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Bayesian Approach for ARMA Process and Its Application

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Abstract

There have been a lot of works relating to time series analysis. In this paper, the Bayesian analysis method for ARMA model is discussed and an application example is given. Firstly, the Bayesian theoretic results about AR model and the determination approach for model order are obtained. Then, the approach are presented for Bayesian analysis of MA and ARMA models. As its application, the forecasting model for Shanghai real estate price index is set up and the digital result shows well performance.

Keywords: Bayesian analysis, ARMA, Economic forecsting, Real estate price index

1. Introduction

The past few decades have seen the main results of research on Bayesian time series analysis method (Chongjun Fan, 1993). It has a short history. However, since its special significance of the method for fitting economics data, research in this aspect has seen a vigorous development and progress very soon.

The Bayesian approach of ARMA model which is given by Broemeling and Shaarawy (1986) is particularly appropriate. With conditional likelihood function and base on normal-Gamma prior distribution, they educed that both the posterior distribution of model parameters and the one-step-ahead predictive distribution of AR model are t-distribution, and then apply the results to MA and ARMA model. Meanwhile, they have shown arithmetic to determinate the order number of the models under generalized prior.

In this paper, above methods are reorganized, simplified and expanded, particularly the problem of the model order number under normal - gamma prior distribution is discussed.

2. The results about AR model

In this section, we consider following p-th auto-regressive (AR) model

$$Y(t)=\Phi_1 Y(t-1)+\cdots+\Phi_p Y(t-p)+e(t) \tag{2.1}$$

Where, we assume that $Y(t)$ is observation at time t, and $e(t), t=0,\pm1,\cdots$ are i.i.d. $N(0,\tau^{-1})$. Let

$$\Phi=(\Phi_1,\cdots,\Phi_p)'$$

Then Φ and τ are unknown parameter of the modeling.

If we suppose on n observation are available, denote Y (1)…Y (n), and let

$$S_i=(Y(1),\ \cdots,Y(i))' \qquad S_{(n-i)}=(Y(i+1),\ \cdots,Y(n))'$$

Then the condition density of $S_{(n-p)}$ when S_p is given is as follows:

$$f(S_{(n-p)}/S_p;\Phi,\tau)\propto\tau^{\frac{n-p}{2}}\exp\{-\frac{\tau}{2}\sum_{i=p+1}^{n}(Y(t)-\sum_{i=1}^{p}\Phi_i Y(t-i)^2\}$$
$$\propto\tau^{\frac{n-p}{2}}\exp\{-\frac{\tau}{2}(S_{(n-p)}-X\Phi)'(S_{(n-p)}-X\Phi)\} \tag{2.2}$$

where

$$X=\begin{bmatrix} Y(p)' & \cdots & Y(1)' \\ \vdots & & \vdots \\ Y(n-1)' & \cdots & Y(n-p)' \end{bmatrix}$$

We can use this condition likelihood function instead of the exact likelihood function when n is large relative to p (Priestley M B., 1981). In this section, the discussions are based on function (2.2).

In the model, we consider the group of conjugated prior distribution: normal-Gamma distribution. It can be proved that the Bayesian analysis method for AR model equates to the following Bayesian multivariate regressive analysis:

$$S_{(n-p)}=X\Phi+E \quad E\sim N(0,\tau^{-1}I) \tag{2.3}$$

with the prior density:

$$\begin{cases}\phi/\tau(\text{condition prior density of }\phi\text{ given }\tau)\sim N_{m,r}(\mu,(\tau\Delta)^{-1}) \\ \tau(\text{marginal prior density of }\tau)\sim\Gamma(\alpha-\frac{p}{2},\beta)\end{cases} \tag{2.4}$$

The following result can be obtained easily by the results of Bayesian multivariate regressive model:

Theorem 1. Combining (2.2) with (2.4), It can be obtained the marginal posterior densities of Φ, τ, and one-step-ahead predictive Y(n+1) are multivariate t-distribution, Gamma distribution and univariate t-distribution respectively, namely

$$\Phi\sim t_p(\hat{\Phi},Q,n+2\alpha-2p) \tag{2.5}$$

$$\tau\sim\Gamma(\frac{n}{2}-p+\alpha,\beta+\frac{1}{2}SS_e) \tag{2.6}$$

$$Y_{(n+1)}\sim t(\hat{Y}_{(n+1)},D,n-2p+2\alpha) \tag{2.7}$$

Where,

$$\hat{\Phi}=(X'X+\Delta)^{-1}(X'S_{(n-p)}+\Delta\mu) \tag{2.8}$$

$$SSe=S'_{(n-p)}S_{(n-p)}+\mu'\Delta\mu+(X'S_{(n-p)}+\Delta\mu)'$$
$$(X'X+\Delta)^{-1}(X'S_{(n-p)}+\Delta\mu)$$

$$Q=(\alpha+\frac{n}{2}-p)^{-1}(\beta+\frac{1}{2}SSe)(X'X+\Delta)^{-1}$$

$$L=(Y_{(n)},\cdots,Y_{(n-p+1)})'$$

$$\hat{Y}_{(n+1)}=L'\hat{\Phi}$$

$$D=[1+L'(X'X+\Delta)^{-1}L](n-2p+2\alpha)^{-1}(2\beta+SS_e) \tag{2.9}$$

The following four corollaries can be obtained easily from theorem 1, that is

Corollary 1. With the square loss, Bayes estimation of Φ is the weighted average of least square estimator

$$\hat{\Phi}_{LS} = (X'X)^{-1} X'S_{(n-p)}$$

and prior mean u, that is

$$\hat{\Phi} = (X'X + \Delta)^{-1} X'X\hat{\Phi}_{LS} + (X'X + \Delta)^{-1} \Delta\mu$$

Proof: It can be proved from equations (2.5) and (2.8)

Corollary 2. With the square loss, the Bayes estimation of random error variance, τ^{-1} is

$$\tau^{-1} = (\frac{n}{2} - p + \alpha - 1)^{-1}(\beta + \frac{1}{2}SS_e) \tag{2.10}$$

Proof: It follows from (2.6) that the marginal posterior densities of τ^{-1} is athwart Gamma distribution, then we can obtain (2.10) from the results about the expected value of athwart- Gamma distribution (See page 395 of Berger J. O., 1980).

Corollary 3. One-step-ahead predictive Y (n+1) can be divided to two parts as:

$$V = \tau^{-1} + L'(X'X + \Delta)^{-1} L\hat{\tau}^{-1} \tag{2.11}$$

where $L'(X'X + \Delta)^{-1} L\hat{\tau}^{-1}$ is posterior variance of $L'\Phi$.

Proof: Combining (2.7) and the results about variance of t distribution (See page 394 of Berger J. O., 1980), we can obtain the equation

$$V = (n - 2p + 2\alpha)[(n - 2p + 2\alpha) - 2]^{-1} D$$

Then we obtain this Corollary from (2.9) and (2.10).

Corollary 4. When suppose Δ=0, α=0, β=0, the equations (2.5), (2.6) and (2.7) are the same with the results of generalize prior density $\pi(\Phi, \tau) \propto \tau^{-1}$ in Jeffreys no-information condition.

Equation (2.7) gives the distribution of one-step-ahead $Y_{(n+1)}$. It can be proved distribution of $Y_{(n+2)}$ is also univariate t-distribution when one-step-ahead $Y_{(n+1)}$ is given. But for the joint predictive distribution of $Y_{(n+1)}$, …, $Y_{(n+k)}$ and sole multi-step-ahead distribution $Y_{(n+k)}$, we need make use of digital algorithm. We will not make further discussion in the paper.

3. The determination of model order number

Inference on the order number of the model, we need use the following results:

Theorem 2. It follows with the result of Theorem 1, that

(1) $$Q_p^{-\frac{1}{2}}(\Phi_p - \hat{\Phi}_p) \sim t(0, 1, n + 2\alpha - 2p) \tag{3.1}$$

where

$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_1 \\ \vdots \\ \hat{\Phi}_p \end{bmatrix} \qquad Q = \begin{bmatrix} Q_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & Q_p \end{bmatrix}$$

(2) Piecemeal make the matrix above

$$\Phi = \begin{bmatrix} \Phi_{(1)} \\ \Phi_{(2)} \end{bmatrix} \qquad \hat{\Phi} = \begin{bmatrix} \hat{\Phi}_{(1)} \\ \hat{\Phi}_{(2)} \end{bmatrix} \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

Where, $\Phi_{(1)}$, $\hat{\Phi}_{(1)}$ are $p_1 \times 1$ order matrix, Q_{11} is $p_1 \times p_1$ order matrix respectively, and $p_1 + p_2 = p$. Then the posterior distribution of $\Phi_{(1)}$ when $\Phi_{(2)} = 0$ is given is as follows:

$$\begin{aligned}&\Phi_{(1)} / \Phi_{(2)} = 0 \sim t_{p_1}(\hat{\Phi}_{(1)} - Q_{12}Q_{22}^{-1}\hat{\Phi}_{(2)}, \\ &(n + 2\alpha - 2p + p_2)^{-1}(n + 2\alpha - 2p + \hat{\Phi}_{(2)}{}' Q_{22}^{-1}\hat{\Phi}_{(2)}) \\ &(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}), n + 2\alpha - 2p + p_2)\end{aligned} \tag{3.2}$$

Proof: It can be obtained from the property of multivariate t-distribution and (2.5).

If the model order number is p_1, not p, the prior distribution (2.4) can be adjusted to express when $\Phi_2 = 0$ is given as follows:

$$\begin{cases} \Phi_{(1)} / \tau \sim N(\mu_{(1)} - \Delta_{11}^{-1}\Delta_{12}\mu_{(2)}, (\tau\Delta_{11})^{-1}) \\ \tau \sim \Gamma(\alpha - \dfrac{p_1}{2}, \beta + \dfrac{1}{2}\mu_{(2)}{}'(\Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12})\mu_{(2)}) \end{cases} \tag{3.3}$$

where

$$u = \begin{bmatrix} u_{(1)} \\ u_{(2)} \end{bmatrix} \qquad \Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$$

and where $u_{(1)}$ is $p_1 \times 1$, Δ_{11} is $p_1 \times p_1$.

Discussion about the order number determination of model AR is as follows.

It can be shown that there are two kinds of recursion algorithms from Theorem 1about the posterior distribution of coefficient parameter, which is

<1> We Suppose that the real order number of the model is not bigger than an integer N. since the posterior distribution of Φ when $p = N$ is given, then we can test by using (3.1). Suppose H_0: $\Phi_p = 0$, and opposing $H_1 : \Phi_p \neq 0$, if accept H_0, it means that the value of p can be descended, so make $p = N - 1$, and then adjust the prior distribution by (3.3). It can be calculated recursively until the value of p can not be descended.

<2> When $\Phi_p = 0$ is accepted, we can use (3.2) to take the next test directly, then also by this way we can determine the order number finally.

It can be shown the relation between two kinds of recursion algorithms by the following Theorem 3.

it is similar with the FPE rule of traditional method. By using the Corollary 3 to set up a step of predictive error criterion to determinate order number:

Suppose that N is bigger than or equal to the model real order number. Equation (2.11) shows that we can calculate the value of V when p =1, … , N, and we also can adjust to the prior distribution using by (3.3). If the value of V is the smallest at $p = \hat{p}$, then we can make $\hat{p}$ as the estimator of p.

Theorem 3. If the order number of AR model is p_1, let $Y_{(p_2+1)}, \ldots, Y_{(n)}$ denote available observations, with the prior distribution (3.3), then the posterior distribution of $\Phi_{(1)}$ is (3.2).

4. The analysis method of MA model

Generally q-th moving average (MA) model can be represented as

$$Y(t) = e(t) - \theta_1 e(t-1) - \cdots - \theta_q e(t-q) \tag{4.1}$$

where we also assume that $e(t)$, $t = 0, \pm 1, \ldots$, are $i.i.d. N(0, \tau^{-1})$ correspondingly with likelihood function

$$L(\theta, \tau / S_n) \propto \tau^{\frac{n}{2}} \exp\left\{ -\frac{\tau}{2} \sum_{i=1}^{n} \left[Y(t) + \sum_{j=1}^{q} \theta_j e(t-j) \right]^2 \right\} \tag{4.2}$$

If $e(t)$ is given, then equations (4.2) and (2.2) are the same in the form. Then we known all equitation above is also suitable in the model. Replaced $\theta = (\theta_1, \ldots \theta_q)$ by Φ, n by $n-p$, S_n by S_{n-p}, Correspondingly we can recast the model. For example we can obtain

$$X = \begin{bmatrix} e(0) & e(-1) & \cdots & e(-q+1) \\ e(n-1) & e(n-2) & \cdots & e(n-q) \end{bmatrix}$$

and so on.

Therefore, the key of analysis method of MA model is how to assess the value of $\{e(t)\}$. In Shaarawy and Broemeling (1984), the Box-Jenkins analysis method particularly appropriated. Estimation $\hat{\theta}_* = (\hat{\theta}_{*1}, \ldots, \theta_{*q})'$ of θ can be calculated firstly, then we can get the estimation by using the following formulas.

$$\begin{cases} \hat{e}(t) = Y(t) + \sum_{j=0}^{q} \hat{\theta}_{*j} \hat{e}(t-j) & t = 1, 2, \ldots, n \\ \hat{e}(t) = 0 & t = 0, -1, \ldots, -q+1 \end{cases} \tag{4.3}$$

Then in the context of the Bayesian analysis approach above we can deduced the posterior distributions of Φ θ and τ

Obviously, the analysis approach of MA model combined traditional methods with Bayesian approach and the results are approximately. A common approach assessment for $e(t)$ shown by when we see estimation as observations, There is, however, on how to get the estimation of $e(t)$ only by Bayesian analysis approach is relatively little research and unsolved.

5. The analysis method of ARMA model

The analysis method of ARMA model ARMA similar with the analysis method of MA model.ARMA model can be recast into the following

$$Y(t) = \Phi_1 Y(t-1) + \cdots + \Phi_p Y(t-p) + e(t)\sqrt{2} - \theta_1 e(t-1) - \cdots - \theta_q e(t-q) \tag{5.1}$$

Now firstly we also consider about the likelihood function of S_p

$$L(\Phi, \theta, \tau / S_n) \propto \tau^{\frac{n-p}{2}} \exp\left\{ -\frac{\tau}{2} \sum_{i=p+1}^{n} \left[Y(t) - \sum_{i=1}^{p} \Phi_i Y(t-i) + \sum_{j=1}^{q} \theta_j e(t-j) \right]^2 \right\}$$

The difficulties are also how to get $\{e(t)\}$. Firstly we can get $\hat{\Phi}_*$ and $\hat{\theta}_*$ by Box-Jenkins analysis approach, then from following equation we can estimate $\hat{e}(t)$:

$$\begin{cases} \hat{e}(t) = Y(t) - \sum_{i=1}^{p} \hat{\Phi}_{*i} Y(t-i) + \sum_{j=0}^{q} \hat{\theta}_{*j} \hat{e}(t-j) & t = p+1, \ldots, n \\ \hat{e}(t) = 0 & t = -q+p+1, -q+p+2, \ldots, p \end{cases}$$

Therefore, It can be obtained the value by equation (5.1). Base on above equation (5.1), we can obtain the posterior distributions of Φ θ and τ .

Similarly Box-Jenkins analysis method helps to get the estimation of $e(t)$. And we can draw the conclusion from the discussion above that it is the point to make well the model of MA.

6. An application

As we known, economical system is a complex system and in change frequently (Chongjun Fan, 2008). There is always not enough history data for setting up the forecast model. Therefore, how to make use of Bayesian approach to improve the forecast precision by the relevant data and information is a meaningful topic. Here we discuss the application of Bayes analysis approach to the following case.

The general analysis on real estate by using price index has been carried out for many years in China. The price index is composed by the data from the market research. These data which keep track of the quotations on the market all the times are come from different estates and become a dynamic graph used to observe the quotation on the market. The quantitative research and the precise description and forecast on the orbit track of price index play a major role in real estate research. This paper constructs the forecast modeling on Shanghai Composite Index of Chinese Restate from January in 2005 to May in 2008. The data are deseasonized and normalized first.

First, we get the prior distribution by using Beijing Composite Index of Chinese Restate from October in 2005 to March in 2008, then set Shanghai model up.

The calculations: the order number of the model is p=3, and we can see Shanghai composite index of Chinese restate by Table 1 as below. It is can be seen from table 1 that error rate for one step forecasting of April, May and June in the year 2008 respectively is -0.49%, -0.34%, 0.09%. Better precision and well performance can be drawn form results. (Insert Table 1 here)

7. Conclusions

This paper mainly at studying the Bayesian approach for ARMA models and illustrating the methods for forecasting Shanghai composite index of Chinese restate as an application. Since our insufficient economy data, it takes us difficulties to forecast just by setting up a model, however, in the view of an introduction of a new solution thought which we can use the prior distributed, it has common significance. The application results do show that the methods we proposed are effective.

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Table 1. Predicted result of Shanghai composite index of Chinese restate

Year/Month	Shanghai composite index of Chinese restate	One step forecasting	Error rate for one step forecasting
08/04	2008	1998	-0.49%
08/05	2034	2027	-0.34%
08/06	2068	2070	0.09%