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Alternative Measures for Modeling Risk and Expected Utility Theory (Risk Adjustment, Measurement and Attitude)

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Abstract

In this paper, we propose alternative measures for modeling risk to be used in Expected Return (ER) calculations instead of "Utility" of Expected Utility Theory (EUT). These new measures are based on the idea that: First, there is a need to make a Risk-Adjustment to make the risky-alternative comparable with the risk-free alternative. Second, the new measures can be more in congruence with actual risk attitude existing in the market. This is possible by making the Risk-Measurement to be included in the probability function and the expected return to be based on the market portfolio on the Capital Market Line (CML) tangent to the Markowitz Bullet as defined in CAPM of finance theory. Finaly, these new measures may also make it possible to include Risk-Attitude in the "Probability" or "Returns" functions and there may no longer be need for a "Utility" function. The two measures introduced in the paper can be used with certain advantages as a substitute for expected utility theory of economics, game theory and decision theory in predicting choice behavior under risk, and in a more correct financial derivatives pricing using binomial models of finance.

Keywords: expected utility theory, utility function, risk, risk aversion, risk-including probability measure, capital market line, capital asset pricing model, sharp ratio, risk-adjusted probability, risk-adjusted returns

1. Introduction

Utility theory, to be used together with the income budget of individuals, is important in Microeconomics for analyzing consumer behavior. It is one of the two theories used in determining individual demand, which determines market demand when aggregated across individuals. Meanwhile, the other theory used in determining individual demand is the budget line and indifference curve framework. Utility by itself doesn't have any units, but is intended to represent the satisfaction different individuals get from consuming a good or service. According to utility representation, the purpose of individuals is to maximize their total utility when making consumption choices, as a result of which their demand for different goods and services is determined. On the supply side, the firm's objective is to maximize profits, and firm behavior determines the market supply function when aggregated over all the firms producing in a certain market. Then, the demand and supply functions together determine the equilibrium quantity of output and price under free market conditions.

Utility functions are also used in modeling individual behavior under risk. For example, in a lottery setting (which is used in modeling many different real world situations, e.g. the capital budgeting decision of firms or the choice of action of individuals under uncertainty), the probability of occurrence and the utility obtained form each outcome in an expected utility setting is used to predict behavior. Here, the von Neumann-Morgenstern (v.N-M) expected utility is a function with expected utility form such that, there is a utility assignment to the N outcomes. Also, every simple lottery is represented by probability of occurrence of different outcomes. A utility function has expected utility form if it assigns a real number to each lottery in a linear functional form for given probabilities and utilities (Mas-Collel et al., 1995), and further books, articles given in the references section). Our purpose in this paper is to examine alternative approaches for this second use of the utility function for predicting behavior under risk in a probabilistic setting.

As a new approach for examining behavior under risk, we propose a "**risk-adjusted probability measure**" (Seber, 2011), and a "**risk-adjusted returns measure**" in place of "utility". In the original approach, the risk attitude of an agent for a given probability is represented with different functional forms of utility. Agents are classified as "**risk-neutral**", "**risk-averse**" or "**risk-seeking**", depending on their utility (as a function of the lottery outcome) being linear, concave or convex, respectively. In this paper, we propose risk to be represented "in the probability measure" (risk-adjusted probability), or "in the returns" (risk-adjusted returns), rather than "in the utility function". This risk-adjusted probability and risk-adjusted returns is in line with "the certainty equivalent" and "the probability premium" concept described in advanced Microeconomics books. The non-linear concavity or convexity of expected outcome estimations with expected utility no longer exists in linear expected outcome estimations with risk-adjusted probability and return functions. The risk-adjustment enables the proper comparison of the risky-alternative with the risk-free alternative; it is possible to include "risk-attitude" in the new measures; furthermore, the risk-adjustment to be included in both the probability and returns measures may be considered "**natural**" since they are based on the market portfolio on the Capital Market Line (CML) in CAPM of finance theory (Capinski & Zastawniak, 2003), which models actual behavior of agents towards risk existing in the market.

2. Method

In this section we first represent a model of risky project evaluation. The initial wealth of the individual is x_o. He has to make a choice of investing in a risky project with success probability p of high value $x_H = x_o(1+u)$; and a (1-p) probability of low value $x_L = x_o(1+d)$. Meanwhile, the risk-free alternative reaches the value of $x_F = x_o(1+r_F)$. Here, the project completion time can be assumed as 1 yr. for simpliciy (to be able to make it comatible with the CAPM analysis; and the adjustment for the analysis for any t would not be difficult), therefore, r_F represents the annual risk-free interest rate prevailing in the market. The model outcomes may be better visualized as in Figure 1.

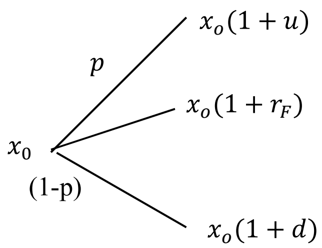

Figure 1. The returns for risky and risk-free projects

Before going into detail about the representation of risk-adjusted measures, we need to understand the basic equation of expected utility theory for evaluating the risky project:

$$E(U_R)=pU(x_H)+(1-p)U(x_L) \quad (1)$$

Here, $E(U_R)$ represents expected utility of the risky project, whereas $U(x_H)$ and $U(x_L)$ represent utility of high and low outcome, respectively. The individual evaluating the risky project may have a "risk-averse", "risk-neutral" or "risk-seeking" attitude towards risk, possibly represented by different functional forms of utility. For example, it is possible to represent utility by the function $U(x) = x^{(1/\gamma)}$, where γ represents the degree of risk aversion. According to this definition, $\gamma=1$, $1/2$ or 2 would represent "risk-neutral", "risk-seeking" and "risk-averse" behaviors, respectively. Furthermore, the absolute risk aversion of the individual may be represented by $U''(x)/U'(x)$ (the curvature of the utility function); and for the above specific utility function, this would be equal to $\left(\frac{1-\gamma}{\gamma}\right)x^{-1}$.

Referring to our representation of the risky project, it is not right as is done in traditional analysis to compare the risky-alternative with the risk-free alternative, since the mean return of the risk-alternative has a positive standard deviation that represents risk. Therefore, there is a need for a risk-adjustment to making the expected return of the risky-alternative with the risk-free return. In order to achieve this, we first need to estimate the risk and return of the risky project given by the following equations:

$$\mu=pu+(1-p)d \quad (2)$$

$$\sigma=(u-d)\sqrt{p(1-p)} \quad (3)$$

Given the fact that the market already has a determined price for risk represented by the Market Portfolio (M) according to the Capital Asset Pricing Model (CAPM) on the Capital Market Line (CML), we also need to find a convinient risk measure to include in the risk-adjusted probability and returns:

$$r_R = r_F + k\left(\frac{\mu_M - r_F}{\sigma_M}\right)\sigma \tag{4}$$

In the equations above, μ is the expected return for the project; σ, the risk of the project represented by its standard deviation; k, the risk attitude of the individual with respect to the market (in case it is equal to 1, it would mean the same attitude towards risk as the market–"not risk-neutral", but "risk-normal"; $k > 1$ or $k < 1$ would in this case may mean "risk-sensitive" or "risk-insensitive" behavior, respectively); and μ_M and σ_M, the risk and return of the market portfolio. In portfolio theory, $(\mu_M - r_F)/\sigma_M$ represents the Sharp Ratio–SR, and therefore, the expected return for the risky project in congruence with the market attitude towards risk as given in equation (4) may further be simplified as $r_R = r_F + k\,SR\,\sigma$. Furthermore, if we define $q = k\,SR\sqrt{p(1-p)}$, the expected return for the risky project would simplify to $r_R = r_F + (u-d)q$.

Equations (2)–(4) are important for determining the amount of risk included in the probabilities and returns. In order to estimate the amount of risk, the expected return for the risky project in equation (2) needs to be equal to the required return as defined by the CML equation of (4). The following equations are used in defining the relevant variables:

$$p_R u + (1-p_R)d = r_F + (u-d)q \tag{5}$$

$$p u_R + (1-p)d = r_F + (u_F - d)q \tag{6}$$

In equations (5) and (6), the probability p_R and return u_R represent values that would equate the expected return of the project to the market expected return for the project, given specific variables about the project (p, u and d), about the individual (k), and the market (r_F, SR). In order to find the risk-adjusted (risk-free) measures of these variables, we need to make sure that when $q=0$ (that is with no risk, no risk sensitivity, or zero SR), $p_R = p_F$ and $u_R = u_F$. Therefore, we obtain the following equations after simplifications:

$$p_R = q + \left(\frac{r_F - d}{u-d}\right);\ p_F = p_R - q \tag{7}$$

$$p u_F + (1-p)d = r_F;\ u_F = u_R\left(\frac{p}{p+q}\right) + d\left(\frac{q}{p+q}\right) \tag{8}$$

What is defined here is that, the original p (success probability–represented in equation (7) as p_R) and u (success return–represented in equation (8) as u_R) needs to be adjusted for risk inherent in the project σ, the sensitivity of the individual towards risk k, and the return-risk behavior represented by SR prevailing in the market, to obtain risk-free probability and risk-free return measures of p_F and u_F (which are less than their corresponding original values).

To better visualize the effect of risk-inclusion on the new variables, we also define the quality of the risky project variable α as given below:

$$\alpha = \left(\frac{p}{1-p}\right)\left(\frac{u-r_F}{r_F-d}\right) \tag{9}$$

According to this definition, the higher p, u, as well as d, represents higher quality for the risky project. The effect of risk-adjustment is to reduce α through reduction in p or u. We can try to better understand the effect of risk adjustment on the risk-adjusted probability, risk-adjusted returns and the quality of the risky project by looking at the following graph of q(p):

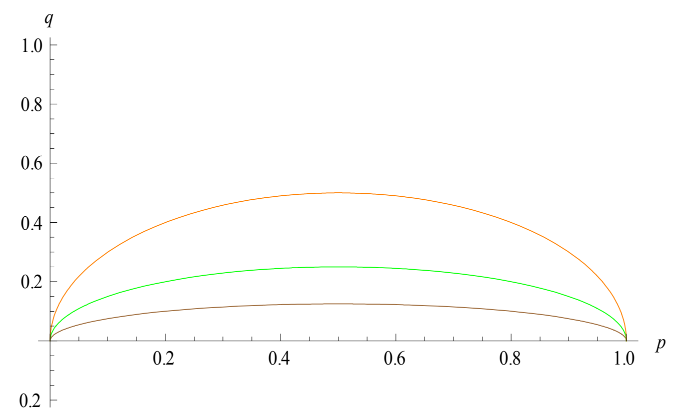

Figure 2. $q(p)$ for $k\,SR$ = 1, 0.5 and 0.25, consequetively, with $k\,SR$ = 1 given by the highest line

The Expected Return (ER) of the project after adjusting for risk in the probability and return measures may now be estimated by the following equations, using "risk-adjusted probability–p_F and risk-adjusted returns–u_F:

$$ER_P = p_F u + (1 - p_F) d \quad (10)$$

$$ER_U = p u_F + (1 - p) d \quad (11)$$

3. Applications

The estimation of SR is important for our model and may be different in different markets. For example S&P 500 estimation of 3 and 5 year SR until the market collapse of 2007 were 0.33, 0.35, respectively. A more recent report IESE Business School estimates the market risk premium $(\mu_M - r_F)$ in different country markets between 4% - 8%. The risk of the market portfolio σ_M needs to be stated in these markets to find the SR. Furthermore, some limitations of the SR also need to be considered for necessary corrections as stated in Harding of Winston Capital Management.

In order to understand the new risk measures and compare with expected utility calculations, we can look at the following example with different outcomes and probabilities:

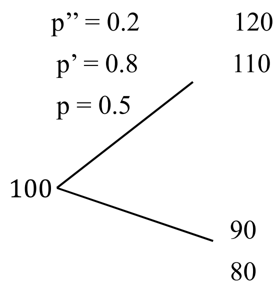

Figure 3. Alternative outcomes and probabilities for the risky project

Using equations (1), (7), (8), (10), (11) and the definition of q, assuming an SR value of 0.35, a risk-normal person with k =1, and a square root risk-averse concave utility function, $x^{1/2}$, we can calculate the expected outcome using different risk measures:

Table 1. Expected outcomes when p = 0.5

Probabilities	Outcomes			
p = 0.5, q = 0.175	(110, 90)	(120, 80)	(110, 80)	(120, 90)
Expected outcome, E[x]	100	100	95	105
Expected outcome estimated with expected utility, E[U] 2	99.75	98.98	94.4	104.46
Expected outcome estimated with risk-adjusted probability, x_0 (1 + ER_P)	96.5	93	89.75	99.75
Expected outcome estimated with risk-adjusted returns, x_0 (1 + ER_u)	97.4	94.8	91.1	101.1

Table 2. Expected outcomes when p = 0.8

Probabilities	Outcomes			
p = 0.8, q = 0.14	(110, 90)	(110, 90)	(110, 90)	(110, 90)
Expected outcome, E[x]	106	112	104	114
Expected outcome estimated with expected utility, E[U] 2	105.84	111.35	03.62	113.66
Expected outcome estimated with risk-adjusted probability, x_0 (1 + ER_P)	103.2	106.4	99.8	109.32
Expected outcome estimated with risk-adjusted returns, x_0 (1 + ER_u)	103.6	107.2	100.4	110.4

Table 3. Expected outcomes when p = 0.2

Probabilities	Outcomes			
p = 0.2, q = 0.14	(110, 90)	(110, 90)	(110, 90)	(110, 90)
Expected outcome, E[x]	94	88	86	96
Expected outcome estimated with expected utility, E[U] 2	93.84	87.35	85.62	95.66
Expected outcome estimated with risk-adjusted probability, x_0 (1 + ER_P)	91.2	82.4	81.8	91.8
Expected outcome estimated with risk-adjusted returns, x_0 (1 + ER_u)	92.36	84.72	83.54	93.54

Examining the results, we first observe that "expected outcome estimated with expected utility" is very close to "expected outcome" calculations. Actually, so close that "expected outcome" may be assumed to give approximately the same estimate as this measure. Second, we observe that "risk-adjusted probability" and "risk-adjusted returns" measures give different estimations than "expected outcome". The next estimate to expected utility measure close to "expected outcome" is the one estimated with risk-adjusted returns followed by the risk-adjusted probability measures.

4. Conclusion

The Vikipedia Free Encyclopedia describes "Expected Utility Hypothesis" as follows:

"In economics, game theory, and decision theory, expected utility hypothesis is a theory of utility in which "betting preferences" of people with regard to uncertain outcomes (gambles) are represented by a function of the payouts (whether in money or other goods), the probabilities of occurrence, risk aversion, and the different utility of the same payout to people with different assets or personal preferences. This theory has proved useful to explain some popular choices that seem to contradict the expected value criterion (which takes into account only the sizes of the payouts and the probabilities of occurrence), such as occur in the contexts of gambling and insurance."

Therefore, we might state that the alternative approaches of risk-adjusted probability and returns measures presented in this paper have applicability in all the areas suitable for application of EUT and as a substitute for EUT. The new measures have the advantage that non-linearity of utility functions in expected outcome estimations with expected utility is replaced by the linearity of expected outcome estimations with risk-adjusted probability and returns. Furthermore, the risk-adjustment made in the new measures may be considered "natural", since they are based on the capital market line (CML) of the capital asset pricing model (CAPM) which represents individual behavior towards risk existing in the market. Additionally, the new measures introduced in this paper can be used in financial derivatives pricing with binomial models. The resulting calculations will be more correct than risk-neutral probability based models currently used in financial derivatives pricing.

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