# Uncertain Models for Bed Allocation

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**Abstract**

The purpose of this paper is to develop a methodology for modeling sickbed allocation problems with uncertain length of stay for each patient. Two uncertain bed allocation models are presented. Hybrid intelligent algorithm is employed for solving these models. Finally, numerical example is provided to demonstrate the feasibility of the proposed algorithm.

**Keywords:** Uncertain variable, Sickbed, Allocation, Uncertain programming

## 1. Introduction

Queue is frequently encountered in daily life. Queueing theory is the mathematical study of waiting lines (or queues) and generally considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide service. It is applicable in a wide variety of situations that may be encountered in business, commerce, industry, public service, engineering and healthcare.

In supermarkets and in banks queues form when there are insufficient server units to meet the demand for service. Similarly, in hospitals, the queues form when there are insufficient beds available to admit ill people. In supermarkets and in banks customers can go elsewhere. But sick people have no alternative options: they just have to wait. When the lack of sickbeds occurs patients wait to be admitted. The bed crisis was considered as a queuing system and discrete event simulation was employed to evaluate the model numerically by C. Vasilakis and E. El-Darzi.

Other researches have existed in healthcare service. A methodology that uses system simulation combined with optimization to determine the optimal number of doctors, lab technicians and nurses required to maximize patient throughput and to reduce patient time in the system subject to budget restrictions was presented by Mohamed and Talal.

Real-life decisions are usually made in the state of uncertainty. By uncertain programming we mean the optimization theory in uncertain environments.

In many cases, fuzziness and randomness simultaneously appear in a system. In order to describe these phenomena, a fuzzy random variable was introduced by Kwakernak as a random element taking “fuzzy variable” values. A random fuzzy variable was proposed by Liu as a fuzzy element taking ”random variable” values. More generally, a hybrid variable was introduced by Liu as a measurable function from a chance space to the set of real numbers. Fuzzy random variable and random fuzzy variable are instances of hybrid variable. In order to

measure hybrid events, a concept of chance measure was introduced by Li and Liu. After that, a general framework of hybrid programming was proposed by Liu.

Uncertainty theory was founded by Liu in 2007 as a branch of mathematics based on normality, monotonicity, selfduality, and countable subadditivity axioms. Uncertain programming was presented by Liu in 2008 as the optimization theory in uncertain environments. It has been applied in system reliability design, project scheduling problem, vehicle routing problem, facility location problem, and machine scheduling problem.

In this paper, we will introduce two uncertain models for bed allocation problem. The rest of the paper is organized as follows: in Section 2, we will briefly review the concepts of uncertainty. Then we describe the assumptions and notations for uncertain bed allocation models in Section 3. After that, two uncertain models for bed allocation are formulated in Section 4. In Section 5, a hybrid intelligent algorithm designed for solving the models is described. In Section 6, a numerical example is given to show the effectiveness of the proposed algorithm. Finally, we discuss the conclusions and provide future direction for research.

## 2. Preliminaries

Let $\Gamma$ be a nonempty set, and let $\mathrm{L}$ be a σ-algebra over $\Gamma$. Each element $\Lambda \in \mathrm{L}$ is called an *event*. $\mathrm{M}$ is a set function which follows the four axioms given by Liu:

**Axiom 1**. *(Normality)* $\mathrm{M}\{\Gamma\}=1$.

**Axiom 2**. *(Monotonicity)* $\mathrm{M}\{\Lambda_1\} \le \mathrm{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \subset \Lambda_2$.

**Axiom 3**. *(Self-Duality)* $\mathrm{M}\{\Lambda\}+\mathrm{M}\{\Lambda^C\}=1$ *for any event* $\Lambda$.

**Axiom 4**. *(Countable Subadditivity) For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$\mathrm{M}\left\{\bigcap_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathrm{M}\{\Lambda_i\}.$$

Then $\mathrm{M}$ is called an uncertain measure, the triplet $(\Gamma,\mathrm{L},\mathrm{M})$ is called an *uncertainty space*.

A measurable function defined on $\Gamma$ into $\square$ is called an *uncertain variable*.

### *2.1 Identification Function*

A random variable may be characterized by a probability density function, and a fuzzy variable may be described by a membership function. Liu introduce an identification function to characterize an uncertain variable.

**Definition 1** (*Liu) An uncertain variable* $\xi$ *is said to have a first identification function* $\lambda$ *if*

*(i)* $\lambda(x)$ *is a nonnegative function on* $\mathcal{R}$ *such that*

$$\sup_{x\neq y}\left(\lambda(x)+\lambda(y)\right)=1;$$

*(ii) for any set* $B$ *of real numbers, we have*

$$\mathrm{M}\{\xi \in B\}=\begin{cases}\sup\limits_{x\in B}\lambda(x), & \text{if } \sup\limits_{x\in B}\lambda(x)<0.5\\ 1-\sup\limits_{x\in B^C}\lambda(x), & \text{if } \sup\limits_{x\in B}\lambda(x)\ge 0.5\end{cases}$$

**Example 1:** By a *trapezoidal uncertain variable* we mean the uncertain variable fully determined by the quadruplet $(a;b;c;d)$ of crisp numbers with $a<b<c<d$, whose first identification function is

$$\lambda(x)=\begin{cases}(x-a)/2(b-a), & \text{if } a\le x\le b;\\ 0.5, & \text{if b}\le x\le c;\\ (x-d)/2(c-d), & \text{if } c\le x\le d;\\ 0, & otherwise.\end{cases}$$

### *2.2 Expected Value*

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

**Definition 2** (*Liu) Let* $\xi$ *be an uncertain variable. Then the expected value of* $\xi$ *is defined by*

$$E[\xi] = \int_0^{+\infty} \mathrm{M}\ \{\xi \geq r\} dr - \int_{-\infty}^{0} \mathrm{M}\ \{\xi \leq r\} dr$$

*provided that at least one of the two integrals is finite.*

## 3. Assumptions and notations

Bed crisis sometimes appears in hospitals. The crisis is usually attributed to factors such as the bad weather, influenza, older people, geriatricians, and lack of cash or nurse shortages. As mentioned above, in hospitals, the queues form when there are insufficient beds available to admit ill people, and sick people have to wait until free bed is available.

When a patient arrives, he/she enter a queuing system. We will divide the service system into three phases including waiting sickbed, preoperative preparation and postoperative recovery. After that, the patient leaves the system.

For optimize the beds allocation during the gusty busy period, we describe assumptions and notations as follows:

*3.1 Assumptions*

1) treatment received across different sickbeds is identical;

2) the total account of sickbeds is fixed and each bed is occupied by one patient;

3) the out-patients arrivals randomly and independently of each other;

4) LOS(length of stay) for each patient is considered to be a uncertain variable.

*3.2 Notations*

$n$ : the number of out-patients need to be hospitalized during a period of $t$ days;

$m$ : the number of fixed sickbeds;

$j = 1, 2, \cdots, m$ : the sickbeds;

$m_j$ : the number of patients cared for in sickbed $j$ ;

$i = 1, 2, \cdots, m_j$ : the patients cared for in sickbed $j$ ;

$k(j,i)$ : the patient $i$ cared for in sickbed $j$ ;

$\boldsymbol{x}$ : the decision vector, where

$$\boldsymbol{x} = \left(k(1,1), k(1,2), \cdots, k(1,m_1), k(2,1), k(2,2), \cdots, k(2,m_2), \cdots, k(j,1), k(j,2), \cdots, k(j,m_j)\right),$$

and the sequence $\{k(j,i)\}$ is the rearrangement of $\{1, 2, \cdots, n\}$ ;

$D\left(k(j,i)\right)$ : the arriving time of patient $k(j,i)$, $\left(i = 1, 2, \cdots, m_j; j = 1, 2, \cdots, m\right)$;

$T\left(k(j,i)\right)$ : the LOS(length of stay) of patient $k(j,i)$, $\left(i = 0, 1, 2, \cdots, m_j; j = 1, 2, \cdots, m\right)$, and $T\left(k(j,0)\right)$ is the occupation time of initial patient in sickbed $j$, so patient $k(j,i)$ will leave the hospital at the time $\sum_{i_0=0}^{i} T\left(k(j,i_0)\right)$;

$\boldsymbol{\xi}$ : the uncertain vector where $\boldsymbol{\xi}_i = \mathrm{T}\left(\boldsymbol{x}_i\right)$.

Thus the waiting time of patient $k(j,i)$ is $\left(\sum_{i_0=0}^{i-1} T\left(k(j,i_0)\right) - D\left(k(j,i)\right)\right) \vee 0$.

We denote the total waiting times and the longest staying time in hospital of all the $n$ patients by $f_1(\boldsymbol{x}, \boldsymbol{\xi})$ and $f_2(\boldsymbol{x}, \boldsymbol{\xi})$, respectively. Then we have

$$f_1(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{j=1}^{m} \sum_{i=1}^{m_j} \left(\left(\sum_{i_0=0}^{i-1} T\left(k(j,i_0)\right) - D\left(k(j,i)\right)\right) \vee 0\right)$$

$$f_2(\boldsymbol{x}, \boldsymbol{\xi}) = \max_{1 \leq j \leq m} \sum_{i=0}^{m_j} \left(T\left(k(j,i)\right)\right)$$

## 4. Uncertain Bed Allocation Model

With assumption of the uncertain $\xi$, we introduce two uncertain models for bed allocation.

*4.1 Expected Time Goal Model*

In order to balance the above conflicting objectives $f_1(\boldsymbol{x},\boldsymbol{\xi})$ and $f_2(\boldsymbol{x},\boldsymbol{\xi})$, we may have the following target levels and priority structure:

At the first priority level, the expected total waiting time $E\left[f_1(\boldsymbol{x},\boldsymbol{\xi})\right]$ should not exceed the target value $b_1$. Thus we have a goal constraint

$$E\left[f_1(\boldsymbol{x},\boldsymbol{\xi})\right]-b_1=d_1^+,$$

in which $d_1^+ \vee 0$ will be minimized.

At the second priority level, the expected longest staying time in hospital $E\left[f_2(\boldsymbol{x},\boldsymbol{\xi})\right]$ should not exceed the target value $b_2$. That is, we have a goal constraint

$$E\left[f_2(\boldsymbol{x},\boldsymbol{\xi})\right]-b_2=d_2^+,$$

in which $d_2^+ \vee 0$ will be minimized.

Then we have the following expected time goal programming model for the bed allocation problem:

$$\begin{cases} \text{lexmin}\left\{d_1^+ \vee 0, d_2^+ \vee 0\right\} \\ s.t. \\ E\left[f_1(\boldsymbol{x},\boldsymbol{\xi})\right]-b_1=d_1^+ \\ E\left[f_2(\boldsymbol{x},\boldsymbol{\xi})\right]-b_2=d_2^+ \\ 1\le k(j,i)\le n \quad j=1,2,\cdots,m, i=1,2,\cdots,m_j, \text{integers} \\ k(j_1,i_1)\ne k(j_2,i_2), \text{if } (j_1,i_1)\ne(j_2,i_2) \end{cases}$$

where lexmin represents lexicographically minimizing the objective vector.

*4.2 Chance-Constrained Goal Programming*

We assume the following priority structure. At the first priority level, the total waiting time $f_1(\boldsymbol{x},\boldsymbol{\xi})$ should not exceed the target value $b_1$ with confidence level $\alpha_1$. Thus we have a goal constraint

$$\text{M}\left\{f_1(\boldsymbol{x},\boldsymbol{\xi})-b_1\le d_1^+\right\}\ge\alpha_1$$

in which $d_1^+ \vee 0$ will be minimized.

At the second priority level, the longest staying time in hospital $f_2(\boldsymbol{x},\boldsymbol{\xi})$ should not exceed the target value $b_2$ with confidence level $\alpha_2$. Thus we have a goal constraint

$$\text{M}\left\{f_2(\boldsymbol{x},\boldsymbol{\xi})-b_2\le d_2^+\right\}\ge\alpha_2$$

in which $d_2^+ \vee 0$ will be minimized.

Then we have the following chance-constrained goal programming model for the bed allocation problem:

$$\begin{cases} \text{lexmin}\left\{d_1^+ \vee 0, d_2^+ \vee 0\right\} \\ s.t. \\ \text{M}\left\{f_1(\boldsymbol{x},\boldsymbol{\xi})-b_1\le d_1^+\right\}\ge\alpha_1 \\ \text{M}\left\{f_2(\boldsymbol{x},\boldsymbol{\xi})-b_2\le d_2^+\right\}\ge\alpha_2 \\ 1\le k(j,i)\le n \quad j=1,2,\cdots,m, i=1,2,\cdots,m_j, \text{integers} \\ k(j_1,i_1)\ne k(j_2,i_2), \text{if } (j_1,i_1)\ne(j_2,i_2) \end{cases}$$

**5. Intelligent Algorithm Based on Uncertain Simulation**

In order to solve the models, we integrate uncertain simulation, neural network and GA to produce a hybrid intelligent algorithm.

**Step 1.** Generate training input-output data for uncertain functions like

$$U_1 : \boldsymbol{x} \to E[f(\boldsymbol{x}, \boldsymbol{\xi})]$$

$$U_2 : \boldsymbol{x} \to \mathrm{M}\ \{f(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\}$$

for any given decision vector $\boldsymbol{x}$.

**Step 2.** Train a neural network to approximate the uncertain functions by the generated training data.

**Step 3.** Initialize *pop_size* chromosomes whose feasibility may be checked by the trained neural network.

**Step 4.** Update the chromosomes by crossover and mutation operations and the trained neural network may be employed to check the feasibility of offspring.

**Step 5.** Calculate the objective value for all chromosomes by the trained neural network.

**Step 6.** Compute the fitness of each chromosome by rank-based evaluation function based on the objective values.

**Step 7.** Select the chromosome by rank-based evaluation function based on the objective values. Here, we adopt the following rank-based evaluation function $Eval(V_i) = a(1-a)^{i-1}$, $i = 1, 2, \cdots, pop_size$ where parameter $a \in (0,1)$, $i = 1$ means the best individual, and $i = pop_size$ the worst individual.

**Step 8.** Repeat the fourth to seventh steps a given number of cycles.

**Step 9.** Report the best chromosome as the optimal solution.

**6. Numerical Experiments**

In this section, we give a simple example to illustrate the effectiveness of the proposed hybrid intelligent algorithms employing MATLAB. Here the parameters are set as follows: the population size is 100, the probability of crossover $P_c$ is 0.6, the probability of mutation $P_m$ is 0.3, and the parameter $a$ in the rank-based evaluation function is 0.05.

**Example 2.** Let us consider 10 patients arrive in one day and 3 sickbeds are available. The estimated lengths of stay for each patient are trapezoidal uncertain variables as table 1.

At the first priority level, the expected total waiting time $E\left[f_1(\boldsymbol{x}, \boldsymbol{\xi})\right]$ should be as little as possible. Then we have a goal constraint

$$E\left[f_1(\boldsymbol{x}, \boldsymbol{\xi})\right] - d_1^+ = 0,$$

in which $d_1^+$ will be minimized.

At the second priority level, the expected longest staying time in hospital $E\left[f_2(\boldsymbol{x}, \boldsymbol{\xi})\right]$ should not exceed the target value18. That is, we have a goal constraint

$$E\left[f_2(\boldsymbol{x}, \boldsymbol{\xi})\right] - d_2^+ = 18,$$

in which $d_2^+$ will be minimized.

Then we have the following expected time goal programming model for the bed allocation problem:

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$$\begin{cases} \text{lexmin}\left\{d_1^+ \vee 0, d_2^+ \vee 0\right\} \\ s.t. \\ E\left[f_1(\boldsymbol{x},\boldsymbol{\xi})\right] - d_1^+ = 0 \\ E\left[f_2(\boldsymbol{x},\boldsymbol{\xi})\right] - d_2^+ = 18 \\ 1 \le k(j,i) \le n \quad j = 1,2,3, i = 1,2,\cdots,m_3, \text{integers} \\ k(j_1,i_1) \ne k(j_2,i_2), \text{if } (j_1,i_1) \ne (j_2,i_2) \end{cases}$$

A run of the hybrid intelligent algorithm (3000 cycles in fuzzy simulation, 500 generation in GA) shows that the optimal allocation is

Sickbed 1: $4 \to 5 \to 3$;

Sickbed 2: $2 \to 6 \to 9 \to 7$;

Sickbed 3: $8 \to 1 \to 10$.

**7. Conclusions**

In this paper, with the introduced uncertain variable, the bed allocation problem in hospital is discussed under uncertain environment; thereby two uncertain models are presented. This model effectively reduce the patients' time in queue, thus can improve the public satisfaction to the health service. The realization of this method will open up new research field of the application of uncertainty theory. In the future research, dynamic bed allocation is concerned and we will improve the efficiency of algorithm.

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Table 1. The estimated length of stay for each patient

| Patient | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Length of Stay | (6,9,11,14) | (4,7,9,12) | (3,6,8,11) | (1,4,6,9) | (1,3,5,8) |
| Patient | 6 | 7 | 8 | 9 | 10 |
| Length of Stay | (0,3,5,8) | (1,3,5,7) | (0,1,3,4) | (0,1,3,4) | (0,1,3,4) |