

Wavelet Method for Nonlinear Partial Differential Equations of Fractional Order

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Abstract

A wavelet method to the solution for time-fractional partial differential equation, by which combining with Haar wavelet and operational matrix to discretize the given functions efficaciously. The time-fractional partial differential equation is transformed into matrix equation. Then they can be solved in the computer oriented methods. The numerical example shows that the method is effective.

Keywords: Operational matrix, Haar wavelet, Numerical solution, Partial differential equations of fractional order

1. Introduction

Wavelet is a function which meets some specific properties. Their compression and translation can generate a group base of space, because Mallat makes wavelet have more meticulous video analysis ability transforms than traditional Fourier analysis (Zhengxing Cheng and Shouzhi Yang, 2007; Lizhi Cheng and Hongxia Wang, 2004). At present, it has been widely applied to signal analysis, image processing, numerical analysis (Yiming Chen and Yongbing Wu, 2010), etc. Wavelet analysis mainly researches expression of function, that is to say that it can decompose function to the sum of "basis functions", and "the basis function" is composed of wavelet functions by translating and compressing. Wavelet function has the very good properties of partial and smooth, which makes people describe function by decomposition coefficient and can analyze the properties of partial and global of function. It's efficient to get high-precision solution.

Fractional integral is the expansion from derivative and integral to the non-integer order, derivative and integral in partial differential equation is non-integer, it is called the fractional order partial differential equation (I. Podlubny, 1999). Compared with an integer order differential equation, the advantage of fractional order differential equation is that it can mimic natural physical process and dynamic system process for better. Therefore, the fractional order differential equation has been drawn more and more extensive attention in applied research. However, not only the given conditions and quantities of solving analytical solution are limited, but also it's very complicated and difficult. Literatures (Congcong Tian and Mei Zhang, 2010; Shuqin Zhang and Depin Lan, 2006) discuss the boundary value problems of fractional order partial differential equation; literatures (J.L.Wu, 2009; Abbas Saadatmandi and Mehdi Dehghan, 2010) discuss solution of linear fractional order partial differential equations and nonlinear ordinary differential equations. In this paper, using operational matrix of orthogonal function, we make a unified framework which generate operational matrix. Based on solving numerical solution of linear fractional order partial differential equation, we provide the algorithm for a class of nonlinear partial differential equation method. In this method, we use Haar wavelet to establish a matrix equation and obtain the algebraic equation which is suitable for computer programming. It simplifies problems and accelerates the calculation speed.

In this paper, we use the related properties of Haar wavelet and the operational matrix and consider time fractional order partial differential equation as follows:

$$D_{*t}^{\alpha} u(x, t) = f(u, u_x, u_{xx}) + g(x, t), m-1 < \alpha \leq m \quad (1)$$

Where $D_{*t}^{\alpha} = \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}$ is the Caputo fractional derivative of order α , $m \in N$, f is a nonlinear function and g is the source function.

2. Preparation knowledge

The Haar wavelets have the following features: (1) orthogonal and normalization, (2) having close support, and (3) the simple expression. They are single rectangle wavelets in the support region; accordingly, the Haar wavelets are widely used in dealing with practical problems.

For $t \in [0, 1]$, Haar wavelet function is defined as follows:

$$h_0(t) = \frac{1}{\sqrt{m}} \quad (2)$$

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, \frac{k-1}{2^j} \leq t < \frac{k-1/2}{2^j} \\ -2^{\frac{j}{2}}, \frac{k-1/2}{2^j} \leq t < \frac{k}{2^j} \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Where, $i = 0, 1, 2, \dots, m-1$, $m = 2^l$, l is a positive integer. j and k represent integer decomposition of the index i , $i = 2^j + k - 1$, $j \geq 0$.

In the development of fractional order partial differential equation, based on different basis, purpose and application field, which form the various defines of fractional operator, such as Riemann–Liouville, Grunwald–Letnikov, Riemann–Liouville, Capotu fractional differential–integral definition and Riesz potentials operator, etc.

We define relevant fractional differential - integral as follows:

(1) A real function, $f(x)$, $x > 0$, is said to be in the space C_{μ} , $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty]$, and it is said to be in the space C_{μ}^m if $f^{(m)} \in C_{\mu}$, $m \in N$.

(2) The Riemann - Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \alpha > 0, t > 0 \quad (4)$$

$$I^0 f(t) = f(t) \quad (5)$$

(3) For $f \in C_p$, $p > 0$, $\alpha > 0$, $\beta > -1$:

- $I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x)$;
- $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x)$;
- $I^{\alpha} x^{\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}$;

$$\int_0^t \frac{t^n}{\sqrt{x-t}} dt = \sqrt{\pi} \frac{x^{n+1/2} \Gamma(n+1)}{\Gamma(n+3/2)}$$

$$d. \quad I^\alpha D^\alpha f(x) = f(x) - f(0).$$

(4) Grünwald - Letnikov definition:

For any real α , we define derivative of fractional order α as follows:

$${}_a^G \mathbf{D}_t^\alpha f(t) = \lim_{\substack{h \rightarrow 0 \\ mh=t-a}} h^{-\alpha} \sum_{k=0}^m g_k f(t-kh) \quad (6)$$

Where: $g_k = \frac{(-\alpha)(-\alpha+1)(-\alpha+2)\cdots(-\alpha+k-1)}{k!}$ is constant.

(5) Caputo definition:

$${}_a^C \mathbf{D}_t^\alpha f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & \alpha = m \in \mathbb{N}; \\ \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \leq m-1 < \alpha < m. \end{cases} \quad (7)$$

3. Proposed method

Chen and Hsiao (Zaid Odibat and Shaher Momani, 2008) raised the ideology of operational matrix in 1975, and Kilicman and Al Zhour (C.F.Chen and C.H.Hsiao, 1975) investigated the generalized integral operational matrix, that is, the integral of matrix $\Phi(t)$ can be approximated as follows:

$$\int_0^t \Phi(t) dt \cong Q_\phi \Phi(t) \quad (8)$$

Where Q_ϕ is an operational matrix of one-time integral of matrix $\Phi(t)$, similarly, we can get operational matrix Q_ϕ^n of n-time integral of $\phi(t)$. Wu and Hsiao (J.L. Wu and C.H. Hsiao, 1997) proposed a uniform method to obtain the corresponding integral operational matrix of different basis. For example, the operational matrix of $\Phi(t)$ can be expressed by following:

$$Q_\Phi = \Phi(t) Q_B \Phi^{-1}(t) \quad (9)$$

Here Q_B is the operational matrix of the block pulse function:

$$Q_{B_m} = \frac{1}{m} \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{2} & 1 & \cdots & 1 \\ 0 & 0 & \frac{1}{2} & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix} \quad (10)$$

If $\Phi(t)$ is a unitary matrix, then $Q_\Phi = \Phi Q_B \Phi^T$, Q_Φ is a matrix with characteristic of briefness and profound utility.

For arbitrary function $y(x, t) \in L^2(R)$, it can be expanded into Haar series by

$$y(x, t) \approx \sum_{i=1}^m \sum_{j=1}^m c_{i,j} h_i(x) h_j(t) \quad (11)$$

Where $c_{i,j} = \int_0^1 y(x,t)h_i(x)dx \cdot \int_0^1 y(x,t)h_j(t)dt$ ($i, j = 0, 1, 2, \dots, m-1$) are coefficients, discrete $y(x,t)$ by choosing the same step of x and t , we obtain

$$Y(x,t) = H^T(x)CH(t) \quad (12)$$

Here $y(x,t)$ is the discrete form of $y(x,t)$, and

$$H = \begin{bmatrix} \vec{h}_0 \\ \vec{h}_1 \\ \vdots \\ \vec{h}_{m-1} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \vdots & \vdots & & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}, \quad C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1} \\ \vdots & \vdots & & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1} \end{bmatrix}$$

C is the coefficient matrix of Y , and it can be obtained by formula: $C = (H^T)^{-1} \cdot Y \cdot H^{-1}$, H is an orthogonal matrix, then,

$$C = H \cdot Y \cdot H^{-1} \quad (13)$$

Consider the time-fractional partial differential equation:

$$D_*^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = f(u, u_x, u_{xx}) + g(x,t), m-1 < \alpha \leq m$$

$$u(x,0) = h(x), 0 < \alpha \leq 1$$

$$g(x,t) \in D, D = [0,1] \times [0,1]$$

Since $u(x,t) \in L^2(R)$, we suppose

$$u(x,t) \approx \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} c_{ij} h_i(x) h_j(t) \quad (14)$$

Then we can obtain the discrete form of Eq. (13) by taking step $\Delta = \frac{1}{m}$ of x, t , there is

$$U(x,t) = H^T(x)CH(t) \quad (15)$$

Then combining Eq. (15) with Eq. (9), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{\partial^\alpha U}{\partial t^\alpha} = H^T(x)C \frac{\partial^\alpha}{\partial t^\alpha} H(t) = H^T(x)CQ_H^{-\alpha} H(t) \quad (16)$$

$$\frac{\partial u(x,t)}{\partial u} \approx H^T(x)(Q_H^{-1})^T CH(t) \quad (17)$$

$$\frac{\partial^2 u(x,t)}{\partial u^2} \approx H^T(x)(Q_H^{-2})^T CH(t) \quad (18)$$

Here, $g(x,t)$ is a known function, discrete it, then we have

$$D = (g(x_i, t_j)), \quad i, j = 0, 1, 2, \dots, m-1 \quad (19)$$

Substitute Eqs. (16) - (19) into Eq. (11), there is

$$H^T(x)CQ_H^{-\alpha} H(t) = f(H^T(x)CH(t), H^T(x)(Q_H^{-1})^T CH(t), H^T(x)(Q_H^{-2})^T CH(t)) + D \quad (20)$$

Eq. (20) is a nonlinear matrix equation. We can solve with MATLAB software.

4. Numerical examples

Consider time - fractional order partial differential equations as follows

$$D_{*t}^\alpha u(x,t) = u_{xx}(x,t) + 6u(x,t)(1-u(x,t)), 0 < t \leq 1, 0 \leq x \leq 1, 0 < \alpha \leq 1, \quad (21)$$

$$u(x,0) = \frac{1}{(1+e^x)^2}.$$

When $\alpha = 0.5$, then

$$D_{*t}^{1/2} u(x,t) = \frac{\partial^{1/2} u}{\partial t^{1/2}} \approx \frac{\partial^{1/2} U}{\partial t^{1/2}} = H^T(x)C \frac{\partial^{1/2}}{\partial t^{1/2}} H(t) = H^T(x)CQ_H^{-1/2} H(t) \quad (22)$$

$$u_{xx}(x,t) = \frac{\partial^2 u(x,t)}{\partial u^2} \approx H^T(x)(Q_H^{-2})^T CH(t) \quad (23)$$

$$u(x,t) \approx U(x,t) = H^T(x)CH(t) \quad (24)$$

Substituting (22) - (24) into (21),

$$\begin{aligned} & H^T(x)CQ_H^{-1/2}H(t) - H^T(x)(Q_H^{-2})^T CH(t) - 6H^T(x)CH(t) \\ & - 6H^T(x)CH(t)H^T(x)CH(t) = 0 \end{aligned} \quad (25)$$

$$\text{We obtain } CQ_H^{-1/2} - (Q_H^{-2})^T C - 6C - 6C^2 = 0 \quad (26)$$

Comparison of the Numerical solutions and exact solution shown in Table 1, Figure 1, Figure 2, Figure 3 and Figure 4.

The calculation results show that, by combining with wavelet matrix, the above method can obtain the numerical solution of time - fractional order partial differential equations including fractional derivative. Numerical example shows the effectiveness and feasibility of this method. From the result, numerical solution of the method in this paper approximates the exact solution. It's more simple and convenient than difference format.

5. Conclusion

This paper presents a numerical method by combining wavelet function with operational matrix and dispersing given function effectively. Taking advantage of good properties of orthogonal and sparseness of wavelet matrix, we transform time fractional differential equation into matrix equation. It's easy to be solved. Numerical example shows that as m increases, the numerical solution approximates exact solution and the method above is an efficient algorithm.

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Table 1. Let $m = 8$, $m = 16$, $m = 32$, $m = 64$, by calculating with software MATLAB we can obtain
Let $\alpha = 0.5$ and $m=64$, the comparisons of numerical solution and exact solution are as follows:

t	x	$m = 64$	u_{Exact}
0.1	0.25	0.289700	0.316042
	0.50	0.231977	0.250000
	0.75	0.181667	0.191689
	1.0	0.139471	0.142537
0.2	0.25	0.437823	0.461284
	0.50	0.377540	0.387456
	0.75	0.320821	0.316042
	1.0	0.268941	0.250000
0.3	0.25	0.516463	0.604195
	0.50	0.458744	0.534447
	0.75	0.402726	0.461284
	1.0	0.349724	0.387456

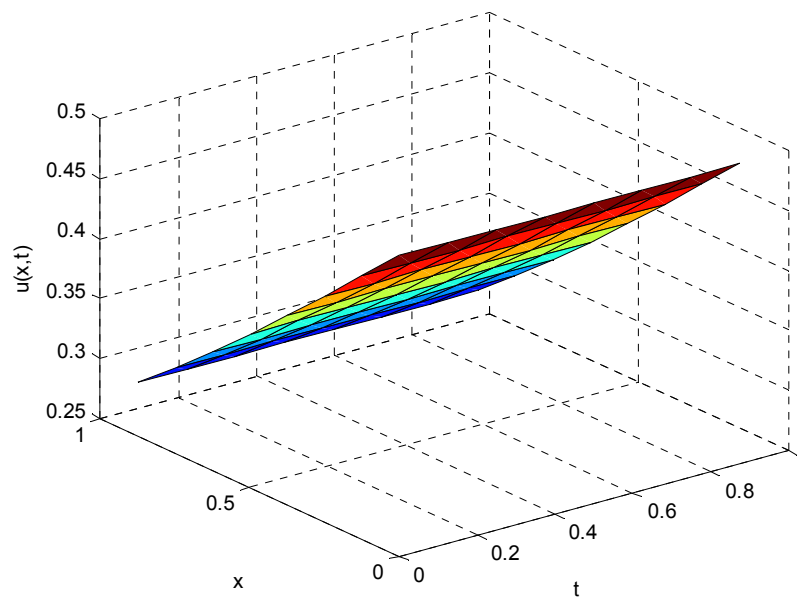
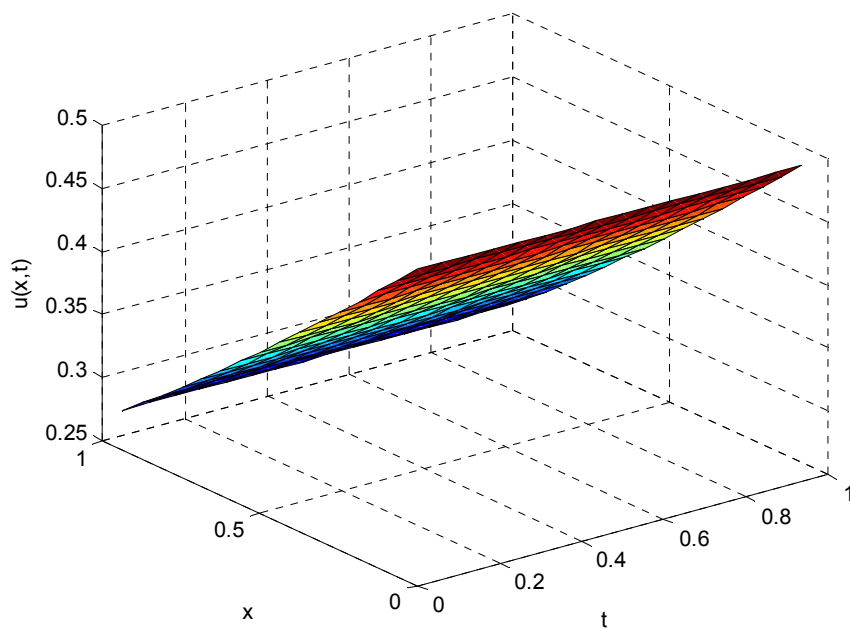
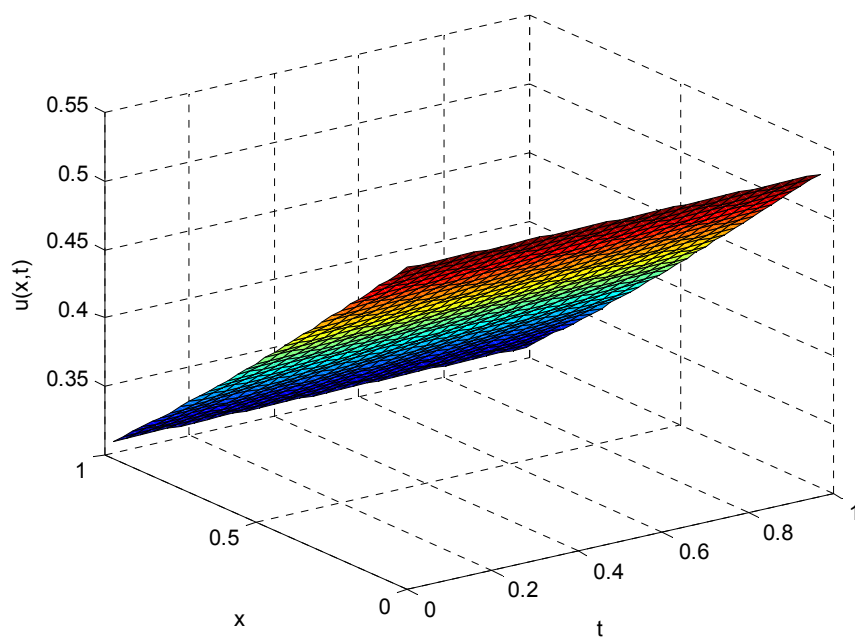


Figure 1. Numerical solution of $m=8$.

Figure 2. Numerical solution of $m=16$.Figure 3. Numerical solution of $m=32$.

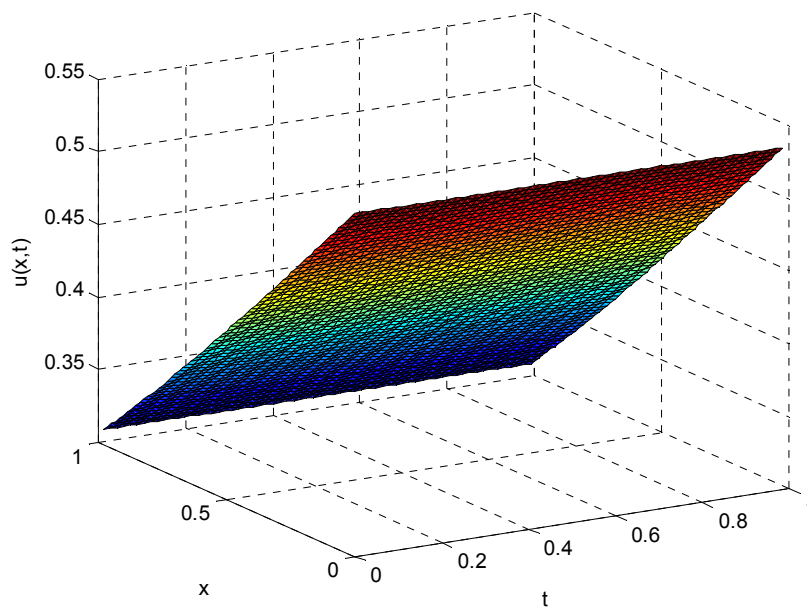


Figure 4. Numerical solution of $m=64$.