# High-Accuracy Integral Equation Approach for Pricing American Options with Stochastic Volatility

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**Abstract**

The paper concerns high-order collocation implementation of the integral equation approach for pricing American options with stochastic volatility. As shown in Detemple and Tian (2002) , the value of American options can be written as the sum of the corresponding European option price and the early exercise premium (EEP). This EEP representation results in a nonlinear Volterra integral equation for the optimal exercise boundary. There are no efficient and reliable numerical methods for solving the integral equations in the literature. The aim of this paper is to develop a high-order collocation method for solving the nonlinear integral equations. Collocation methods are widely studied in the area of numerical integral equations. After the exercise boundary is resolved, the value of the American options is obtained by evaluating the EEP representation.

**Keywords:** American put options, Optimal exercise boundary, Collocation methods, Integral equations

## 1. Introduction

The major difficulty and key to pricing American options are to identify the optimal exercise boundary. It is not possible to get the closed-form of the optimal exercise boundary. So the numerical methods are the necessary tools in resolving the optimal exercise boundary. For the underlying asset price following a lognormal process, Kim (1990), Jacka (1991), and Carr et al. (1992) show that the American option price is equal to the corresponding European option price plus an Early Exercise Premium (EEP) which captures the benefits from exercising prior to maturity. Detemple and Tian (2002) give the EEP representation of American options under general diffusion process with stochastic volatility and interest rate. From the EEP representation, the optimal exercise boundary satisfies an integral equation whose analytical and numerical solutions are difficult to find. To this end, Ma et al. (2010) construct a high-order collocation method on non-uniform meshes for solving the integral equation arising in the EEP representation of American put options under lognormal process. In this paper, we extend the collocation methods to solution of the integral equations for the general diffusion process with stochastic volatility. Compared to the case of the lognormal process, the implementation of collocation methods is much more difficult for the case of general diffusion process with stochastic volatility.

In the history, a few of works, cf. Huang et al. (1996), Ju (1998), Detemple and Tian (2002) have studied the implementations of the EEP methods for pricing the American put options. However their approaches are based on low-order approximations.

The remaining parts are arranged as follows: In the next section the high-order collocation methods are described to solve the integral equations that the optimal exercise boundary satisfies; In Section 3, an algorithm for the valuation of the American options is constructed and implemented through a variety of numerical examples; Concluding remarks are given in the final section.

## 2. Collocation methods for the integral equations

Assume that the risk-neutralized underlying asset price follows a diffusion process

$$dS_t / S_t = (r(S_t,t) - q(S_t,t))dt + \sigma(S_t,t)dW_t, \tag{1}$$

where $r$ denotes the interest rate, $q$ the dividend yield and $\sigma$ the volatility. Following the ideas in Detemple and Tian (2002), the early exercise premium can be expressed as an integral form which depends on the optimal exercise boundary $B(t)$ :

$$V(S_t,t;B(\cdot)) = V_E(S_t,t) + \prod(S_t,t;B(\cdot)), \tag{2}$$

where

$$\prod(S_t,t;B(\cdot)) = E_t^*\left[\int_t^T e^{-\int_t^T r(S_v,v)dv}(r(S_v,v)K - q(S_v,v)S_v)1_{\{S_v \le B(v)\}}dv\right]$$

is the early exercise premium, $V_E(S_t,t)$ the European option price, and $1_A$ the indicator of the set $A$ , $K$ is the strike price. The second term indicates that the incremental gain over the time period $[t,t+dt]$ from exercising the option at time $t$ is $(r(S_t,t)K - q(S_t,t)S_t)dt$ . Since immediate exercise is optimal when $S_t = B(t)$ , we obtain an integral equation for the optimal exercise boundary:

$$K - B(t) = V_E(B(t),t) + \prod(B(t),t;B(\cdot)) \tag{3}$$

subject to the boundary condition $B(T) = \min\{K,(r(T,B(T))/q(T,B(T)))K\}$ . It is well-known that the implied volatility of the market often shows a smile structure.

The model displays this property is the constant elasticity of variance (CEV) model

introduced by Cox and Ross (1976), in (1): $r, q$ are constants and $\sigma = \sigma_0 S_t^{\theta/2-1}$,

where $\sigma_0, \theta$ are positive constants. Following Detemple and Tian (2002) (Note 1), an explicit expression of (3) is given by

$$K - B(t) = Ke^{-r(T-t)}\phi_2(B(t),K,T) - B(t)e^{-q(T-t)}\phi_1(B(t),K,T)$$
$$+\int_t^T (rKe^{-r(v-t)}\phi_2(B(t),B(v),v) - qB(t)e^{-q(v-t)}\phi_1(B(t),B(v),v))dv \tag{4}$$

where $\phi_1, \phi_2$ are given by

$$\phi_1(\delta_1,\delta_2,\delta_3) = \begin{matrix} \chi^2(2y(\delta_2,\delta_3);2+\frac{2}{2-\theta},2x(\delta_1,\delta_3)) & if\theta < 2 \\ \chi^2(2x(\delta_1,\delta_3);\frac{2}{\theta-2},2y(\delta_2,\delta_3)) & if\theta > 2 \end{matrix}, \tag{5}$$

$$\phi_2(\delta_1,\delta_2,\delta_3) = \begin{matrix} 1-\chi^2(2x(\delta_1,\delta_3);\frac{2}{2-\theta},2y(\delta_2,\delta_3)) & if\theta < 2 \\ 1-\chi^2(2y(\delta_2,\delta_3);2+\frac{2}{\theta-2},2x(\delta_1,\delta_3)) & if\theta > 2 \end{matrix}, \tag{6}$$

and functions $x$ , $y$ are defined by

$$x(v_1,v_2) = \frac{2(r-q)}{\sigma_0^2(2-\theta)(e^{(r-q)(2-\theta)(v_2-t)}-1)}v_1^{2-\theta}e^{(r-q)(2-\theta)(v_2-t)}, \tag{7}$$

$$y(v_1,v_2) = \frac{2(r-q)}{\sigma_0^2(2-\theta)(e^{(r-q)(2-\theta)(v_2-t)}-1)}v_1^{2-\theta}. \tag{8}$$

The function $\chi^2(\xi_1;\xi_2,\xi_3)$ is the complementary noncentral chi-square distribution function (Note 2) evaluated at $\xi_1$, with $\xi_2$ degrees of freedom and noncentral parameter $\xi_3$.

The integral equation is subject to the boundary condition $B(T) = \min\{K,(r/q)K\}$ ·

To compute the optimal exercise boundary, we transform (4) into an initial value

Problem, by the coordinate transformations, $\tau = T - t$ , $B(t) = \widetilde{B}(\tau)$ ,

$$\widetilde{B}(\tau)-K=\widetilde{B}(\tau)e^{-q\tau}\phi_1(\widetilde{B}(\tau),K,T)-Ke^{-r\tau}\phi_2(\widetilde{B}(\tau),K,T)$$

$$+q\widetilde{B}(\tau)\int_0^{\tau}e^{-q(\tau-v)}\phi_1(\widetilde{B}(\tau),\widetilde{B}(v),T-v)dv-rK\int_0^{\tau}e^{-r(\tau-v)}\phi_2(\widetilde{B}(\tau),\widetilde{B}(v),T-v)dv, \tag{9}$$

with $\widetilde{B}(0)=\min\{K,(r/q)K\}$.

Now we develop a collocation method (Note 3) to solve the integral equation (9). As proved for the case of constant Volatility (see the summary in Ma et al. (2010)), the solution of the integral equation is singular in general. So using uniform meshes cannot get an optimal accuracy in the numerical solution of the integral equation (5).

We use the graded meshes which is developed in Brunner (1985) (cf. Brunner (2004)).

The graded meshes are defined as $\tau_i=T(i/L)^2$, $i=0,1,...,L$. Obviously mesh points are clustered around the origin. Based on graded meshes, we introduce the collocation methods:

Define a piecewise polynomial space $\prod_3^{-1}(0,T]=\{p:p(\tau)\in P_3,\tau\in(\tau_i,\tau_{i+1}],i=0,1,...,L-1\}$,

where $P_3$ is the set of polynomials of degree 3. The high-order collocation method is defined by: Find $Y(\tau)\in\prod_3^{-1}$ such that

$$Y(\tau)-K=Y(\tau)e^{-q\tau}\phi_1(Y(\tau),K,T)-Ke^{-r\tau}\phi_2(Y(\tau),K,T)$$

$$+qY(\tau)\int_0^{\tau}e^{-q(\tau-v)}\phi_1(Y(\tau),Y(v),T-v)dv-rK\int_0^{\tau}e^{-r(\tau-v)}\phi_2(Y(\tau),Y(v),T-v)dv. \tag{10}$$

holds exactly at the collocating points

$$\tau_{i,j}=\tau_i+\frac{j}{4}(\tau_{i+1}-\tau_i),\quad j=1,2,3,4;\quad i=0,1,...,L-1.$$

At time interval ($\tau_i,\tau_{i+1}$], polynomial $Y(\tau)$ can be written by

$$Y(\tau)=\sum_{j=1}^{4}Y_j^i l_j^i(\tau), \tag{11}$$

where $l_j^i(\tau)$ is the Lagrange basis at points $\tau_{i,j}$, $j=1,2,3,4;$ i.e.,

$$l_j^i(\tau)=\prod_{k\neq j}\frac{\tau-\tau_k^i}{\tau_j^i-\tau_k^i}.$$

Inserting (11) into (10) gives the computational form

$$F_j^i(Y_1^i,Y_2^i,Y_3^i,Y_4^i)=0,\quad j=1,2,3,4;\quad i=0,1,...,L-1 \tag{12}$$

for the unknowns $Y_1^i,Y_2^i,Y_3^i,Y_4^i$, $i=0,1,...,L-1$, where the function $F_j^i$ is given by

$$F_j^i=Y_j^i-K-Y_j^ie^{-q\tau_j^i}\phi_1(Y_j^i,K,T)+Ke^{-r\tau_j^i}\phi_2(Y_j^i,K,T)$$

$$-qY_j^i\int_0^{\tau_j^i}e^{-q(\tau_j^i-v)}\phi_1(Y_j^i,Y(v),T-v)dv+rK\int_0^{\tau_j^i}e^{-r(\tau_j^i-v)}\phi_2(Y_j^i,Y(v),T-v)dv. \tag{13}$$

To solve the nonlinear algebraic equations (13), the Newton's method is used. The Jacobians (Note 4): $\frac{\partial F_j^i}{\partial Y_j^i}$ and $\frac{\partial F_j^i}{\partial Y_k^i}$, for $k\neq j$ are given in the appendix.

Denote $Y^i = (Y_1^i, Y_2^i, Y_3^i, Y_4^i)^T$ and $Z^{(k)} = (Z_1^{(k)}, Z_2^{(k)}, Z_3^{(k)}, Z_4^{(k)})^T$.

Then the Newton's method is given by

$$Z_j^{(k+1)} = Z_j^{(k)} - \frac{F_j^i(Z^{(k)})}{\lambda^{(k)}}. \quad j = 1,2,3,4; \quad k = 0,1,\ldots,M, \tag{14}$$

where

$$\lambda^{(k)} = \max_{1\le j\le 4} \frac{\partial F_j^i(Z^{(k)})}{\partial Z_j^{(k)}}$$

The initial value $Z^{(0)}$ is taken as the extrapolation of $Y^{i-1}$ at the interval $(\tau_i, \tau_{i+1}]$. The stopping value of Newton's method gives the solution at interval $(\tau_i, \tau_{i+1}]$, i.e., $Y^i = Z^{(M)}$. All the involved integrals are approximated by four-points Gauss-Legendre quadrature.

## 3. Numerical implementations

In the examples, we use collocation method (10) to find the solution of integral equation (4). The result gives the value of the optimal exercise boundary. Then inserting the value to EEP Representation (2) gives the value of the American options. Two examples for the cases $r < q$ and $r > q$ are computed. For each example, two figures are drawn to illustrate the alidity and efficiency of our algorithm.

**Example 1** Consider the free boundary value problem for pricing American put options with

$r = 4\%$, $q = 6\%$, $K = 100$, $T = 1$, $\sigma = \sigma_0 S_t^{\theta/2-1}$

with $\sigma_0 = 0.2\sqrt{10}$ and $\theta$=1, 1.3, 1.6, 1.9. (the case $r < q$).

Figure 1 gives the price of the American options with $\theta$=1.6. The figure shows that the curve goes to the pay-off when t approaching to the maturity T. Figure 2 gives the optimal exercise oundaries for $\theta$=1, 1.3, 1.6, 1.9. Figure 2 shows that the optimal exercise boundaries are ecreasing with $\theta$ increasing, and in fact converge to the exercise boundary of lognormal model ($\theta$=2). Actually this phenomenon is reasonable, since the volatility of the underlying asset price raises with $\theta$ increasing and henceforth the value of the American option increases.

So the exercise boundary for the American put options will decrease when $\theta$ increasing. As $\theta$ approaches 2, the underlying asset price process converges to the lognormal process and the convergence of the optimal exercise boundary and the value of American options follows.

**Example 2** Consider the free boundary value problem for pricing American put options with

$r = 6\%$, $q = 4\%$, $K = 100$, $T = 1$, $\sigma = \sigma_0 S_t^{\theta/2-1}$

with $\sigma_0 = 0.2\sqrt{10}$ and $\theta$=1, 1.3, 1.6, 1.9. (the case $r > q$).

Figures 3 and 4 show the case $r > q$. The discussions are similar to that for Example 1.

## 4. Concluding remarks

In this paper we have developed the high-order collocation method for solving the nonlinear integral equations arising in the EEP representation of the American options with stochastic volatility. The optimal Exercise boundaries and the value of the American options are calculated with high accuracy and efficiency. The high-order implementation is essential to the development of the EEP or integral equation approaches for pricing American options.

Since the quasi-Monte-Carlo methods can be applied easily for the integral forms, the EEP or integral equation approaches are especially important for pricing multi-dimensional American options.

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**Appendix 1:** The density function of chi-square distribution is given by

$$f_X(x;k,\lambda)=\frac{1}{2}e^{-\frac{(x+\lambda)}{2}}\left(\frac{x}{\lambda}\right)^{\frac{k}{4}-\frac{1}{2}}I_{\frac{k}{2}-1}(\sqrt{\lambda x}),$$

where $I_a(y)=(\frac{y}{2})^a\sum_{p=0}^{\infty}\frac{(\frac{y^2}{4})^p}{p!\Gamma(a+h+1)}$ and $\Gamma(m)=\int_0^{\infty}t^{m-1}e^{-t}dt$ is Gamma function.

With the definitions of functions $I_a$ and $\Gamma$, the density function $f_X$ can be re-written into a more concrete form:

$$f_X(x;k,\lambda)=\frac{1}{2}e^{-\frac{(x+\lambda)}{2}}\sum_{p=0}^{\infty}\frac{(\frac{\lambda}{2})^p(\frac{x}{2})^{\frac{k}{2}+p-1}}{p!\Gamma(\frac{k}{2}+p)}.$$

The accumulative function of the chi-square distribution is defined by:

$$P(x;k,\lambda)=\sum_{p=0}^{\infty}e^{-\lambda/2}\frac{(\lambda/2)^p}{p!}Q(x;k+2p),$$

where $Q(x;k)=\frac{\gamma(k/2,x/2)}{\Gamma(k/2)}$ and $\gamma(m,x)=\int_0^{x}t^{m-1}e^{-t}dt$ is an incomplete Gamma function. The accumulative function can be written into the following form:

$$P(x;k,\lambda)=\sum_{p=0}^{\infty}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^p}{p!}\cdot\frac{\int_0^{\frac{x}{2}}t^{\frac{k+2p}{2}-1}e^{-t}dt}{\int_0^{\infty}t^{\frac{k+2p}{2}-1}e^{-t}dt}=\sum_{p=0}^{\infty}e^{-\lambda/2}\frac{(\lambda/2)^p}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})}.$$

Now derive the derivatives of the cumulative functions with the first and third inputs.

$$\frac{\partial \mathrm{P}(x;k,\lambda)}{\partial x} = f_X(x;k,\lambda) \tag{15}$$

and

$$\frac{\partial P(x;k,\lambda)}{\partial \lambda} = \sum_{p=0}^{\infty}\left(-\frac{1}{2}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^p}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})} + e^{-\frac{\lambda}{2}}\cdot\frac{\frac{p}{2}(\frac{\lambda}{2})^{p-1}}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})}\right)$$

$$= \sum_{p=0}^{\infty}\left(-\frac{1}{2}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^p}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})} + \frac{p}{2}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^{p-1}}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})}\right)$$

$$= -\frac{1}{2}\sum_{p=0}^{\infty}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^p}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})} + \frac{1}{2}\sum_{p=1}^{\infty}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^{p-1}}{(p-1)!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})}$$

$$= -\frac{1}{2}\sum_{p=0}^{\infty}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^p}{p!}\cdot\frac{\gamma(\frac{k+2p}{2},\frac{x}{2})}{\Gamma(\frac{k+2p}{2})} + \frac{1}{2}\sum_{q=0}^{\infty}e^{-\frac{\lambda}{2}}\cdot\frac{(\frac{\lambda}{2})^{q}}{q!}\cdot\frac{\gamma(\frac{(k+2)+2q}{2},\frac{x}{2})}{\Gamma(\frac{(k+2)+2q}{2})}$$

$$= -\frac{1}{2}P(x;k,\lambda) + \frac{1}{2}P(x;k+2,\lambda) \,. \tag{16}$$

**Appendix 2:** The Jacobians are derived as follows:

$$\frac{\partial F_j^i}{\partial Y_j^i} = 1 - e^{-q\tau_j^i}[\phi_1(Y_j^i,K,T) + Y_j^i\frac{\partial \phi_1(Y_j^i,K,T)}{\partial Y_j^i}] + Ke^{-r\tau_j^i}\frac{\partial \phi_2(Y_j^i,K,T)}{\partial Y_j^i}$$

$$-q[\int_0^{\tau_j^i} e^{-q(\tau_j^i-v)}[\phi_1(Y_j^i,Y(v),T-v) + Y_j^i\frac{d\phi_1(Y_j^i,Y(v),T-v)}{dY_j^i}]dv$$

$$+rK\int_0^{\tau_j^i} e^{-r(\tau_j^i-v)}\frac{d\phi_2(Y_j^i,Y(v),T-v)}{dY_j^i}dv \,. \tag{17}$$

The derivatives of $\phi_1$ and $\phi_2$ are derived in the following. Noting that $\chi^2$ is denoted by $P$ and using (17) and the expression of function $x$, we obtain that: for $\theta < 2$,

$$\frac{\partial \phi_1(Y_j^i,K,T)}{\partial Y_j^i} \tag{18}$$

$$= P_3(2y(K,T);2+\frac{2}{2-\theta},2x(Y_j^i,0))\frac{d}{dY_j^i}(2x(Y_j^i,T))$$

$$= (2-\theta)\frac{x(Y_j^i,T)}{Y_j^i}[P(2y(K,T);4+\frac{2}{2-\theta},2x(Y_j^i,T)) - P(2y(K,T);2+\frac{2}{2-\theta},2x(Y_j^i,T))];$$

for $\theta > 2$, we derive that

$$\frac{\partial \phi_1(Y_j^i,K,T)}{\partial Y_j^i} \tag{19}$$

$$= P_1(2x(Y_j^i,T);\frac{2}{\theta-2},2y(K,T))\frac{d}{dY_j^i}(2x(Y_j^i,T))$$

$$= 2(2-\theta)\frac{x(Y_j^i,T)}{Y_j^i}f_X(2x(Y_j^i,T);\frac{2}{\theta-2},2y(K,T)).$$

Similarly, we can get

$$\frac{\partial\phi_2(Y_j^i,K,T)}{\partial Y_j^i} = \tag{20}$$

$$\begin{cases}(\theta-2)\frac{x(Y_j^i,T)}{Y_j^i}[P(2y(K,T);4+\frac{2}{\theta-2},2x(Y_j^i,T))-P(2y(K,T);2+\frac{2}{\theta-2},2x(Y_j^i,T))], & \theta>2\\ 2(\theta-2)\frac{x(Y_j^i,T)}{Y_j^i}f_X(2x(Y_j^i,T);\frac{2}{2-\theta},2y(K,T)), & \theta<2\end{cases}.$$

Now we calculate, for $v\in[0,\tau_j^i]$: for $\theta<2$,

$$\frac{d}{dY_j^i}\phi_1(Y_i^j,Y(v),T-v) \tag{21}$$

$$= P_1(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))\frac{d}{dY_j^i}(2y(Y(v),T-v))$$

$$+ P_3(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))\frac{d}{dY_j^i}(2x(Y_j^i,T-v))$$

$\equiv I_1+I_2$, where $I_1$ and $I_2$ are calculated as follows:

$$I_1 = 2(2-\theta)f_X(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))\frac{y(Y(v),T-v)}{Y(v)}\frac{dY(v)}{dY_j^i}$$

$$= 2(2-\theta)f_X(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))\frac{y(Y(v),T-v)}{Y(v)}l_j^i(v),$$

and similarly,

$$I_2 =$$

$$(2-\theta)\frac{x(Y_j^i,T-v)}{Y_j^i}[P(2y(Y(v),T-v);4+\frac{2}{2-\theta},2x(Y_j^i,T-v))-P(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))].$$

Analogously for $\theta>2$, we have

$$\frac{d}{dY_j^i}\phi_1(Y_i^j,Y(v),T-v)\equiv R_1+R_2 \tag{22}$$

with

$$R_1 = 2(2-\theta)f_X(2x(Y_j^i,T-v);\frac{2}{\theta-2},2y(Y(v),T-v))\frac{x(Y_j^i,T-v)}{Y_j^i},$$

and

$$R_2 =$$

$$(2-\theta)\frac{y(Y(v),T-v)}{Y(v)}[P(2x(Y_j^i,T-v);2+\frac{2}{\theta-2},2y(Y(v),T-v))-P(2x(Y_j^i,T-v);\frac{2}{\theta-2},2y(Y(v),T-v))]l_j^i(v).$$

In a similar manner, we can obtain that: for $\theta < 2$,

$$\frac{d}{dY_j^i}\phi_2(Y_i^j,Y(v),T-v) = \tag{23}$$

$$2(\theta-2)f_X(2x(Y_j^i,T-v);\frac{2}{2-\theta},2y(Y(v),T-v))\frac{x(Y_j^i,T-v)}{Y_j^i} +$$

$$(\theta-2)\frac{y(Y(v),T-v)}{Y(v)}[P(2x(Y_j^i,T-v);2+\frac{2}{2-\theta},2y(Y(v),T-v))-P(2x(Y_j^i,T-v);\frac{2}{2-\theta},2y(Y(v),T-v))]l_j^i(v);$$

and for $\theta > 2$,

$$\frac{d}{dY_j^i}\phi_2(Y_i^j,Y(v),T-v) = \tag{24}$$

$$2(\theta-2)f_X(2y(Y(v),T-v);2+\frac{2}{\theta-2},2x(Y_j^i,T-v))\frac{y(Y(v),T-v)}{Y(v)}l_j^i(v) +$$

$$(\theta-2)\frac{x(Y_j^i,T-v)}{Y_j^i}[P(2y(Y(v),T-v);4+\frac{2}{\theta-2},2x(Y_j^i,T-v))-P(2y(Y(v),T-v);2+\frac{2}{\theta-2},2x(Y_j^i,T-v))].$$

Now we calculate, for $k \neq j$,

$$\frac{\partial F_j^i}{\partial Y_k^i} = -qY_j^i\int_0^{\tau_j^i} e^{-q(\tau_j^i-v)}\frac{d\phi_1(Y_j^i;Y(v),v)}{dY_k^i}dv + rK\int_0^{\tau_j^i} e^{-r(\tau_j^i-v)}\frac{d\phi_2(Y_j^i;Y(v),v)}{dY_k^i}dv, \tag{25}$$

where for $\theta < 2$,

$$\frac{d}{dY_k^i}\phi_1(Y_i^j,Y(v),T-v) \tag{26}$$

$$= 2(2-\theta)f_X(2y(Y(v),T-v);2+\frac{2}{2-\theta},2x(Y_j^i,T-v))\frac{y(Y(v),T-v)}{Y(v)}l_k^i(v),$$

$$\frac{d}{dY_k^i}\phi_2(Y_i^j,Y(v),T-v) = \tag{27}$$

$$(\theta-2)\frac{y(Y(v),T-v)}{Y(v)}[P(2x(Y_j^i,T-v);2+\frac{2}{2-\theta},2y(Y(v),T-v))-P(2x(Y_j^i,T-v);\frac{2}{2-\theta},2y(Y(v),T-v))]l_k^i(v);$$

and for $\theta > 2$,

$$\frac{d}{dY_k^i}\phi_1(Y_i^j,Y(v),T-v) = \tag{28}$$

$$(2-\theta)\frac{y(Y(v),T-v)}{Y(v)}[P(2x(Y_j^i,T-v);2+\frac{2}{\theta-2},2y(Y(v),T-v))-P(2x(Y_j^i,T-v);\frac{2}{\theta-2},2y(Y(v),T-v))]l_k^i(v),$$

$$\frac{d}{dY_k^i}\phi_2(Y_i^j,Y(v),T-v) \tag{29}$$

$$= 2(\theta-2)f_X(2y(Y(v),T-v);2+\frac{2}{\theta-2},2x(Y_j^i,T-v))\frac{y(Y(v),T-v)}{Y(v)}l_k^i(v).$$

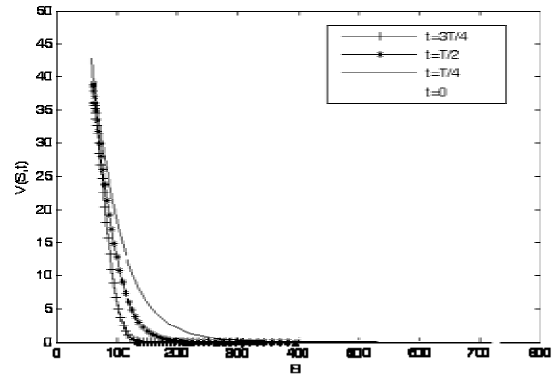


Figure 1. Values of underlying assets (2D drawing) for Example 1 with $\theta = 1.6$

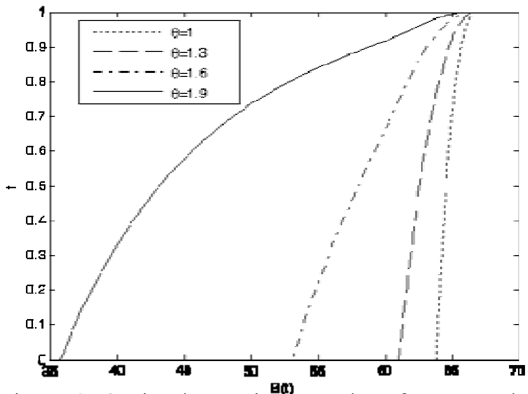


Figure 2. Optimal exercise boundary for Example 1

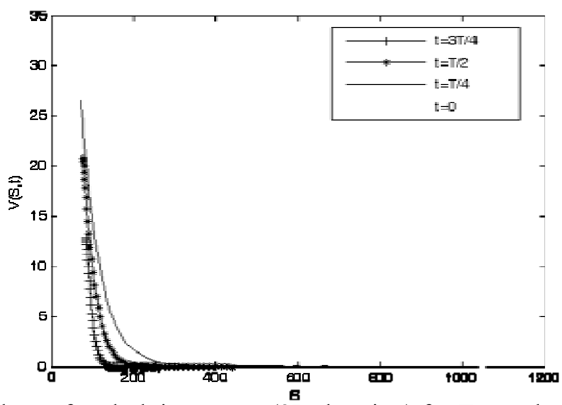


Figure 3. Values of underlying assets (2D drawing) for Example 2 with $\theta = 1.6$

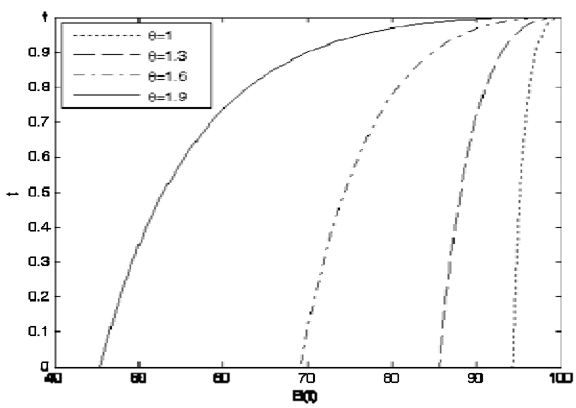


Figure 4. Optimal exercise boundary for Example 2