An Analytical and Numerical Investigation of the Optical Vortex Solitons: A Computational Study

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Abstract

In this paper, we investigate the possibility of forming spatiotemporal vortex solitons in the dispersive inhomogeneous non-linear optical fibers using a graded-index Kerr medium as an example. Also, we use a variation approach to solve the multidimensional inhomogeneous, the non-linear Schrödinger equation (NLSE) and show that spatiotemporal vortex solitons can be stabilized under certain conditions.

Keywords: Singularity, Soliton, Light bullet, Vortex

1. Introduction

Over the past decade, the optical temporal-spatial vortex solitons called "Vortex Optical Bullets" have attracted a great deal of interest. They are multidimensional pulses whose shape will be remained unchanged during the time and in a spatial dimension due to the balance among the group-velocity dispersion, diffraction, and phase modulation; furthermore, they possess phase unity (Russell, 1844; Schiesser, 1994; Wadati, 2001; Mollenauer, et al., 1980; Desyatnikov, et al., 2005; Kosterlitz & Thouless, 1973).

Studying the optical vortexes is of importance in terms of pure and applied physics. It is expected that the unique nature of vortex fields result in applications in various fields such as storing the optical data and processing them. Because the vortex solitons can turn during the propagation in the non-linear medium, this motion may be used for making the optical rotational switching based on the vortex solitons' concept (Carlsson, 2000). Owing to the temporal-spatial solitons' extremely fast all-optical switching ability in a mass medium, they are expected to appear as a new fundamental physical creature in the photonic. Among other applications, the optical bullets are of great significance in the all-optical digital logic (Kivshar & Agrawal, 2003; Young, 1802; Whewell, 1833; Winfree, 1995; Soskin & Vasnetsov, 2001; Nye & Berry, 1974; Pismen, 1999).

The optical vortexes can be applied for trapping tiny particles. Also, another special application of optical temporal-spatial vortex solitons includes conducting the waves as the waveguide. Take the step-index waveguides for instance, due to their variable refraction index in different points of the medium resulted from its structure, they lead to light conduction and formation of waveguide. In the presence of optical vortex, its propagation in the medium can induce the inclined refraction index in the medium, and consequently, as it is conducted in the medium, it conducts other waves and creates a waveguide bringing about producing all-optical circuits (Carlsson, 2000).

The optical temporal-spatial vortex solitons in the third-order non-linear and dispersive media have been previously studied. Here, we assume that the studied medium is heterogeneous besides having those properties. Therefore, we want to investigate the possibility of forming the optical temporal-spatial vortex solitons in the

dispersive heterogeneous optical fibers with the third-order non-linearity (Kerr medium). To solve the multidimensional non-linear Schrödinger equation, we utilize a variational method, and indicate that the optical temporal-spatial vortex solitons can be stable under certain conditions (Mihalache, et al., 2006; 2002; Novikov, 1984; Hasegawa & Tappert, 1973; Iannone, et al., 1998; Agrawal, 1989; Bergé, 1998; Jackiw & So-Young, 1990; Nore, et al., 1993; Velchev, 1997; Chen & Atai, 1992; Tikhonenko, et al., 1998; Kivshar, et al., 1998; Królikowski, et al., 1998; Kivshar & Yang, 1994; Rubinstein & Pismen, 1994; Luther-Davies, et al., 1994; Staliunas, 1994; Butylkin, et al., 1989; Kuznetsov, et al., 1986; Kuznetsov & Turitsyn, 1988; Kivshar & Yang, 1994; Mamaev,et al., 1996; Lugomer, et al., 1998; Law & Swartzlander, 1994; Tilkhonenko, et al., 1996; Swartzlander, 1992; Luther-Davies, et al., 1997;Mamaev, et al., 1996; Kivshar & Luther-Davies, 1998; Neshev, et al., 2004; Kivshar & Peyrard, 1992; Berry, 1984; Desyatnikov & Kivshar, 2001).

2. The Non-linear Schrödinger Equation (NLSE)

In order to study the vortex optical bullets in the fiber non-linear medium, we assume that the medium possesses Kerr non-linearity, and in contrast with the studied cases, we consider the medium to be heterogeneously. Consequently, the refraction index of such medium can be given by:

$$n(r,\omega) = n_0(\omega) + n_1(x^2 + y^2) + n_2 |E|^2 \tag{1}$$

where the homogenous part $n_0(\omega)$ results from the color dispersion, n_2 results from the non-linear response of the self-focusing or self-defocusing medium, and n_1 is the changes in the refraction index in the transverse dimensions of x and y. We assume that the light propagates along the axis z. When $n_1 > 0$, the medium is conductive, and when $n_1 < 0$, the medium is anti-conductive.

We begin by analyzing Eq. (1) using the Maxwell equations. The oscillating electric field in the frequency ω_0 can be considered as:

$$E(r,t) = \frac{1}{2}\hat{e}A'(r,t)\exp\left[i(\beta_0 z - \omega_0 t)\right] + c.c. \tag{2}$$

where $\hat{e}$ is the unit vector, $\beta_0 = n_0(\omega_0)k_0$, the wave number $k_0 = \omega/c$, and $A'(r,t)$ is the field envelope. Applying the slowly varying envelope approximations, we will attain the equation governing the envelope. We begin with the wave equation:

$$\nabla^2 \vec{E} - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{3}$$

From Eq. (1), we have:

$$n^2(r,\omega) = \left[n_0(\omega) + n_1(x^2 + y^2) + n_2 |E|^2\right]^2$$

$$= n_0^2(\omega_0) + n_1^2(x^2 + y^2)^2 + n_2^2 |E|^4 + 2n_0(\omega_0)n_1(x^2 + y^2) + 2n_0(\omega_0)n_2 |E|^2 + 2n_1(x^2 + y^2)n_2 |E|^2 \tag{4}$$

In the above-mentioned terms, we can ignore the orders higher than n_1 and n_2. The wave equation's derivations are calculated as follows:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{2}\hat{e}\left\{\frac{\partial^2 A}{\partial x^2}\exp i(\beta_0 z - \omega_0 t) + C.C\right\}$$

$$\frac{\partial^2 E}{\partial y^2} = \frac{1}{2}\hat{e}\left\{\frac{\partial^2 A}{\partial y^2}\exp i(\beta_0 z - \omega_0 t) + C.C\right\}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{2}\hat{e}\left\{2i\beta_0\frac{\partial A}{\partial z}\exp i(\beta_0 z - \omega_0 t) - \beta_0^2 A\exp i(\beta_0 z - \omega_0 t) + C.C\right\}$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{2}\hat{e}\left\{\frac{\partial^2 A}{\partial t^2}\exp i\left(\beta_0 z - \omega_0 t\right) - 2i\omega_0 \frac{\partial A}{\partial t}\exp i\left(\beta_0 z - \omega_0 t\right) - \omega_0^2 A \exp i\left(\beta_0 z - \omega_0 t\right) + C.C\right\} \tag{5}$$

Inputting Eqs. (4) and (5) to Eq. (3), we have:

$$\frac{1}{2\beta_0}\nabla_\perp^2 A' + i\frac{\partial A'}{\partial z} - \frac{1}{2\beta_0}\frac{n_0^2}{c^2}\frac{\partial^2 A'}{\partial t^2} - \frac{n_0 n_1\left(x^2 + y^2\right)}{\beta_0 c^2}\frac{\partial^2 A'}{\partial t^2} - \frac{n_0 n_2 |E|^2}{\beta_0 c^2}\frac{\partial^2 A'}{\partial t^2} + i\omega_0 \frac{n_0^2}{\beta_0 c^2}\frac{\partial A'}{\partial t}$$

$$+ i\omega_0 \frac{2n_0 n_1\left(x^2 + y^2\right)}{\beta_0 c^2}\frac{\partial A'}{\partial t} + i\omega_0 \frac{2n_0 n_2 |E|^2}{\beta_0 c^2}\frac{\partial A'}{\partial t} + A'n_1\left(x^2 + y^2\right)k_0 + k_0 n_2 |A'| A' = 0 \tag{6}$$

where $\beta_0 = n_0\left(\omega_0\right)k_0$, $k_0 = \omega/c$, and $\nabla_\perp^2$ is the transverse Laplacian operator defined as $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

From Eq. (2), we can pull out the terms including $i\frac{\partial A'}{\partial t}$, and then we have:

$$\left\{\omega_0 \frac{n_0^2}{\beta_0 c^2} + \omega_0 \frac{2n_0 n_1\left(x^2 + y^2\right)}{\beta_0 c^2} + \omega_0 \frac{2n_0 n_2 |E|^2}{\beta_0 c^2}\right\}i\frac{\partial A'}{\partial t} = \left\{\frac{n_0}{c} + \frac{2n_1\left(x^2 + y^2\right)}{c} + \frac{2n_2 |E|^2}{c}\right\}i\frac{\partial A'}{\partial t} \tag{7}$$

We know that:

$$\beta = \frac{n(\omega)\omega}{c}; \quad \beta_1 = \frac{d\beta}{d\omega}\bigg|_{\omega=\omega_0}; \quad \beta_2 = \frac{d^2\beta}{d\omega^2}\bigg|_{\omega=\omega_0}; \quad \beta_1 = \frac{n(\omega)}{c} + \frac{dn}{d\omega}\frac{\omega}{c}\bigg|_{\omega=\omega_0} \tag{8}$$

Now, using the Caushi equation, we have:

$$\frac{dn}{d\omega} = A'\omega \tag{9}$$

It can be written:

$$\beta_1 = \frac{n(\omega)}{c} + \frac{A'\omega^2}{c} = \frac{n_0(\omega) + n_1\left(x^2 + y^2\right) + n_2 |E|^2 + A'\omega^2}{c} \tag{10}$$

Also, we assume the following equation is established:

$$A'\omega^2 = n_1\left(x^2 + y^2\right) + n_2 |E|^2 \tag{11}$$

This assumption is based on this principle that to form the temporal-spatial solitons, in Eq. (10), the spatial and temporal parts of $n_2 |E|^2$ are neutralized with $n_1\left(x^2 + y^2\right)$ and $A'\omega^2$, respectively. Therefore, we have:

$$\beta_1 = \frac{n_0(\omega)}{c} + \frac{2n_1\left(x^2 + y^2\right)}{c} + \frac{2n_2 |E|^2}{c} \tag{12}$$

where this is the coefficient of $i\frac{\partial A'}{\partial t}$ mentioned in Eq. (6).

Accordingly, we pull out the terms including $\frac{\partial^2 A'}{\partial t^2}$ in Eq. (6), and write:

$$\left\{\frac{1}{2\beta_0}\frac{n_0^2}{c^2} + \frac{n_0 n_1\left(x^2 + y^2\right)}{\beta_0 c^2} + \frac{n_0 n_2 |E|^2}{\beta_0 c^2}\right\}\frac{\partial^2 A'}{\partial t^2} = \frac{1}{2\omega_0 c}\left\{n_0 + 2n_1\left(x^2 + y^2\right) + 2n_2 |E|^2\right\}\frac{\partial^2 A'}{\partial t^2} \tag{13}$$

On the other hand:

$$\beta_2 = \frac{2}{c}\frac{dn}{d\omega} + \frac{\omega}{c}\frac{d^2n}{d\omega^2}\Big|_{\omega=\omega_0} \tag{14}$$

Employing the Caushi equation mentioned in Eq. (9), the above equation is rewritten as the form below:

$$\beta_2 = \frac{2}{c}\frac{dn}{d\omega} + \frac{\omega}{c}\frac{d^2n}{d\omega^2}\Big|_{\omega=\omega_0} = \frac{2}{c}\frac{A'\omega^2}{\omega} + A'\frac{\omega^2}{c\omega} \tag{15}$$

Using Eq. (11), we obtain:

$$\beta_2 = \frac{2}{c}\frac{n_1\left(x^2+y^2\right)+n_2|E|^2}{\omega} + \frac{A'\omega^2}{c\omega} \tag{16}$$

The next assumption is as follows:

$$n_0(\omega) = A'\omega^2 \tag{17}$$

Actually, this assumption is correct owing to the fact that the dispersion equation for n is the same as the dispersion equation for n_0 because n_0 is the only function of ω, and any ω functionality is not appeared in n_1 and n_2.

Generally, the Caushi equation is written as the following form:

$$n = A + \frac{B}{\lambda^2} \tag{18}$$

from which Eq. (9) derives, and Eq. (11) is the same as the Caushi equation without the constant A. Consequently, we have:

$$\beta_2 = \frac{1}{\omega c}\left\{n_0(\omega) + 2n_1\left(x^2+y^2\right) + 2n_2|E|^2\right\} \tag{19}$$

It means that the coefficient of $\dfrac{\partial^2 A'}{\partial t^2}$ in Eq. (6) is $\dfrac{\beta_2}{2}$.

Through inserting β_1 and β_2 into Eq. (2), we have:

$$i\left(\frac{\partial A'}{\partial z} + \beta_1\frac{\partial A'}{\partial t}\right) - \frac{\beta_2}{2}\frac{\partial^2 A'}{\partial t^2} + \frac{1}{2\beta_0}\nabla_\perp^2 A' + k_0 n_1\left(x^2+y^2\right)A' + k_0 n_2|A'|^2 A' = 0 \tag{20}$$

The above equation, which will be used in the vortex solitons' equation, is the same as the equation governing the electric field envelope.

3. The Mathematical forMulating of Vortex Optical Bullets

To find the solutions to Eq. (20), we initially normalize this equation. We need to consider some quantities to normalize. We define the transverse (length) scale and transverse-coordinates scale as $h_0 = \left(2k_0|n_1|\beta_0\right)^{-1/4}$ and $(X,Y) = (x,y)/h_0$, respectively. Similarly, the longitudinal (length) scale and longitudinal-coordinates scales are respectively defined as $L_d = \beta_0 h_0^2$ (the diffraction length) and $Z = z/L_d$. Also, the time is scaled as $\tau = (t - \beta_1 z)/T_0$ where $T_0 = \sqrt{|\beta_2|L_d}$.

Regarding the introduced quantities, the derivatives with the following equations are related to the new coordinates:

$$\frac{\partial^2}{\partial x^2} = \left(\frac{1}{\omega_0}\right)^2\frac{\partial^2}{\partial X^2}; \quad \frac{\partial^2}{\partial y^2} = \left(\frac{1}{\omega_0}\right)^2\frac{\partial^2}{\partial Y^2} \tag{21}$$

and also:

$$\frac{\partial}{\partial z} = \frac{1}{L_d}\frac{\partial}{\partial Z} - \frac{\beta_1}{T_0}\frac{\partial}{\partial \tau}; \quad \frac{\partial^2}{\partial t^2} = \left(\frac{1}{T_0}\right)^2 \frac{\partial^2}{\partial \tau^2} \qquad (22)$$

According to the above equations, Eq. (20) is transformed into the normalized form below:

$$i\frac{\partial U}{\partial Z} + \frac{1}{2}\left(\nabla_\perp^2 U\right) + \frac{\delta}{2}\frac{\partial^2 U}{\partial \tau^2} - \frac{s}{2}r^2 U + \upsilon|U|^2 U = 0 \qquad (23)$$

where $U(X,Y,Z,\tau) = \sqrt{k_0 n_2 L_d}\, A'(x,y,z,t)$, and the parameter r is a coordinate of the polar coordinates which being related to the transverse coordinates through $r^2 = X^2 + Y^2$. Moreover, the parameters $\delta = \pm 1$, $s = \pm 1$, and $\upsilon = \pm 1$ respectively denote β_2, n_1, and n_2 which imply the medium with normal or abnormal group-velocity dispersion, conductive or anti-conductive medium, and self-focusing or self-defocusing medium, respectively. This equation is similar to the multidimensional standard non-linear Schrödinger equation (NLSE) and the main difference relates to the heterogeneity of non-linear medium.

It can be indicated that the following Lagrangian can be attributed to the above non-linear Schrödinger equation because the above equation is considered as the motion equation and the same answers can be achieved by both equations:

$$l = -\operatorname{Im}\left(U^* \frac{\partial U}{\partial Z}\right) - (H) \qquad (24)$$

where H is the system's Hamiltonian. We have the Lagrangian opened form below:

$$l = -\operatorname{Im}\left(U^* \frac{\partial U}{\partial Z}\right) + \left(\frac{1}{2}|\nabla U|^2 + \frac{\delta}{2}\left|\frac{\partial U}{\partial \tau}\right|^2 + \frac{s}{2}r^2|U|^2 - \frac{\upsilon}{2}|U|^4\right) \qquad (25)$$

We know that the Euler's theorem is written as follows:

$$\frac{d}{dq_k} = \left(\frac{\partial l}{\partial U/\partial q_k}\right) - \frac{\partial l}{\partial U^*} = 0; \quad q_k = X,Y,Z,\tau \qquad (26)$$

Therefore, utilizing Eq. (25) and applying the Euler's theorem, the following equations can be obtained:

$$\frac{d}{dX}\left(\frac{\partial l}{\partial U/\partial X}\right) = \frac{1}{2}\frac{\partial^2 U}{\partial X^2} \qquad (27)$$

$$\frac{d}{dY}\left(\frac{\partial l}{\partial U/\partial Y}\right) = \frac{1}{2}\frac{\partial^2 U}{\partial Y^2} \qquad (28)$$

$$\frac{d}{dZ}\left(\frac{\partial l}{\partial U/\partial Z}\right) = 0 \qquad (29)$$

$$\frac{d}{d\tau}\left(\frac{\partial l}{\partial U/\partial \tau}\right) = \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} \qquad (30)$$

Through inputting the above-mentioned equations to the Euler's theorem, we reach to Eq. (23) that this case indicates the Lagrange equation's accuracy for this system, and consequently, the Lagrangian's correctness becomes validated.

Selecting the trial solution is the determinant factor in the variation method's success. In Eq. (23), it can be observed that, in the temporal dimension, the combination of group-velocity dispersion and self-defocusing can lead to the sech light soliton. In addition, in the transverse dimensions of waveguide, the gradual index is turned to the Hermite–Gaussian modes such that the basic mode is in the Gaussian form. In the absence of such trapping, the temporal and spatial dimensions must behave symmetrically; however, because the transverse

trapping is strong, we consider the trial function as the product of a chirped sech pulse and a chirped Gaussian beam. Furthermore, we add the singularity with a power such that:

$$U(X,Y,Z,\tau) = A\operatorname{sech}\left(\frac{\tau}{w}\right)r^m \exp\left(-\frac{r^2}{2a^2} + i\Phi\right) \tag{31}$$

where $\Phi = m\theta + \alpha r^2 + \beta\tau^2 + \varphi$ is the phase function. $\alpha(Z)$ and $\beta(Z)$ are the wavefront curvatures (chirps), $\varphi(Z)$ is the free phase, $\theta = \tan^{-1}(Y/X)$ is a coordinate of the cylindrical coordinates, and m is an integer that indicates the phase wrap around the intensity loop, and is called the vortex topological charge. In addition to the radial modes, such a number determines the higher-order directional steady states. We normalize U such that $\int |U|^2 dX dY d\tau = \varepsilon$. By this definition and using the above-mentioned integrals, the pulse energy constant is found that equals $\varepsilon = 2\pi m! A^2 a^{2m+2} w$. Therefore, we replace the constant A in Eq. (31) with its equivalent, $A = \sqrt{\varepsilon/2\pi m! a^{2m+2} w}$.

In accordance with the standard method, the effective Lagrangian is calculated as follows:

$$L = \langle l \rangle = \int l(u,u^*) dX dY d\tau \tag{32}$$

Through inserting the Lagrangian of Eq. (25) into Eq. (32) and integrating, we obtain some integrals such as the temporal integrals:

$$\int \operatorname{sech}^2(\eta\tau) d\tau = \frac{2}{\eta} \tag{33}$$

$$\int \operatorname{sech}^4(\eta\tau) d\tau = \frac{4}{3\eta} \tag{34}$$

$$\int \operatorname{sech}^2(\eta\tau)\tanh^2(\eta\tau) d\tau = \frac{2}{3\eta} \tag{35}$$

$$\int \tau^2 \operatorname{sech}^2(\eta\tau) d\tau = \frac{\pi^2}{6\eta^3} \tag{36}$$

and also, the spatial integrals which have a solution proportional to the Gama function:

$$\int_0^\infty \exp(-u)u^n du = 2\int_0^\infty \exp(-u^2)u^{2n+1} du = n! = \Gamma(n+1) \tag{37}$$

$$\int \exp\left[-\frac{x^2+y^2}{a^2}\right](x^2+y^2)^m dx\, dy = \int \exp\left[-\frac{\rho^2}{a^2}\right](\rho^2)^m \rho\, d\rho\, d\theta = \pi a^{2(m+1)}\Gamma(m+1) \tag{38}$$

$$\int \exp\left[-\frac{x^2+y^2}{a^2}\right](x^2+y^2)^{m-1} dx\, dy = \pi a^{2m}\Gamma(m) \tag{39}$$

$$\int \exp\left[-\frac{x^2+y^2}{a^2}\right](x^2+y^2)^{m+1} dx\, dy = \pi a^{2(m+2)}\Gamma(m+2) \tag{40}$$

$$\int \exp\left[-\frac{2(x^2+y^2)}{a^2}\right](x^2+y^2)^{2m} dx\, dy = \pi \frac{a^{2(2m+1)}}{2^{2m+1}}\Gamma(2m+1) \tag{41}$$

Consequently, using the above-mentioned equations, the effective Lagrangian is obtained as follows:

$$L = \varepsilon\left\{(m+1)a^2 \frac{d\alpha}{dZ} + \frac{w^2\pi^2}{12}\frac{d\beta}{dZ} + \frac{d\varphi}{dZ}\right.$$

$$+\frac{1}{2}\left[4\alpha^2 a^2(m+1)+\frac{(m+1)}{a^2}+\frac{\delta}{3w^2}+\frac{\delta\beta^2 w^2\pi^2}{3}+s(m+1)a^2\right]-\frac{1}{2^{2m+1}}\frac{\upsilon\varepsilon}{6wa^2\pi}\frac{(2m)!}{(m!)^2}\Bigg\}$$ (42)

Then, by the means of the Lagrange–Euler equations, the following equations are achieved:

$$\frac{dw}{dZ}=2\beta\delta w$$ (43)

$$\frac{da}{dZ}=2a\alpha$$ (44)

$$\frac{d\beta}{dZ}=\frac{2\delta}{\pi^2 w^4}-2\delta\beta^2-\frac{2\upsilon\tilde{\varepsilon}}{\pi^2 w^3 a^2}$$ (45)

$$\frac{d\alpha}{dZ}=-2\alpha^2-\frac{s}{2}+\frac{1}{2a^4}-\frac{\upsilon\tilde{\varepsilon}}{3(m+1)wa^4}$$ (46)

where $\tilde{\varepsilon}=\varepsilon(2m)!\Big/2^{2m+2}\pi(m!)^2$

Now, using the above-mentioned results, it can be indicated that the paired equations governing on the spatial width a and pulse durability w include:

$$\frac{d^2 a}{dZ^2}=-sa+\frac{1}{a^3}-\frac{2}{3}\frac{\upsilon\tilde{\varepsilon}}{(m+1)a^3 w}\equiv I(a,w)$$ (47)

$$\frac{d^2 w}{dZ^2}=\frac{4}{\pi^2}\left(\frac{1}{w^3}-\frac{\upsilon\delta\tilde{\varepsilon}}{w^2 a^2}\right)\equiv J(a,w)$$ (48)

The first term at the right side of Eqs. (47) and (48) are respectively related to the diffraction and dispersion, and the last terms being proportional to the pulse energy indicate the effect of non-linearity.

4. Results and Discussion

The solitonic answers w_0 and a_0 can be obtained at the steady limit of Eqs. (47) and (48), when their left side tends to zero. For $s=0$, the replaced transverse answers, $a_0<\infty$, solely exist for $\upsilon=+1$ (self-focusing media). While for $s=1$, ignoring the sign of υ, the field becomes trapped in the waveguide. Particularly for $\varepsilon=0$, the linear conducted mode with $a_0=1$ is appeared as the continuous answer to the work with $w_0=\infty$. In contrast, the solitonic pulses with the final width, $w_0=\dfrac{a_0^2}{\delta\upsilon\tilde{\varepsilon}}<\infty$, exist only when $\delta\upsilon=+1$. In the steady state, the spatial width is given by the positive roots of the equation below:

$$sa'^3-a'+\gamma=0$$ (49)

where $a'=a_0^2$ and $\gamma=2\delta\tilde{\varepsilon}^2/3(m+1)$. For $s=0$, the rotating optical bullets are only the steady answers for $\delta=\upsilon=+1$, and $a_0^2=2\tilde{\varepsilon}^2/3(m+1)$ [50-52]. For $s=1$, the above third-order polynominal has no positive answer in $\gamma>2/3\sqrt{3}$, and the first answer, $a_0^2=1/\sqrt{3}$, appears in $\gamma=2/3\sqrt{3}$. More decreasing the pulse energy, $0<\gamma<2/3\sqrt{3}$, we will have two answers and the vortex soliton will become double steady. Then through passing the linear conducted mode $\gamma=0$ and reversing the sign of non-linearity, $\gamma<0$ and $\delta=-1$, a singular answer similar to the reference (Raghavan & Agrawal, 2000) is obtained.

Here, we consider the non-linear medium to be the self-defocusing medium, and apply it to the illustration of diagrams by considering $\upsilon=-1$. As mentioned, the solitonic pulses with the final width, $w_0=a_0^2/\delta\upsilon\tilde{\varepsilon}$, exist only when $\delta\upsilon=+1$, because the width is always, $w_0<\infty$, a positive value. Therefore, this condition causes $\delta=-1$,

and also, this suggests that the medium is a medium with normal group-velocity dissemination. Moreover, we consider the non-linear medium to be a conductive medium which it requires applying the condition $s = +1$.

In Fig. 1, the vortex solitons for different topological charges are illustrated. In the part (a), it is observed that, for the small r (representing the vortex core), the intensity is zero. This case is clearly shown in the intensity distribution graph, Fig. 1b, for one of the topological charges. This intensity graph for each topological charge is as the following form. It can be observed that, for the higher order of m, the vortex soliton resembles loops around the center, the same as donut.

Next, we study the changes in the spatial width and the durability of optical temporal-spatial vortex soliton in the steady state and according to the pulse energy. Fig. (2) indicates that the rate of changes in the pulse spatial width according to the energy decline by increasing the order of topological charge, and inclines towards the lim $a_0 \to 1$ which is the same as the limit of linear conducted mode or continuous mode of work.

Fig. 3 indicates the changes in the pulse durability according to the pulse energy for different topological charges. As it is evident, the energy threshold corresponding to the continuous state of work, $w_0 \to \infty$, rises with increasing the topological charge.

To investigate the stability of temporal-spatial solitons, it is necessary for Eqs. (47) and (48) to become linear around the steady point. For this purpose, we consider the answers as $a(Z) = a_0 + p(Z)$ and $w(Z) = w_0 + q(Z)$ such that $p(Z) \Box a_0$ and $q(Z) \Box w_0$ are considered as the disturbance terms.

For the above small disturbances, Eqs. (47) and (48) can be indicated as the following matrix:

$$\frac{d^2}{dZ^2}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} I_a & I_w \\ J_a & J_w \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{50}$$

where the quantities of J_w, J_a, I_w, and I_a are the partial derivatives of J, I with respect to a, w, spatial width, and pulse durability in the steady values. The eigenvalues of above-mentioned matrix include:

$$\lambda = \pm \frac{1}{2}\left(I_a + J_w \pm \sqrt{(I_a + J_w)^2 - 4(I_a J_w - I_w J_a)} \right) \tag{51}$$

The soliton is stable when the eigenvalues have no positive actual value. It can be indicated that the equations governing the above disturbances are as follows:

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}\exp\left(\sqrt{\lambda}Z\right) \tag{52}$$

where $p_0 = p(Z = 0)$ and $q_0 = q(Z = 0)$.

The impact of disturbances on the pulse width, a, for different topological charges is demonstrated in Fig. 4. It is observed that the pulse width oscillates due to the energy and specific disturbances while propagating around the steady position. This oscillation has higher frequency for the higher topological charge.

As indicated in Fig. 5, the effect of disturbance on the pulse durability is similar to its impact on the pulse width for different topological charges. This similarity can be clearly found in the above figures.

5. Conclusion

Such stability in the pulse durability and pulse width results in the formation of temporal-spatial vortex solitons during the propagation in the fiber. Previously, the temporal-spatial solitons called optical bullets have been introduced. Here, since the intensity is zero in the center of vortex, we can see the hollow optical bullets like what indicated in Fig. 6.

Acknowledgments

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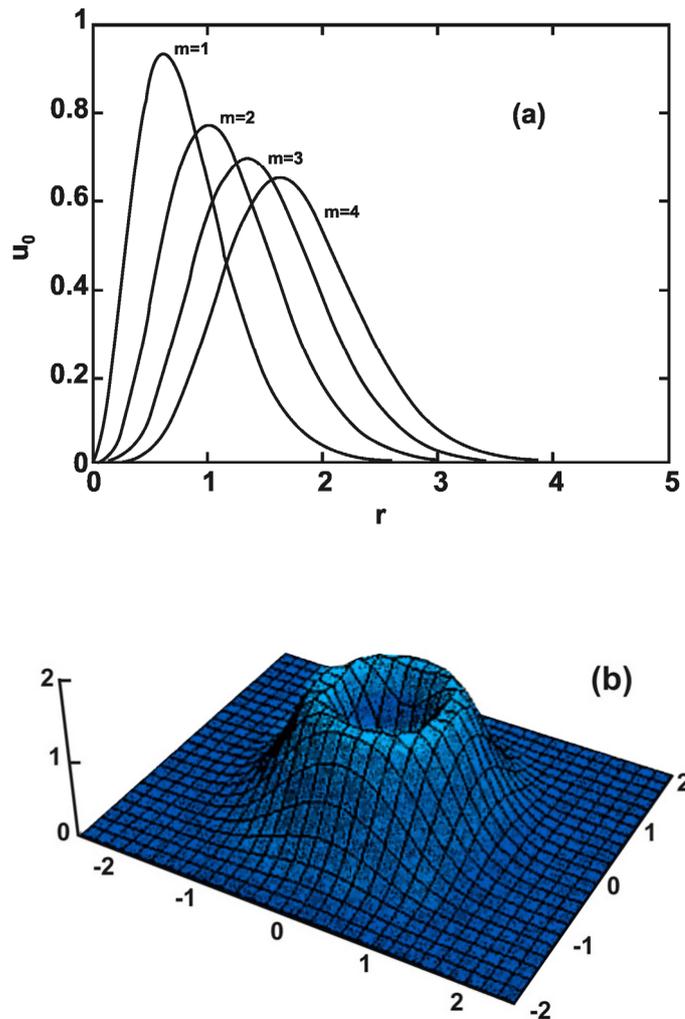
Figure 1. (a) The radial profile of the vortex solitons for different topological charges m; (b) the intensity distribution of vortex soliton for each topological charge
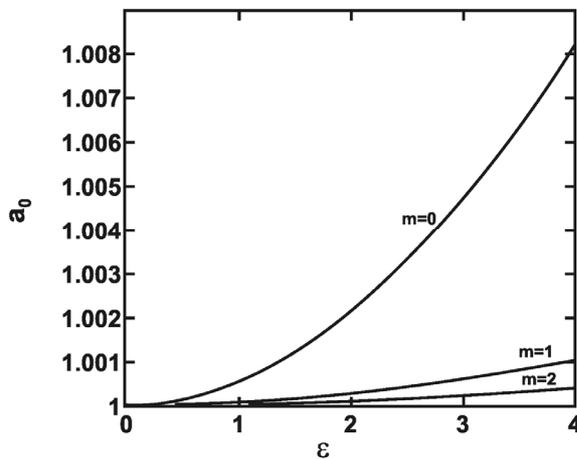
Figure 2. The changes in the spatial width according to the energy for different topological charges *m*
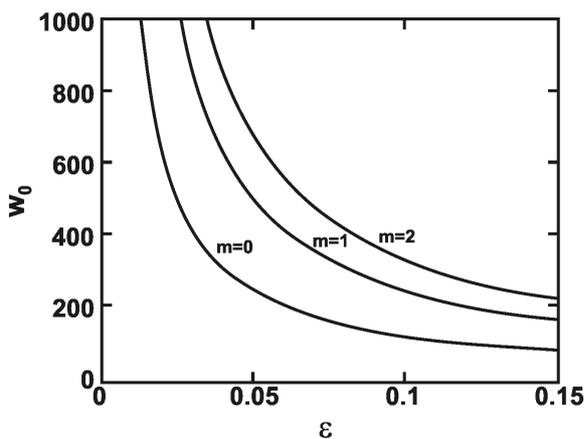
Figure 3. The changes in the pulse durability according to the energy for different topological charges *m*
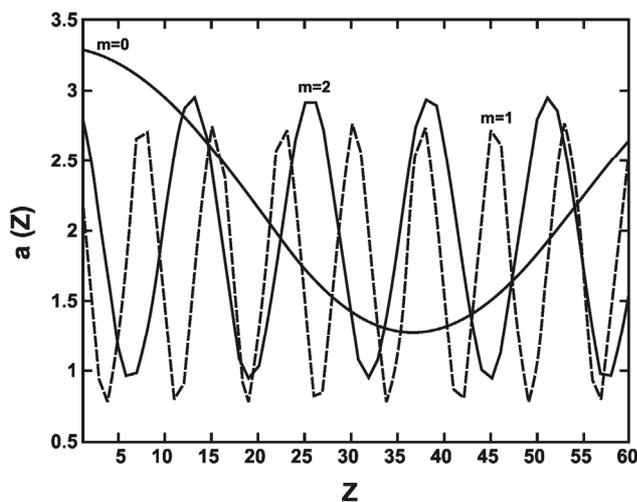
Figure 4. The effect of disturbance on the propagated pulse width in the direction Z for different *m*
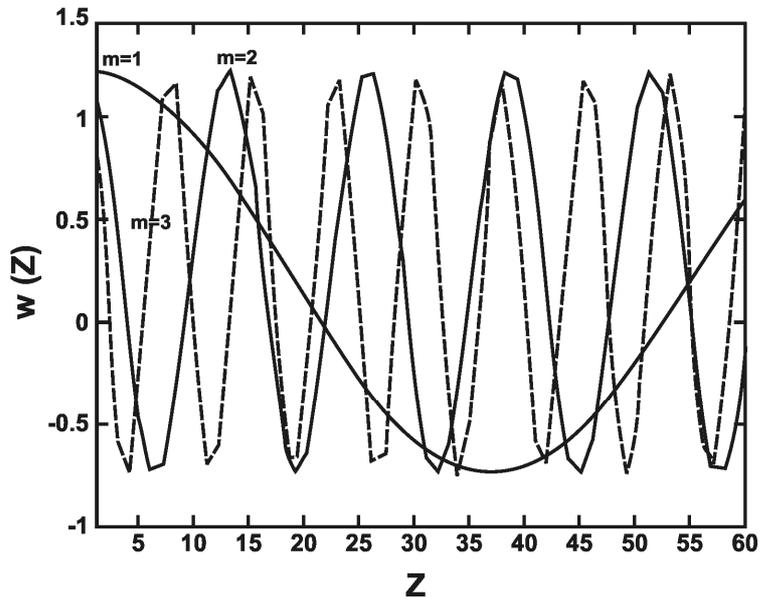
Figure 5. The effect of disturbance on the propagated pulse durability in the direction Z for different m

Figure 6. The vortex optical bullet