

Proposition 2: Along the optimal path, the saving rate $s_{[t]}$ is a decreasing function of $k_{(t)}$ with $\frac{\partial s_{[t]}}{\partial k_t} < 0$:

i) $gs_{[t]} < 0$ when $gk_t > 0$ ii) $gs_{[t]} = 0$ when $gk_t = 0$ iii) $gs_{[t]} > 0$ when $gk_t < 0$

Given the basic properties of the production function, this result implies that the incentives to save are more important as the interest rate is high (or in parallel, incentives to invest are more important as productivity of capital is high). In fact, during capital accumulation along the transition path, the level of interest rate decreases as well as the level of the saving rate. The monotonic property of this variation coincides with recent works of Barro and Sala-i-Martin (2004) under the general case of a Cobb Douglas production function (and a CRRA Utility function). In our case, the sense of this variation means that the inter-temporal substitution effect, which consists in transferring current to future consumption when the interest rate increases, always dominates the income effect (which consists in the reverse).

If the growth rate of the saving rate is monotonic in this model, this result is no longer the case for the level of savings s_t . Formally,

Proposition 3: Along the transition path towards steady state, the level of savings S_{t} :

- i) increases if : $gk_t (\alpha . y_t c_t) > 0$
- ii) decreases if: $gk_t (\alpha . y_t c_t) < 0$

Proof: The level of s_t is given by: $s_t = y_t - c_t$ Differentiating with respect to time: $s_t .gs_t = y_t .gy_t - c_t .gc_t = y_t .\alpha .gk_t - c_t .gk_t$ Assuming $s_t \neq 0$ we deduce the growth rate of s_t as a function of gk_t :

$$gs_t = gk_t \cdot \frac{\alpha \cdot y_t - c_t}{y_t - c_t}$$
(16)

and thus $gs_t > 0$ if $gk_t . (\alpha . y_t - c_t) > 0$. QED.

Note also that $gs_t = 0$ if $gk_t = 0$ or if $c_t = \alpha y_t$

3.3 The golden rule

The similarities with the exogenous model of Solow (1956) are interesting to investigate. As seen previously, the relation between the two models consists in defining a rate of preference for consumption such that the equilibrium saving rates coincide. The question becomes to determine the saving rate such that consumption is maximized on the balanced growth path (Phelps, 1961). The problem to solve is:

Max
$$c_t^* = f(k_t^*) - (n+\delta)k_t^*$$
 (17)

yielding
$$f'(k_t^*) = (n+\delta)$$
 (18)

which gives for result:

$$s(\theta)^* = \alpha \qquad \Rightarrow \qquad \theta^* = \frac{2.(1-\alpha)}{2-\alpha}$$
(19)

Hence, the discussion of the two models is very simple: they exhibit identical balanced growth path (BGP) values if the golden rule of capital accumulation holds, and if not, the closest saving rate to the golden value insures a higher level of consumption on BGP. The only remaining difference concerns the transition golden path towards BGP, with an endogenous version in this paper of the following remarkable closed form:

$$c_t = \frac{1 - \alpha}{\alpha} (n + \delta) k_t \tag{20}$$

The consumption golden path is given by the ratio of labor and capital output elasticities multiplied by the total depreciation of capital.

5. Conclusion

Following directly from the general definition of utility, the possible assumption of an agent gaining satisfaction from the fulfillment of his desires to consume and save, leads to the nice results derived in this standard model. The utility function introduced in this paper can also describe the capitalist spirit that H.Zou (1994) considers necessary in optimal growth models. This one shows that countries might get higher levels of growth and per capital income given a high degree of thriftiness or capitalist spirit of agents, but they might also get lower levels of consumption (everything equal). It remains to know if agents (or capitalists) really want to maximize consumption on BGP, rather than income and capital subject to a constraint on consumption...

A goal of this paper is to provide a simple functional model which can be tested or calibrated on empirical macroeconomic data. Indeed, the exogenous version of R.Solow (1956) was already well adapted in the explanation of the world income distribution, and the presented model appears as the analogue under an optimization context. The endogenous saving rate path should make it more appropriate for fitting data, knowing that the described dominating inter-temporal substitution effect is more observed in developed countries.

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Notes

Note 1. This part of the subject deals with determination and feasibility of optimal trajectories; for more details on this pure mathematical framework, see L.S.Pontryagin in "Ordinary Differential Equations" (1962), "The Mathematical Theory of Optimal Processes" (1962), or even examples in Economics with K.Shell in "Application of Pontryagin's maximum principle to economics" (1969)...

t	gkt	k t	y t	$f'(k_t) = r_t$	ct	St	C (Prop.)	S (Prop.)
0	0.69	1	1	0.3	0.16	0.84	0.16	0.84
1	0.31	2.0	1.23	0.18	0.33	0.90	0.27	0.73
2	0.19	2.74	1.35	0.15	0.45	0.90	0.33	0.67
3	0.13	3.31	1.43	0.13	0.54	0.89	0.38	0.62
4	0.09	3.77	1.49	0.12	0.62	0.87	0.41	0.59
00	0	5.49	1.67	0.09	0.90	0.77	0.54	0.46
œ	0	5.49	1.67	0.09	0.90	0.77	0.54	0.46

Table 1. Numerical simulation using ($\theta = 0.7$; n = 0.02; $\delta = 0.12$; $\alpha = 0.3$; $k_0 = 1$)

Table 2. Simulation in Aggregated units (starting from $L_0 = 100$)

t	K _t	L _t	Y _t	Ct	$\mathbf{S}_{\mathbf{t}}$	S (Prop.)	r _t	$W_{t (wage)}^{*}$
0	100	100	100	16.33	83.67	0.84	0.3	70
1	204,76	102,02	125,73	33,44	92,29	0,73	0,18	88,01
2	285,02	104,08	140,81	46,55	94,25	0,67	0,15	98,56
3	351,87	106,18	152,11	57,47	94,64	0,62	0,13	106,48
4	408,38	108,33	161,30	66,70	94,60	0,59	0,12	112,91
5	456,62	110,52	169,15	74,58	94,57	0,56	0,11	118,40

*The wage level $W_t = w_t L_t$ where $w_t = f(k_t) - r_t k_t$

Note that labor augmenting technical progress can be used to adjust the model on real data.



Figure 1. Phase diagram Graphical resolution of the differential system