The Inventory Problem with A Non-Symmetric Loss Function

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Abstract

The paper deals with the inventory problem with uncertain demand and a non-symmetric loss function using normal quantile estimation theory The future demand is predicted via Bayesian theory by using data on previous demands. The paper minimizes an inter temporal loss function by choosing 'new' orders. The micro approach of this paper begins by specifying why and how can an inter temporal loss function be minimized and thus explains a richer variation of inventory behavior than an production smoothing model.

Keywords: Uncertainty, Quantile estimation, Loss function

1. Introduction

The univariate normal quantile estimation theory is applied in many operations research problems. Long ago, Edgeworth applied this technique to determine the amount of cash to be kept on hand in a bank. Suppose that there is a large number of depositors, for each of whom the amount to be withdrawn, during a single given period, is a random variable independent of the withdrawals of the other depositors. Then the total amount withdrawn is a normal random variable x, with mean μ and variance σ^2, both unknown. The bank may regard as tolerable a certain probability of not being able to meet all withdrawals. The amount of cash C kept on hand is then $C = \mu + \delta\sigma$ where δ is a standard normal deviate corresponding to an upper tail of θ. The purpose of the paper is to build inventory model in conformity with the basic arguments of Edgeworth model. (Note 1). Here the random demand follows a normal distribution. The future demand for the Nth period is related to the previous demand via the parameters. Thus, the future demand is predicted via Bayesian theory by using the data on previous demands. An inter temporal loss function is minimized in the process.

The motivation for the paper comes from the tension that was developing between the emerging macroeconomic and microeconomic views of investors.. Macroeconomics routinely thought of inventories as a *destabilizing* factor: in theory, the inventory accelerator created cycles that otherwise might not exist; in practice, GNP was more volatile than final sales (GNP less inventory investment). Yet, the prevailing micro theory viewed inventories as a *stabilizing* factor, something cost minimizing or loss minimizing firm could use instead of aiming at smoothing production.

The production smoothing model which has been the basic paradigm within which empirical research on inventories is being conducted, embeds a host of strong restrictions regarding the nature of demand and market structure. Despite the alleged revolution in inventory practices brought about by computerization, the economy- wide ratio of real inventories to real sales has been trendless for the last 50 years. A more promising avenue, therefore, for improving the performance of models should be to choose an inter temporal loss function and then minimize it by choosing 'new' orders. With that end in view, we will follow Edgeworth's banking model, where future demand will be predicted via Bayesian theory using data on previous demands. (Note 2).The plan of the paper will be as follows: Section I will introduce the problem. Section II will formulate the model. Section II will be followed by section III that will contain an analysis of the determination of the optimum. The paper will conclude with a few remarks in section IV.

2. The Model

We will assume that the demand at the beginning of the i th period is : $d_i = h_i'\beta + \varepsilon_i$, where h_i is a known q component vector, β is an unknown q component regression parameter vector, and ε_i is normal with parameters zero and σ. Further we assume that U_i is the quantity to be ordered at the beginning of the i th period and x_i is on-hand inventory at the end of the period i. Then input to the system is the demand d_i and the output is x_i which are related by

$$x_i = x_{i-1} + U_i - d_i = \sum_{j=i} U_j - \sum_{j=i} d_j + x_0 \tag{1}$$

where x_0 is a given constant and denoted the stock at the beginning of the first period. The random variables x_i and U_i have certain normal distribution. The regression parameter vector β is assumed to have a certain prior normal distributions, assuming whether σ is known or unknown.

Let C_1 be the inventory carrying cost per item per period and C_2 be the storage cost per unit per period. Then the specific cost W_i for the i th period is

$$W_i = C_1 x_i, \qquad x_i \geq 0, \qquad W_i = -C_2 x_i, \qquad x_i < 0 \tag{2}$$

Now we define the quantities

$$R_i = E[W_i \mid x_{i-1}, \overline{U}_{i-1}, \overline{d}_{i-1}] \tag{3}$$

where

$$\overline{x}_{i-1} = (x_0, x_1 .... x_{i-1}), \quad \overline{U}_{i-1} = (U_1 .... U_{i-1}), \quad \overline{d}_{i-1} = (d_1 .... d_{i-1}), \text{ and}$$

$$M_i = E(R_i) = \int R_i \, g(\overline{x}_{i-1}, \overline{d}_{i-1}, \overline{U}_{i-1}) \, d\overline{x}_{i-1} \, d\overline{d}_{i-1} \, d\overline{U}_{-1} \tag{4}$$

2.1   Solution

How to derive the optimum , a question that is ultimately related to minimizing the inter temporal

loss function we want to obtain? Now the optimum U_i will be determined by minimizing the controller

$$M_i + \sum_{j=i+1}^{N} M_i$$

where N is the last period.

We will proceed to show that optimum U_N will be determined by minimizing the expected value W_N by using the posterior density of W_N. After determining the optimum U_N, we use this optimum value to find optimum U_{N-1} and so on.

We proceed to show   a la Gleser and Olkin (1970) that optimum U_N will be determined by minimizing the expected values W_N by using the posterior density of W_N. After determining optimum U_N we use this optimum value to find optimum U_{N-1} and so on.

Optimum U_N will be obtained by minimizing M_N, where

$$M_N = \langle E(W_i \mid \overline{\overline{x}}_{N-1}, \overline{\overline{U}}_{N-1}, \overline{\overline{d}}_{N-1})\}$$

$$+ \int W_N \, g \langle x_N \mid \overline{\overline{x}}_{N-1}, \overline{U}_{N-1}, \overline{\overline{d}}_{N-1}) \, g(\overline{x}_{N-1}, \overline{U}_{N-1}, \overline{\overline{d}}_{N-1}\}$$

$$X \quad dx_N \, d\ddot{x}_{N-1} \, d\ddot{U}_{N-1} \, d\ddot{d}_{N-1} \tag{5}$$

To simplify the analysis, we assume that x_N depends only on x_{N-1} and U_N and not on the past values of $\overline{\overline{x}}_{N-1}, \overline{\overline{U}}_{N-1}$. Thus, we may write

$$g \langle \overline{\overline{x}}_N \mid \overline{\overline{x}}_{N-1}, \overline{\overline{U}}_{N-1}, \overline{\overline{d}}_{N-1} \rangle$$

$$= \int \langle g(x_N \mid x_{N-1}, \overline{\overline{U}}_N, \overline{\overline{d}}_{N-1}) \, g \langle U_N \mid x_{N-1}, \overline{\overline{U}}_{N-1}, \overline{\overline{d}}_{N-1} \rangle dU_N$$

$$= \int g \langle x_N \mid x_{N-1}, U_N, \overline{\overline{d}}_{N-1}) \, g \langle U_N \mid x_{N-1}, \overline{U}_{N-1}, \overline{d}_{N-1}\} \, dU_N$$

$$= \int g \langle x_N \mid x_{N-1}, U_N, \beta) \, g \langle \beta \mid \overline{d}_{N-1} \rangle\} \, g \langle U_N \mid \overline{x}_{N-1}, \overline{U}_{N-1}, \overline{d}_{N-1}\} \, dU_N d\beta \tag{6}$$

Where $g \langle \beta \mid \overline{d}_{N-1}\}$ is posterior density of β. This, we have

$$M_N = \int W_N \, g \langle x_N \mid x_{N-1}, U_N, \beta\} \, g \langle \beta \mid \overline{d}_{N-1}\} \, g \langle U_N \mid \overline{x}_{N-1}, \overline{U}_{n-1}, \overline{d}_{N-1}\}$$

$$X \quad g(\overline{x}_{N-1}, \overline{U}_{N-1}, \overline{d}_{N-1}) \, dU_N d\beta \, d\overline{x}_{N-1} d\overline{U}_{N-1} d\overline{d}_{N-1} dx_N \tag{7}$$

However, we must have

$$M_N = \int R_N \, g(\overline{x}_{N-1}, \overline{U}_{N-1}, \overline{d}_{N-1}) \, d\overline{x}_{N-1} \, d\overline{U}_{N-1} \, d\overline{d}_{N-1} \qquad (8)$$

It follows from (7) and (8) that

$$R_N = \int W_N \, g\langle x_N | x_{N-1}, U_N, \beta \rangle \, g\langle \beta | \overline{d}_{N-1} \rangle \, g\langle U_N | \overline{x}_{N-1}, \overline{U}_{N-1}, \overline{d}_{N-1} \rangle$$

$$\text{X} \quad dU_N d\beta dx_N \qquad (9)$$

Obviously M_N is a minimum when R_N is a minimum. Now let $\overline{d}_{N-1} = H\beta + \varepsilon$ where ε is normal with mean vector zero and covariance matrix $\sigma^2 I$, and let β be normal with mean vector β_0 and covariance matrix $\sigma_0^2 I$. Then,

$$g\langle \beta | \overline{d}_{N-1} \rangle = k \exp \{ -\frac{1}{2}(\beta - \beta_1)' \Sigma (\beta - \beta_1) \} \qquad (10)$$

Where

$$\beta_1 = \Sigma^{-1} \left(\frac{H' \overline{d}_{N-I}}{\sigma^2} + \frac{\beta_0}{\sigma_0^2} \right) \qquad (11)$$

Where σ^2 and σ_0^2 are known. Again, by noting that

$$x_N = x_{N-I} + U_N - d_N = x_{N-I} + U_N - h_N'\beta - \varepsilon_N \qquad (12)$$

We find that

$$g\{\langle x | x_{N-1} \rangle, U_N, \beta\} = k \exp \{ \frac{1}{2\sigma^2} (x_{N-I} + U_N - h_N'\beta - x_N)^2 \} \qquad (13)$$

Now we define the quantities

$$T = x_{N-I} + U_N - h_N'\beta_1 \quad A = (h_N' \Sigma^{-1} h_N + \sigma^2)^{-1} \qquad (14)$$

Using (9), (10), and (13). find that

$$R_N = k \int_0^\infty C_1 x_N \exp\{-\frac{1}{2}(x_N - T)^2 A\} dx_N - k \int_0^\infty C_2 x_N \exp\{-\frac{1}{2}(x_N - T)^2 A\} dx_N$$

$$= (C_1 + C_2) f(T \sqrt{A}) + (C_1 + C_2) TF(T\sqrt{A}) - C_2 T \qquad (15)$$

Where F and f are the distribution and density functions at the standard normal variate. Minimizing R_N with respect to U_N yields the optimum U_N^* as

$$U_N^* = Z_p A^{1/2} - x_{N-I} + h_N' \beta_1$$

Where $Z_p, p - C_2/(C_1 + C_2)$ is the p-th quantile of the standard normal density. Note that (15) relates future x_N with respect to T via the parameters.

To find U_{N-I}^*, we minimize $M_{N-I} + M_N^*$ with respect to U_{N-I} where M_N^* is the value of M_N obtained by substituting U_N^* for U_N. Now M_N^* does not contain x_{N-I} and U_{N-I} so that minimum of $M_{N-I} + M_N^*$ is obtained by minimizing M_{N-I} with respect to U_{N-I}. This applies to any intermediate stage and we get a regular strategy for finding the optimum controller (Note 3, Note 4). This analysis for the normal error structure model is given by Sarker (1975).

3. Conclusion

We connected a single future variable or observation to the past data. This is just a Bayesian theory and we have obtained this Bayesian statistical inventory model over N discrete periods of equal length within the framework of normal distribution theory. An expression for optimal order quantity at the beginning of the N-th period is obtained. And the expected value of a certain function of this predicted demand is minimized to obtain the required order quantity. However, we may also relate more than a single independent variable to past data and form a more general model and the methodology remains the same. The merit of our work is that during 2000s economists once again know something they had known in the 1950s: that inventory investment is of the first order importance in business cycles. But, they will be beginning to realize that the standard production smoothing model or the buffer stock model of inventories can be supplanted by a new micro approach.

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Notes

Note 1. It is obvious that the problem of inference of normal quintiles of the type of are interesting problems in several situations. The inference problems of the normal quantiles theory associated with classical regression models have been tackled in the literature by various procedures: classical, Bayesian, structural, minimax, etc. For some of the readily available or easily derivable results, see, e.g., Guttman (1970). Narragon et al (1974) applied Edgeworth theory to a campus parking problem where different class of users exhibited varying demand practices.

Note 2.This model differs from the other uncertainty models in the sense that demand is the main source of uncertainty here unlike in the other works, for example, Grossman and Stiglitz (1976) and Bray (1981), where uncertainty is introduced in the form of random supply that prevents price from acting as a price statistic. The major thrust of this model is not to establish conditions under which prices reveal information which can be equivalent to perfect information but to improve the estimate of the parameters determining demand. See on related views

Blinder (1986), Eichenbaum (1984) West (1987), and Jeffry and Zeldes (1987). There are other types of inventory problems specifically related to inventory control where one makes a numerical study using advance order information in supply-chain inventory control, see, for example, see Marklund (2006) and Youngwon, Sungwon and Juho (2002)

Note 3. When β and σ are both unknown, we have

$$R_N = k \ \int \ W_N g \langle \beta, \sigma \,|\, \overline{d}_{N-1} \rangle \exp\left(-\frac{1}{2\sigma^2}(x_{N-1} + U_N - h_N^{'} \beta - x_N^{'})^2 \right) dx_N d\beta d\sigma \qquad (17)$$

we assume

$$g\, g\langle \beta, \sigma \,|\, \overline{d}_{N-1} \rangle = k \exp\left\{-\frac{1}{2}(\beta-\beta)^{'}(H^{'}H + I)^{-1}(\beta - \beta_1)/\sigma^2 \right. \qquad \text{where} \qquad (18)$$

$$\beta_1 = (H^{'}H + I)^{-1}(H^{'}\overline{d}_{N-1} + \beta_0) \qquad (19)$$

On integrating (17) with respect to β first and then with respect to σ^{-2} we have

$$R_N = k \ \int \ C_1 \ x_N \,[\, I + (x_N - T)^2 / \sigma \,]^{-(m+1^{'2}} dx_N$$

$$- k \int \ C_2 \ x_N \,[\, I + (x_N - T)^2 / \sigma \,]^{-(m+1)/2} dx_N \qquad (20)$$

where $\theta = I + h_N^{'} (H^{'} H + I)^{-1}$. However (20) and (15) have the same form and hence, in this case, we find that

$$U_N^* = Z_p \theta^{\frac{1}{2}} - x_{N\text{-}I} + h_N^{'} \beta_1 \tag{21}$$

Where Z_p now is the p-th quantile of the density

$$f(x) = k(i + x^2)^{-(m+1)/2}$$

If β_1 of (19) s assumed to be known, then (21) is known. Otherwise, we have to use the Bayesian theory for making probability statement about U_N^*, after substituting Bayes' estimate for θ_1 in (21)

Note 4. Another approach where the parameters are unknown is to use Blackwell- Lehman-Scheffe theorem and obtain U_N^* as the quantile of a certain conditional distribution function.